Chapter 5

Galactic dynamo action in presence of stochastic $\alpha$-effect and shear

5.1 Introduction

In the last chapter we studied MFD models using a coherent $\alpha$-effect and shear to explain large-scale fields observed in disc galaxies. We discussed how galactic dynamos based on such coherent $\alpha$-effect and shear are constrained by magnetic helicity conservation. In the present chapter, we explore the possibility of large-scale dynamo action in galaxies based simply on random fluctuations in the kinetic $\alpha$-effect. Vishniac & Brandenburg (1997) first pointed out the possibility of efficient dynamo action resulting from random fluctuations in the $\alpha$-effect in combination with shear. They investigated a reduced mean-field dynamo model appropriate to accretion discs and showed that growth can occur for large enough random fluctuations in alpha. Since then, several authors have elaborated various aspects of this stochastic alpha-shear dynamos (Sokoloff, 1997; Silantev, 2000; Fedotov et al., 2006; Proctor, 2007; Kleeorin & Rogachevskii, 2008). In particular, Sokoloff (1997) examined a model of a disc dynamo with a fluctuating alpha antisymmetric in space but which changes sign randomly with equal probability. He argued that intermittent large-scale magnetic fields can grow. The role of a stochastic $\alpha$ has also been analyzed in the context of solar dynamos (Proctor, 2007; Brandenburg & Spiegel, 2008; Moss et al., 2008).

In spite of it's possible role in a variety of astrophysical contexts, the exact
origin of such an incoherent α-effect is as yet unclear. In any large Reynolds number
system, many degrees of freedom exist, and hence there could always be a stochastic
component of the mean emf. This could lead to an additive or a multiplicative noise
in the MFD equations. Additive noise provides a seed field for the dynamo, whereas
multiplicative noise in say the α-effect, combined with shear, can lead to exponential
growth of the mean field. In the solar context, Hoyng (1993) argued for a fluctuations
\( \sim u_0/ \sqrt{M} \), where \( u_0 \) is the turbulent velocity and \( M \) is the number of cells being
averaged over in defining the mean field. In principle this can even be larger
than any coherent α-effect. Multiplicative noise is also seen in simulations which
measure the α-effect both in both the kinematic regime (see Fig. 2.2 in Chapter 2)
as well as in the nonlinear regime (Cattaneo & Hughes, 2006; Brandenburg et al.,
2008a) and also in direct simulations of the galactic dynamo (Gressel et al., 2008).
In fact Brandenburg et al. (2008a) measure an incoherent α-effect, with a Gaussian
probability density function (PDF), even in turbulence driven with a non-helical
forcing, where one does not expect a coherent α-effect. Combined with shear,
such systems show large-scale dynamo action (Brandenburg et al., 2008a; Yousef
et al., 2008). Here, we simply examine, in the context of galactic dynamos, the
consequence of having an incoherent alpha effect, without considering in detail its
exact origin. The growth of the mean field varies significantly from one realization
of the stochastic process to another, as also pointed out in Sokoloff (1997). It is
therefore necessary to examine a large number of realizations of the stochastic α to
determine the efficiency of the stochastic αω-galactic dynamo.

We first outline the basics of a one-dimensional stochastic αω-dynamo model
appropriate to galaxies. We then present numerical solutions of the above model
with two different probability distribution functions (PDF’s) for the stochastic alpha :
the first as considered in Sokoloff (1997) and the second where the stochastic alpha
has a Gaussian PDF. Possibility of alleviating catastrophic α-quenching in absence
of helicity fluxes by including the effects of a stochastic α is presented next. The
chapter concludes with a summary of our results and the implications of a stochastic
αω-dynamo for galaxies.

5.2 The stochastic alpha-shear dynamo

The one-dimensional stochastic α model for galaxies which we discuss here, is based
on the fundamental equations of the galactic dynamo introduced in Chapter 1.
In the case of the stochastic dynamo, we modify the α-effect to be of the form:
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\( \alpha = \alpha_k = \bar{\alpha}(z) + \alpha_1(z, t) \). Here \( \bar{\alpha}(z) \) is the average \( \alpha_k \), while \( \alpha_1(z, t) \) is the stochastic \( \alpha \) term. Therefore, the total \( \alpha \) is a sum of the standard kinetic alpha \( \bar{\alpha} \) and a stochastic component \( \alpha_1 \). Further, we consider a mean flow consisting of only a differential rotation such that \( \bar{U} = (0, r \Omega(r), 0) \). Then, going to dimensionless variables, one can write down the evolution equations for the azimuthal (\( \overline{B}_\phi \)) and radial (\( \overline{B}_r \)) fields, (see also Vishniac & Brandenburg (1997))

\[
\frac{\partial \overline{B}_r}{\partial t} = -\frac{\partial}{\partial z} \left( \alpha_0 g(z) \overline{B}_\phi + Q_a f(z) N \overline{B}_\phi \right) + \frac{\partial^2 \overline{B}_r}{\partial z^2},
\]

(5.1)

\[
\frac{\partial \overline{B}_\phi}{\partial t} = \rho \overline{B}_r + \frac{\partial}{\partial z} \left( \alpha_0 g(z) \overline{B}_r + Q_a f(z) N \overline{B}_r \right) + \frac{\partial^2 \overline{B}_\phi}{\partial z^2}
\]

(5.2)

Note that the above equations are similar to equations (1.56) and (1.57) except that there is an extra term proportional to \( Q_a \) which represents the effect of the stochastic alpha. As before, the length and time units are \( h \) and \( t_d = h^2/\eta \), respectively, with \( h \) the semi-thickness of the disc. We adopt \( \bar{\alpha} = \alpha_0 g(z) \), and \( \alpha_1 = \alpha_0 f(z) N(t) \) where \( f(z) = g(z) = \sin(\pi z) \) takes care of the symmetry condition. \( N(t) \) is a stochastic function. In our numerical solutions we adopt the following procedure: We split \( t \) into equally spaced intervals \([n \tau_c, (n+1) \tau_c]\), where \( \tau_c \) is the correlation time of the stochastic alpha, and \( n = 0, 1, 2, \ldots \) are integers. And in any such time interval \( N \) is a random number chosen from a Gaussian (or some other) probability distribution, with unit variance. The relevant dynamo control parameters are \( R_a, Q_a \) and \( R_\omega \) defined as

\[
R_a = \frac{\alpha_0 h}{\eta}, \quad Q_a = \frac{\alpha_0 h}{\eta}, \quad \text{and} \quad R_\omega = \frac{G h^2}{\eta}
\]

(5.3)

As introduced earlier in Chapter 4, \( G = rd\Omega/dr = -\Omega \), for a flat rotation curve. Again, from Krause's formula, \( \alpha_0 = l_0^2 \Omega/h \), where \( l_0 \) is the integral scale of interstellar turbulence. Then \( R_a \sim 3 \Omega t_{ed} \), assuming \( \eta \sim l_0 u_0/3 \) and \( \tau \sim t_{ed} = l_0/\eta \), the eddy turnover time. As mentioned earlier in §1.4.3, typical values of the dynamo control parameters in the solar neighborhood of the Milky Way could be \( R_a \sim 1.0 \), and \( |R_\omega| \sim 10^{-15} \), corresponding to a dynamo number \( D = R_a R_\omega \sim -10 \) to \( D \sim -15 \). The strength of \( Q_a \sim 3(h/l_0) M^{-1/2} \), if one uses the estimate of Hoyng (1993). A horizontal average over a scale \( h \) to define \( \bar{B} \) (cf. Brandenburg et al. (2008a)) would suggest \( M \sim (h/l_0)^2 \) and hence \( Q_a \sim 3 \). However since the exact origin of such fluctuations is as yet unclear, we will vary \( Q_a \) around these values. Thus in general, we will have \( |R_\omega| \gg R_a, Q_a \) so that one can make the standard \( \alpha \omega \)-dynamo approximation, where one neglects the terms with co-efficients \( R_a \) and \( Q_a \) in equation (5.2).
Spiral galaxies rotate faster towards the inner regions of the galactic disc. Therefore, \( \Omega \propto 1/r \), and thus one can have larger dynamo parameters towards the disc center, depending also on how \( h \) and \( l_0 \) behave there. The disc height could be smaller, but \( l_0 \) could also be smaller in the denser inner galactic regions, where supernovae are more confined. This could lead to a net increase in \( R_\alpha \propto \Omega h^2/l_0 \).

Any increase in \( R_\alpha \) depends on how much \( l_0 \) decreases compared to the increase in \( \Omega \). Changes in \( Q_\alpha \) depend on the origin of the \( \alpha \) fluctuations. For example, if \( h \) decreases by factor 2 and \( l_0 \) decreases a factor 5 in the inner galaxy, \( R_\alpha \) would increase by a factor 6.25(\( r/2\text{kpc} \))\(^{-1} \) and \( R_\alpha \) or \( Q_\alpha \) would remain about the same, compared to the solar neighborhood. Overall larger dynamo numbers can be expected in the inner regions of disc galaxies. We now turn to the solution of the stochastic \( \alpha \omega \)-dynamo equations.

### 5.3 Numerical Solutions

Our primary interest is in a scenario where large-scale dynamo action is possible in presence of stochastic alpha and shear. Thus we first seek numerical solutions to equations (5.1) and (5.2), in the \( \alpha \omega \)-dynamo approximation, with the coherent part of the \( \alpha \)-effect taken to be zero; that is with \( R_\alpha = 0 \). We extended the one-dimensional mean-field dynamo code used for studying the effects of magnetic helicity fluxes to include a stochastic component to the \( \alpha \)-effect. As earlier, we use vacuum boundary conditions for the fields

\[
\bar{B}_r = \bar{B}_\phi = 0 \quad \text{at} \quad z = \pm h
\]  

(5.4)

As a test case, we numerically implemented the Sokoloff (1997) model with \( \alpha = 0 \) and \( \alpha_1 = \alpha_{s}(z)N(t) \); \( N \) being either +1 or −1 with equal probability in any time interval \( n\tau_c < t < (n+1)\tau_c \). For \( N = 1 \), the system behaves as a standard \( \alpha \omega \)-dynamo with growing solutions, while for \( N = -1 \) we have decaying oscillations. So if the system is evolved over a finite time interval, there would be random instances of growth and decay. If \( \gamma \) is the growth rate of the growing solutions and \( -\zeta \) that of the decaying ones, the ensemble averaged growth rate is \( \Gamma = (\gamma - \zeta)/2 \) (Sokoloff, 1997). Thus when \( \gamma > \zeta \), one obtains \( \Gamma > 0 \) resulting in an overall growth above a critical dynamo number \( D_{cr} \). To estimate \( D_{cr} \), we use the perturbation solutions discussed in Appendix A. This gives \( \gamma \approx -\pi^2/4 + \sqrt{\pi |D|}/2 \) and \( \zeta \approx \pi^2/4 \), and thus \( D_{cr} \approx -\pi^3 \) for the Sokoloff (1997) model. This is somewhat larger in magnitude than
the critical dynamo number $\sim \pi^3/4$, which obtains for the coherent $a\omega$ dynamo (by demanding $\gamma > 0$).

These features are illustrated in Fig. 5.1. Here we have chosen $\tau_c = 2t_d$ so that one can clearly see both the growing and decaying phases and their net effect. Starting with random seed fields $\overline{B}_r, \overline{B}_\phi \sim 10^{-6}$, we find a number of growing as well as decaying realizations for moderate dynamo number $D = -40$. It is evident from Fig. 5.1, that any given realization has periods, $N\tau_c$, of steady growth (when $N = 1$) and periods, $N\tau_c$ of oscillatory decay (when $N = -1$). One gets a net growth of the field in about 65% of the realizations, as roughly expected from the above arguments for $|D| > |D_c|$. For a larger magnitude of the dynamo number $|D|$ one gets a greater probability for growth. We have also examined the opposite limit when $\tau_c < t_d$, and find that the dynamo becomes less efficient. This is shown in Fig. 5.2 where we plotted a subset of realizations for $\tau_c = 0.02t_d$. It is clearly seen from the plot that the qualitative behavior of the stochastic $a\omega$-dynamo changes based on whether the correlation time is longer or shorter than the turbulent diffusion time. These solutions clearly demonstrate the basic idea behind the incoherent $a\omega$-dynamo as discussed in Sokoloff (1997), that one needs to consider many realizations of the stochastic process. Just solving a double averaged version of the MFD equations need not be representative of the actual evolution of the dynamo for a given realization.

Of course the PDF of the stochastic alpha is not expected to be as described in Sokoloff (1997); for example Brandenburg et al. (2008a) found it can be approximated as a Gaussian. Also in general we expect $\tau_c < t_d$. We now present the results obtained by solving equations (5.1) and (5.2) with the random number $N$ for $a_1$ chosen from a Gaussian PDF, and adopting $\tau_c = 0.02t_d$. This is about $1.5t_d$, assuming $h = 500 \text{ pc}, \eta_t = 10^{26} \text{ cm}^2 \text{s}^{-1}, l_0 = 100 \text{ pc},$ and $u_0 \sim 10 \text{ km/s}$. The initial seed fields are random with amplitudes of $O(1)$. The MFD equations were evolved up to 10 turbulent diffusion time scales, $t = 10$, for dynamo numbers $D = -40, -80$ and $-120$ and up to $t = 15$ for $D = -180$. We also considered 1000 realizations of $a_1(t)$ for each $D$ so as to obtain good statistics. Note that to probe the PDF of the dynamo amplification up to a $3\sigma$ level one needs about these many realizations.

Fig. 5.3 shows the time evolution of the RMS (large scale) magnetic field, $\overline{B}$, for a subset of realizations with $R_\alpha = 0.0, Q_\alpha = 1.0$ and $R_\omega = -40$. There is an initial decay of $\overline{B}$, while the system discovers the proper eigenfunction. Further evolution then occurred on the diffusion time-scale $t_d$. In all realizations, $\overline{B}$ shows an oscillatory decay, even though a significant number of realizations showed growth of $\overline{B}$ up to
Figure 5.1: Time evolution of the rms mean magnetic field in the Sokoloff 1997 model for different realizations using a long correlation time for $\alpha_1(t)$. Parameter values used are $R_\alpha = 0.0$, $Q_\alpha = 1.0$, and $R_\omega = -40$. 
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Figure 5.2: Time evolution of the rms mean magnetic field in the Sokoloff 1997 model for different realizations using a short correlation time for $a_1(t)$. Parameter values used are $R_a = 0.0$, $Q_a = 1.0$, and $R_\omega = -40$. 
Figure 5.3: Time evolution of the rms mean magnetic field in different realizations using a short correlation time for $a_1(t)$ and a Gaussian PDF. Plots are obtained for parameter values $R_a = 0.0$, $Q_a = 1.0$, and $R_\omega = -40$. 
Figure 5.4: Frequency distribution of the dynamo amplification $A = \bar{B}/B_0$ at $t = 2$ (blue/thin), $t = 10$ (red/normal) and also at $t = 15$ (black/thick) (in panel d), for 1000 realizations of $\alpha_1(t)$ in each histogram. Here $R_\alpha = 0.0$, $Q_\alpha = 1.0$, and $R_\omega = -40, -80, -120$ and $-180$. Bin size is 0.2.
Figure 5.5: Frequency distribution of the dynamo amplification $A = B/B_0$ at $t = 10$ (red/normal) and at $t = 15$ (black/thick), for 1000 realizations of $a_1(t)$ in each histogram. Here $R_\alpha = 0.0$, $Q_\alpha = 1.0$, and $R_\omega = -200, -240$. Bin size is 0.2.
Figure 5.6: Time evolution of the rms mean magnetic field upto $t_f = 200$ in a realization using a short correlation time for $\alpha_1(t)$ and a Gaussian PDF. Plots are obtained for parameter values $R_\alpha = 0.0$, $Q_\alpha = 1.0$, and $R_\omega = -200$. 
$t = 2$. For higher dynamo numbers, growth is sustained for a longer time and for a larger number of realizations. This is similar to what is found for the Sokoloff (1997) model for short correlation times; (see Fig. 5.2). Thus having $\tau_c \ll t_d$, qualitatively changes the behavior of the dynamo and leaves an imprint of $t_d$ in the system evolution rather than $\tau_c$. In order to have a quantitative measure of how many realizations show net growth, we show in Fig. 5.4 the frequency distribution of the dynamo amplification $A = \frac{\bar{B}}{\bar{B}_0}$ at $t = 2$, $t = 10$ and also at $t = 15$ in panel (d) for 1000 realizations of $\alpha(t)$, at dynamo numbers, $D = -40, -80, -120$ and $-180$. Here $\bar{B}_0 = 0.32-0.35$ is roughly the value to which $\bar{B}$ initially decays in all the realizations. For $D = -40, -80$ and $-120$, we obtain $A > 1$, for respectively 34%, 65% and 82.8% of realizations at $t = 2$. However at a later time $t = 10$, this percentage decreases to 0%, 18% and 24% respectively. This is evident in the gradual shift of the histogram to the left. For $|D| = 160-180$ the PDF of $|A|$ remains stationary at late times; see panel (d). Above this range, the mean amplification secularly increases with time. This is shown in panels (e) and (f) of Fig. 5.5 where the histogram for $t = 15$ gradually shifts to the right of the one at $t = 10$ for higher dynamo numbers. However, if we run each of the realizations for very long times $\approx 200t_d$, all realizations are found to show growth for shear parameter $|\mathcal{R}_\alpha| > 160-180$ as shown in Fig. 5.6. Such long times are more relevant for a star like the Sun than for disc galaxies which has only lived for a time of order $20t_d$.

Thus, our results show that a stochastic $\alpha\omega$-dynamo is reasonably efficient over a few $t_d$ even at $\mathcal{R}_\alpha = -40$, but requires much larger dynamo numbers, as plausible towards galactic centers, to sustain fields for long periods.

5.4 Dynamical alpha quenching of the stochastic dynamo

As discussed in the last chapter, helicity fluxes across the boundaries of the disc provide a possible mechanism to shed small-scale magnetic helicity, and prevent such quenching. This situation could change in the presence of a stochastic component, as the kinetic alpha can undergo frequent sign reversals. Hence by the time the $\alpha_m$ grows to cancel $\alpha_k$, the kinetic alpha itself might have changed sign. It is then of interest to ask whether addition of a stochastic component to the kinetic alpha can stem the catastrophic quenching. This would then naturally provide a mechanism for healthy dynamo action even in the absence of helicity fluxes. The numerical analysis of the previous section is therefore extended by including an $\alpha_m$ contribution to $\alpha$ in equations (5.1) and (5.2) supplemented with an evolution
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equation for $a_m$. Following equation (4.1) and using equation (4.2) in Chapter 4, the evolution of $a_m$ in the absence of helicity fluxes, i.e $F = 0$ is,

$$\frac{\partial a_m}{\partial t} = -C \left[ \left( g(z) + \frac{Q_a}{R^2} f(z)N + a_m \right) B^2 - \frac{\vec{J} \cdot \vec{B}}{R^2} + \frac{a_m}{R_m} \right]$$

(5.5)

where $g(z)$ and $f(z)$ are as defined in §5.2, $R_m = \eta_1/\eta$, $C = 2\pi^2 (k_0/k_1)^2$, $k_1 = \pi/h$ and we take $k_1/k_0 = 5$. Note that the second term within first brackets in equation (5.5) represents the stochastic $\alpha$-effect. Further, $\vec{J} \cdot \vec{B}$ is the current helicity density of the large-scale field and is given by

$$\vec{J} \cdot \vec{B} = B_\phi \frac{\partial B_r}{\partial z} - B_r \frac{\partial B_\phi}{\partial z}.$$ 

(5.6)

We adopt $a_m = 0$ at $t = 0$ and random initial fields of $O(10^{-6})$. The system of equations (5.1) and (5.2) and (5.5) are then solved numerically in the $\alpha\omega$-dynamo approximation. Note that due to the magnetic part of the $\alpha$-effect, there is an extra term $-\partial (R_a a_m B_\phi)/\partial z$ in equation (5.1). We have also not added any helicity flux term to the r.h.s of equation (5.5).

Fig. 5.7 shows the time evolution of the RMS large-scale field in a number of realizations with $R_a = 1.0, Q_a = 0.0 - 4.0, |R_\omega| = 40 - 50$ and $R_m = 10^5$. Note that for $Q_a = 0.0, R_a = 1.0$ and $R_\omega = -40$ (shown by a dashed line), we recover the standard result that the magnetic field is catastrophically quenched to very low values. The catastrophic quenching still obtains in some realizations for a moderate value of $Q_a = 1.0$ (shown in dotted line in the figure). But in other realizations (shown by dash-dotted lines in the figure), a stochastic kinetic alpha alleviates this quenching to obtain fields of order $0.01 - 0.001 B_{eq}$. A detailed analysis shows that, for these dynamo parameters, $\vec{B}$ has a net growth in about 13% of all the realizations, even till $t = 20$. In fact, stronger values of $Q_a$ and $R_\omega$ can even amplify the field to near equipartition values. Such an example, adopting $Q_a = 4.0$ and $|R_\omega| = 50$ is shown by the solid line in the above figure. A space-time diagram for this realization, between times $t = 8 - 14$, is shown in Fig. 5.8. Both the radial and azimuthal fields have quadrupolar symmetry and show several reversals in sign during this period. We recall that high values of the dynamo control parameters are plausible towards the central regions of a galaxy. Therefore a stochastic $\alpha\omega$-dynamo is more likely to grow coherent magnetic fields efficiently towards the central regions of disc galaxies, and possibly even without helicity fluxes.
Figure 5.7: Time evolution of the rms mean magnetic field for different realizations in the dynamical $\alpha$-quenching model with parameter values $R_\alpha = 1.0$, $Q_\alpha = 0.0 - 4.0$, $|R_\omega| = 40 - 50$ and $R_m = 10^5$. 
Figure 5.8: Space-time diagrams of the radial and azimuthal components of the large-scale field for a realization with parameter values \( R_a = 1.0, Q_a = 4.0 \) and \( |R_w| = 50 \). The color bars on the left panel shows the magnitude of the field.