Chapter 2

The k-edge geodetic number of a graph* †

In this chapter, we introduce the concept of k-edge geodetic number of a graph G and investigate its properties [29, 31]. The k-edge geodetic numbers of certain classes of graphs are determined. Graphs of order n having edge geodetic number n are characterized. Also, we characterize trees of diameter d for which the d-edge geodetic number and the edge geodetic number are equal. It is shown that for each triple a, b, k of integers with $2 \leq a \leq b$ and $k \geq 2$, there is a connected graph G with $g_k(G) = a$ and $eg_k(G) = b$. Also, it is shown that for integers a, b, c and $k \geq 2$ with $3 \leq a \leq b \leq c$, there exists a connected graph G such that $g(G) = a$, $eg(G) = b$ and $eg_k(G) = c$. Further, we invesigate how the edge geodetic number and the k-edge geodetic number of a graph are affected by adding a pendant edge to the graph. It is proved that if G' is a graph obtained from G by adding a pendant edge, then $eg(G) \leq eg(G') \leq eg(G) + 1$ and $eg_2(G) \leq eg_2(G') \leq eg_2(G) + 1$. For any integer $k \geq 2$, it is also proved that $eg_k(G') \leq eg_k(G) + 2$. Also, we characterize graphs

*A part of this chapter has been accepted for publication in *Utilitas Mathematica* [31]

†A part of this chapter has been published in the *Inter. J. of Math. Combin.* [29]

for which the bounds are attained. It is shown that for any integer $k \geq 4$ and for every pair a, b of integers with $4 \leq a \leq b+2$, there is a connected graph G such that $eg_k(G) = b$ and $eg_k(G') = a$.

The k-edge geodetic number of a Graph

Definition 2.1 Let $G = (V, E)$ be a connected graph with at least two vertices and k a positive integer. A set $S \subseteq V$ is called a *k-edge geodetic set* of G if each edge e in $E - E(< S >)$ lies on a k-geodesic of vertices in S. The minimum cardinality of a k- edge geodetic set of G is its *k-edge geodetic number* $eg_k(G)$. A k-edge geodetic set of cardinality $eg_k(G)$ is called a *eg_k-set* of G.

Example 2.2 For the graph G given in Figure 2.1, it is easy to see that the set $S = \{v_1, v_2, v_5, v_6\}$ of end vertices is a g_2-set and so $g_2(G) = 4$. Since the edge v_3v_4 does not lie on any 2-geodesic of vertices in S, S is not a 2-edge geodetic set of G. It is easily seen that $S_1 = \{v_1, v_2, v_3, v_5, v_6\}$ is a minimum 2-edge geodetic set of G so that $eg_2(G) = 5$. Also, $S_2 = \{v_1, v_2, v_4, v_5, v_6\}$ is another eg_2-set of G.

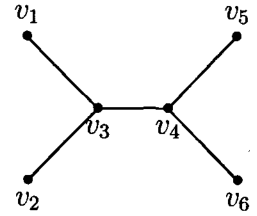

Figure 2.1: G

Example 2.3 For the graph G given in Figure 2.2, it is easily verified that $S = \{v_1, v_7, v_8, v_9\}$ is a g_6-set of G. Since the edge v_8v_4 does not lie on any 6-geodesic of vertices in S, it is not a 6-edge geodetic set of G. It is easily seen that $S_1 = \{v_1, v_4, v_7, v_8, v_9\}$ is a eg_6-set of G and so $eg_6(G) = 5$.

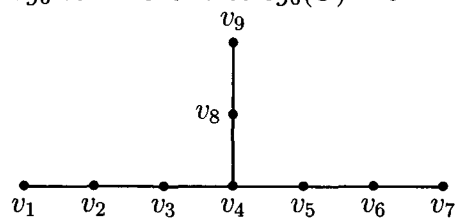

Figure 2.2: G

Proposition 2.4 *For any connected graph G of order $n \geq 2$, $2 \leq eg_k(G) \leq n$.*

Proof. This follows from the fact that any k-edge geodetic set needs at least two vertices and V is a k-edge geodetic set of G. ∎

Proposition 2.5 *For any connected graph G of order $n \geq 2$,*

$2 \leq g(G) \leq eg(G) \leq eg_k(G) \leq n$ and $2 \leq g_k(G) \leq eg_k(G) \leq n$.

Proof. This follows from the fact that every edge geodetic set is a geodetic set and every k-edge geodetic set is a k-geodetic set as well as an edge geodetic set. ∎

Theorem 2.6 *Every k-edge geodetic set contains all the k- extreme vertices of G. In particular, if the set W of all k-extreme vertices is a k-edge geodetic set, then W is the unique eg_k- set of G.*

Proof. A k-extreme vertex is not the internal vertex of any k-geodesic of vertices of G and so the results follows. ∎

Definition 2.7 Let G be a connected graph and $k \geq 2$ be an integer. An edge e of G is called *k-extreme edge* if e does not lie on any k-geodesic of vertices of G.

Example 2.8 For the graph G given in Figure 2.3, $e_1 = v_3v_7$ and $e_2 = v_6v_7$ are 3-extreme edges. However, since e_1 and e_2 lie on the v_3, v_7, v_6 geodesic of length 2, e_1 and e_2 are not 2-extreme edges.

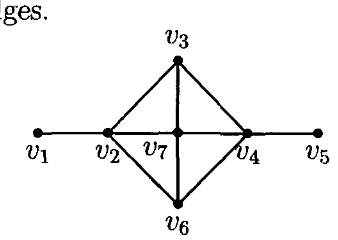

Figure 2.3: G

Note 2.9 If an edge of a connected graph G is k-extreme, then it is l-extreme for all $l \geq k$. Also the ends of a k-extreme edge need not be k-extreme vertices. For the graph G given in Figure 2.4, the edge v_1v_5 is 2-extreme, whereas v_1 and v_5 are not 2-extreme.

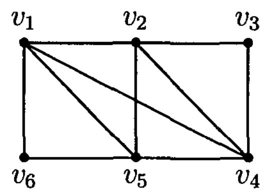

Figure 2.4: G

Theorem 2.10 *Every k-edge geodetic set contains both the ends of each k-extreme edge. If the set W of the ends of all the k-extreme edges together with the set of k-extreme vertices is a k-edge geodetic set, then W is the unique eg_k-set of G and so $eg_k(G) = |W|$.*

Proof. Let $e = xy$ be a k-extreme edge of G. Let S be a k-edge geodetic set of G such that $x \notin S$ or $y \notin S$. Then $e \notin E(< S >)$. Since S is a k-edge geodetic set, e lies on a k-geodesic of vertices in S, which is a contradiction to e being a k-extreme edge of G. Hence every k-edge geodetic set contains both the ends of every k-extreme edge. The second part of the statement now follows from Theorem 2.6. ∎

Definition 2.11 An edge of a connected graph G is said to be *extreme* if it is k-extreme for all $k \geq 2$.

Example 2.12 (i) Every edge of the complete graph $K_n (n \geq 2)$ is extreme.

(ii) No edge of a tree of order at least 3 is extreme.

(iii) For the graph G given in Figure 2.4, the edge v_1v_5 is extreme and the edge v_1v_2 is not extreme.

Proposition 2.13 *For a connected graph G, an edge $e = uv$ is extreme if and only if $N[u] = N[v]$.*

Proof. Suppose that e is extreme. Assume that there exists $w \in N[u]$ such that $w \notin N[v]$. Then it is clear that $w \notin \{u, v\}$. Hence $d(w, v) = 2$ and so the edge e lies on the geodesic $P : w, u, v$ of length 2, which is a contradiction to the fact that e is extreme. Hence $N[u] \subseteq N[v]$. Similarly, we can prove that $N[v] \subseteq N[u]$. Conversely, suppose that $N[u] = N[v]$. Then it is clear that $e = uv$ does not lie on any geodesic of length ≥ 2 and so e is extreme. ∎

Proposition 2.14 *Let G be a connected graph and v a vertex of G. If there exists a vertex u in $N(v)$ such that u is adjacent to all other vertices in $N(v)$, then v belongs to every edge geodetic set of G.*

Proof. Assume that there exists an edge geodetic set S of G such that $v \notin S$. Then uv lies on a geodesic P joining a pair of vertices x, y in S. Since $v \notin S$, we have $v \notin \{x, y\}$. Now, let w be a vertex of P such that $w \neq u$ and $vw \in E$. By hypothesis, $wu \in E$ and this implies that P is not a geodesic of G, which is a contradiction. Hence v belongs to every edge geodetic set of G. ∎

Corollary 2.15 *Let G be a connected graph and v a vertex of G. If there exists a vertex u in $N(v)$ such that u is adjacent to all other vertices in $N(v)$, then v belongs to every k-edge geodetic set of G.*

Proof. Since every k-edge geodetic set is an edge geodetic set, the result follows from Proposition 2.14. ∎

Corollary 2.16 *For an integer $n \geq 4$, $eg(K_1 + C_n) = n$.*

Proof. By Proposition 2.14, all the vertices of the cycle C_n belong to every edge geodetic set of $K_1 + C_n$. Since these vertices form an edge geodetic set of $K_1 + C_n$, it follows that $eg(K_1 + C_n) = n$. ∎

Theorem 2.17 *Every edge geodetic set contains the ends of each extreme edge. If the set W of the ends of all the extreme edges together with the set of extreme vertices is an edge geodetic set, then W is the unique eg-set of G and so $eg(G) = |W|$.*

Proof. Let $e = uv$ be an extreme edge of G. Then, by Proposition 2.13, $N[u] = N[v]$. It follows that $v \in N(u)$ and v is adjacent to all other vertices in $N(u)$. Hence it follows from Proposition 2.14 that the vertex u belongs to every edge geodetic set of G. Similarly, the vertex v belongs to every edge geodetic set of G. Now, the second part follows from Theorem 1.19. ■

Theorem 2.18 *Let G be a connected graph of order n. Then $eg(G) = n$ if and only if for each vertex v of G, $N(v) \subseteq N[u]$ for some $u \in N(v)$.*

Proof. For any vertex v of G, suppose that there exists $u \in N(v)$ such that u is adjacent to all other vertices in $N(v)$. By Proposition 2.14, the vertex v belongs to every edge geodetic set of G and so the result follows. Conversely, suppose that $eg(G) = n$. Assume, to the contrary, that there exists a vertex v such that $N(v) \nsubseteq N[u]$ for any $u \in N(v)$. Let $S = V - \{v\}$. We claim that S is an edge geodetic set of G. Let $e = xy$ be any edge in G. If $v \notin \{x, y\}$, then $x, y \in S$ and so there is nothing to prove. Now, assume without loss of generality that $v = y$. Then $x \in N(v)$. Since $N(v) \nsubseteq N[x]$, there exists a $u \in N(v)$ such that $u \notin N[x]$. It follows that $u \neq x$ and the edge xv lies on the geodesic $P : x, v, u$ with $x, u \in S$. Hence S is an edge geodetic set of G and so $eg(G) \leq |S| = n - 1$, which is a contradiction. Thus the result follows. ■

Corollary 2.19 *For the complete graph $K_n(n \geq 2)$, $eg(K_n) = n$.*

Theorem 2.20 *If G is a connected graph of diameter 2, then $eg_2(G) = eg(G)$.*

Proof. Let S be an edge geodetic set of G. Let $e = xy$ be any edge of G such that $e \notin E(< S >)$. Then either $x \notin S$ or $y \notin S$. Since S is an edge geodetic set of G, the edge e lies on a $u - v$ geodesic with $u, v \in S$. Since the diameter of G is 2, it follows that $d(u, v) = 2$. Thus, S is a 2-edge geodetic set and so $eg_2(G) \leq eg(G)$. Since every 2-edge geodetic set is also an edge geodetic set, we have $eg(G) \leq eg_2(G)$. Hence the result follows. ∎

Remark 2.21 The converse of Theorem 2.20 is not true. For the graph G given in Figure 2.5, the diameter is 4. The edges v_2v_8 and v_4v_6 are 2-extreme and the vertices v_1 and v_5 are 2-extreme in G. Since the set $S = \{v_1, v_2, v_4, v_5, v_6, v_8\}$ is a 2-edge geodetic set of G, it follows from Theorem 2.10 that S is a eg_2-set of G. Similarly, it follows from Theorem 2.17 that S is a eg-set of G.

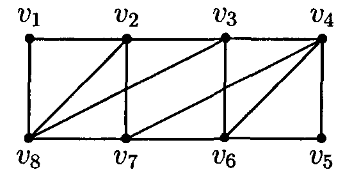

Figure 2.5: G

Theorem 2.22 *Let G be a connected graph of order $n \geq 3$ with $diam(G) \geq 3$ and $eg(G) = 2$. Then $eg_k(G) = eg(G)$ if and only if $diam(G) = k$*

Proof. Let $diam(G) = d$ and let $S = \{x, y\}$ be any eg-set of G. Then, by Theorem 1.23, x and y are antipodal and so $d(x,y) = d$. Hence S is an eg_d-set of G so that $eg_k(G) = eg(G)$ for $k = d$. Conversely, suppose that $eg_k(G) = eg(G)$. Let $S = \{x,y\}$ be a eg_k-set of G. Then S is also an eg-set of G and $d(x,y) = k$ But, by Theorem 1.23, x and y are antipodal and so $d(x,y) = d$. Hence $k = d$. ∎

Theorem 2.23 *Let T be a tree of diameter d and let v be a cut vertex of T such that v is not an end of a d-extreme edge. Then v does not belong to any eg_d-set of G.*

Proof. Suppose that there exists an eg_d-set S of T such that S contains the vertex v. Let $T_1, T_2, \ldots, T_r$ $(r \geq 2)$ be the components of $G - v$. Let $S' = S - \{v\}$. We show that S' is a d-edge geodetic set of T. Let $e = uw$ be an edge of T such that $e \notin E(< S' >)$. We consider two cases.

Case 1. Suppose that $e \notin E(< S >)$. Then, since S is an eg_d-set of G, e lies on an $x - y$ d-geodesic P of T with $x, y \in S$. We claim that $v \notin \{x,y\}$. Assume to the contrary that $v = y$ (say). Now, without loss of generality, assume that x belongs to T_1. Since v is adjacent to at least one vertex in each $T_i (i \leq i \leq r)$, assume that v is adjacent to some vertex z in $T_k (1 < k \leq r)$. Since v is a cut vertex of T, it follows that all the vertices in P except v lie in T_1. Thus the d-geodesic P together with the edge xz is a $(d+1)$-geodesic in T, which is a contradiction to the fact that d is the diameter of T. This shows that $v \notin \{x,y\}$. Hence $x, y \in S'$ and so S' is a d-edge geodetic set of T of cardinality less than $eg_d(G)$, which is a contradiction.

Case 2. Suppose that $e \in E(< S >)$. Since $e \notin E(< S' >)$, it follows that $w = v$ (say). Hence the edge $e = uv$ is not d-extreme so that e lies on an $x - y$ d-geodesic of T. Since T is a tree, it is clear that x and y are end vertices of T and so $v \notin \{x,y\}$.

Now, by Theorem 2.17, $x, y \in S$ and it follows that $x, y \in S'$. Hence S' is a k-edge geodetic set of G, which is a contradiction to the minimality of S. Thus the proof is complete. ∎

Corollary 2.24 *Let T be a tree of diameter d with no d-extreme edges. Then no cut vertex of T belongs to any eg_d-set of G.*

Proof. This follows from Theorem 2.23. ∎

Theorem 2.25 *Let T be a tree of diameter $d \geq 2$. Then $eg_d(T) = eg(T)$ if and only if T has no d-extreme edges.*

Proof. Assume that T has no d-extreme edges. Then, by Corollary 2.24, no cut vertex of T belongs to any eg_d-set of T. Also, by Theorem 2.6, each end vertex of T belongs to every eg_d-set of G. Hence it follows from Theorems 1.19 and 1.24 that $eg_d(T) = eg(T)$. Conversely, assume that $eg_d(T) = eg(T)$. Let k be the number of end vertices of T. Then, by Theorem 1.21, $eg(T) = k$. Let S be any eg_d-set of T. Then $|S| = k$. Hence it follows from Theorem 2.6 that S consists of all the end vertices of T. Since $diam(T) \geq 2$, we have $T \neq K_2$. If T has a d-extreme edge, say $e = uv$, then, by Theorem 2.10, both $u, v \in S$. Since one of u or v is a cut vertex of T, this leads to a contradiction. Hence T has no d-extreme edges. ∎

In view of Proposition 2.5, we have the following two realization theorems.

Theorem 2.26 *Let $k \geq 2$ be an integer. For each pair of integers a, b with $2 \leq a \leq b$, there exists a graph G with $g_k(G) = a$ and $eg_k(G) = b$.*

Proof. Case 1. $a = b$. Let G be the graph obtained from the path $P_k : v_1, v_2, \ldots, v_k$ by joining $a-1$ new vertices $u_1, u_2, \ldots, u_{a-1}$ to v_1. Since the set $\{v_k, u_1, u_2, \ldots, u_{a-1}\}$ of end vertices is a k-edge geodetic set of G, it follows from Theorems 2.6 and 1.28 that $eg_k(G) = g_k(G) = a$.

Case 2. $2 \leq a < b$. We consider two subcases.

Subcase 2.1. Let $b = a + 1$.

First let $a = 2$. Then $b = 3$. Let $C_{2k+1} : v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}, v_{k+3} \ldots, v_{2k+1}, v_1 (k \geq 2)$ be an odd cycle of order $2k+1$. Let G be the graph in the Figure 2.6 obtained from C_{2k+1} by joining the edges $v_k v_{k+2}$ and $v_{k+1} v_{k+3}$. Then it is clear that $S = \{v_k, v_{2k+1}\}$ is a k-geodetic set of G so that $g_k(G) = 2$. Now, since $N[v_{k+1}] = N[v_{k+2}]$, it follows from Proposition 2.13 that the edge $v_{k+1}v_{k+2}$ is an extreme edge of G and so it is k-extreme. Hence, by Theorem 2.10, the vertices v_{k+1} and v_{k+2} belong to every k-edge geodetic set of G. It is clear that $S_1 = \{v_{k+1}, v_{k+2}\}$ is not a k-edge geodetic set of G. Since $S_2 = \{v_1, v_{k+1}, v_{k+2}\}$ is a k-edge geodetic set of G, it follows that $eg_k(G) = 3$.

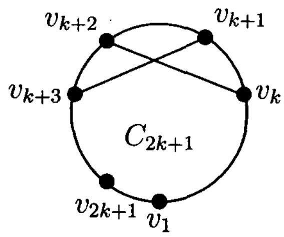

Figure 2.6: G

Now, let $a \geq 3$. Let $P : u_0, u_1, u_2, \ldots, u_{k-1}$ and $Q : v_0, v_1, \ldots, v_{k-1}$ be two vertex disjoint paths. Let H be the graph obtained from P and Q by adding a new vertex w and joining it to both u_{k-1} and v_{k-1} and also joining u_{k-1} and v_{k-1}. Now, let

G be the graph obtained from H by adding $a-3$ new vertices $w_1, w_2, \ldots, w_{a-3}$ and joining them to v_1. The graph G is given in Figure 2.7. Since the set $S = \{w_1, w_2, \ldots, w_{a-3}, u_0, v_0, w\}$ of all k-extreme vertices of G is a k-geodetic set of G, it follows from Theorem 1.28 that $g_k(G) = a$. Every k-edge geodetic set of G is also a k-geodetic set and so it follows from Theorem 1.28 that S is contained in every k-edge geodetic set. Since the edge $u_{k-1}v_{k-1}$does not lie on any k-geodesic of vertices in S, it follows that S is not a k-edge geodetic set. Now, $S \cup \{u_{k-1}\}$ is a k-edge geodetic set of G and so $eg_k(G) = a + 1 = b$.

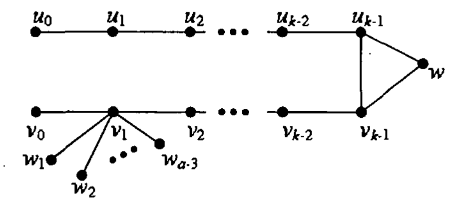

Figure 2.7: G

Subcase 2.2. $b = a + j$, where $j \geq 2$. Then $b \geq 4$. First we consider the case $k = 2$. Let H be the graph obtained from the path $P : u_1, u_2, u_3$ by adding $b - a$ new vertices $v_1, v_2, \ldots, v_{b-a}$ and joining each $v_i (1 \leq i \leq b - a)$ with u_1, u_2, u_3. Let G be the graph obtained from H by adding $a - 2$ new vertices $w_1, w_2, \ldots, w_{a-2}$ and joining each $w_i (1 \leq i \leq a - 2)$ with u_2. The graph G is shown in Figure 2.8. Let $S = \{w_1, w_2, \ldots, w_{a-2}\}$. It is clear that S is not a geodetic set of G. Also, it is easily seen that $S \cup \{v\}$, where $v \in V - S$, is not a geodetic set of G. Now, since $S \cup \{u_1, u_3\}$ is a geodetic set of G, it follows from Theorem 1.19 that $g(G) = a$. Since u_2 is the only full degree vertex of G, by Theorem 1.26, $eg(G) = b$. Also, since the graph G has diameter 2, it follows from Theorems 1.29 and 2.20 that $g_2(G) = g(G) = a$ and $eg_2(G) = eg(G) = b$.

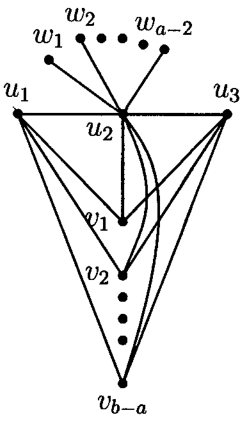

Figure 2.8: G

Now, we consider the case $k \geq 3$. Let H be the graph obtained from the path $P : u_0, u_1, u_2, \ldots, u_k$ by adding $j - 1$ new vertices $v_1, v_2, \ldots, v_{j-1}$ and joining each $v_i (1 \leq i \leq j - 1)$ with u_1, u_2 and u_3. Now, let G be the graph obtained from H by adding $a - 2$ new vertices $w_1, w_2, \ldots, w_{a-2}$ and joining each $w_i (1 \leq i \leq a - 2)$ to u_1. The graph G is shown in the Figure 2.9.

First assume that $k = 3$. Let $S = \{u_0, w_1, w_2, \ldots, w_{a-2}\}$. Then S is the set of k-extreme vertices of G and it is not a k-geodetic set of G. Now, $S_1 = S \cup \{u_3\}$ is a k-geodetic set of G so that by Theorem 1.28, S_1 is a g_k-set of G. Hence $g_k(G) = |S_1| = a$.

The edges $v_i u_2 (1 \leq i \leq j - 1)$ are the only k-extreme edges of G. Let $S_2 = \{v_1, v_2, \ldots, v_{j-1}, u_2\}$. Then S_2 is the set of ends of all the k-extreme edges of G. Let $S_3 = S \cup S_2$. It is clear that S_3 is not a k-edge geodetic set and $S_4 = S_3 \cup \{u_k\}$ (Note that $u_k = u_3$, in this case) is a k-edge geodetic set of G. Hence it follows from

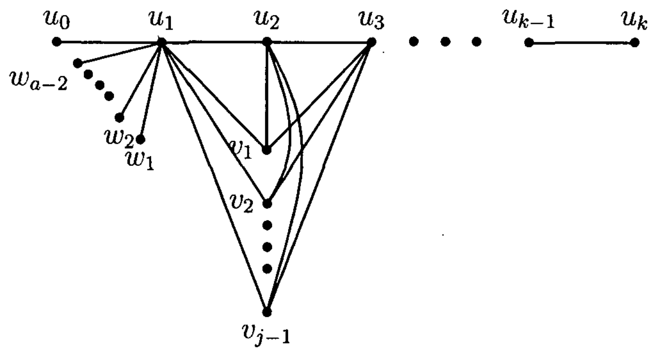

Figure 2.9: G

Theorem 2.10 that S_4 is an eg_k-set of G and $eg_k(G) = a + j = b$.

Next assume that $k \geq 4$. Now, the set $T = \{u_0, w_1, w_2, \ldots, w_{a-2}, u_k\}$ of all k-extreme vertices of G is a k-geodetic set and so by Theorem 1.28, $g_k(G) = a$. As for the case $k = 3$, $S_2 = \{v_1, v_2, \ldots, v_{j-1}, u_2\}$ is the set of ends of all the k-extreme edges of G. Since $T_1 = T \cup S_2$ is a k-edge geodetic set of G, it follows from Theorem 2.10 that $eg_k(G) = |T_1| = |T| + j = a + j$. Thus the proof is complete. ∎

Theorem 2.27 *Let $k \geq 2$ be an integer. For each triple a, b, c of integers with $3 \leq a \leq b \leq c$, there exists a graph G with $g(G) = a$, $eg(G) = b$ and $eg_k(G) = c$.*

Proof. Case 1. $a = b = c$. Let G be the graph obtained from the path P : $v_0, v_1, \ldots, v_k$ by adding $a - 2$ new vertices $w_1, w_2, \ldots, w_{a-2}$ and joining each $w_i (1 \leq i \leq a - 2)$ to v_1. Then G has no k-extreme edges and $S = \{v_1, w_1, w_2, \ldots, w_{a-2}, v_k\}$ is the set of all k-extreme vertices of G. Also S is the set of all extreme vertices of G. Since S is a k-edge geodetic set, it is also a geodetic set as well as an edge geodetic set. Hence it follows from Theorems 2.10 and 1.19 that $eg_k(G) = g(G) = eg(G) = a$.

Case 2. $3 \leq a < b < c$.

Subcase 2.1 Suppose that $k = 2$. Let $l = c - b + 1$. Then $l \geq 2$. Let H be the graph obtained from the path $P : v_0, v_1, \dots, v_{2l}$ by adding $a-2$ new vertices $w_1, w_2, \dots, w_{a-2}$ and joining each $w_i (1 \leq i \leq a-2)$ to v_1. Let K_{b-a} be the complete graph with vertex set $V(K_{b-a}) = \{u_1, u_2, \dots, u_{b-a}\}$ such that it is vertex disjoint with H. Let G be the graph obtained from H and K_{b-a} by joining each $u_i (1 \leq i \leq b-a)$ to the vertices v_0, v_1 and v_2. The graph G is shown in Figure 2.10.

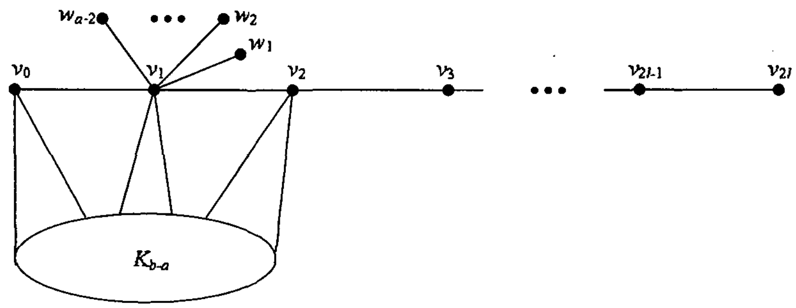

Figure 2.10: G

It is clear that the set $S = \{v_0, v_{2l}, w_1, w_2, \dots, w_{a-2}\}$ of all extreme vertices of G is a geodetic set and so by Theorem 1.19, $g(G) = a$. It is clear that the edges $v_1u_i (1 \leq i \leq b-a)$ and $u_iu_j (i \neq j)$ do not lie on any geodesic of vertices in S and so S is not an edge geodetic set of G. For each $u_i (1 \leq i \leq b-a)$, the vertex v_1 is such that $v_1 \in N(u_i)$ and is adjacent to all other vertices in $N(u_i)$. Hence, by Proposition 2.14, all the vertices $u_i (1 \leq i \leq b-a)$ belong to every edge geodetic set of G. Let $S_1 = S \cup \{u_1, u_2, \dots, u_{b-a}\}$. Then it is clear that S_1 is an edge geodetic set of G and it follows from Theorem 1.19 that $eg(G) = |S_1| = b - a + a = b$. Since S_1 is is contained in every edge geodetic set of G and every 2-edge geodetic set of

G is also an edge geodetic set, it follows that S_1 is also contained in every 2-edge geodetic set of G. Now, the edge $v_{2l-1}v_{2l}$ does not lie on any 2-geodesic of vertices in S_1 and so S_1 is not a 2-edge geodetic set of G. It is easily seen that every edge in a graph has at least one end in every 2-edge geodetic set. In particular, each edge in $M = \{v_2v_3, v_4v_5, \ldots, v_{2l-2}v_{2l-1}\}$ has at least one end in every 2-edge geodetic set. Hence $eg_2(G) \geq |S_1| + |M| = b + l - 1 = b + c - b + 1 - 1 = c$. Now, the set $S_2 = S_1 \cup \{v_2, v_4, \ldots, v_{2l-2}\}$ is a 2-edge geodetic set of G and so $eg_2(G) \leq |S_2| = |S_1| + l - 1 = b + l - 1 = b + c - b + 1 - 1 = c$. Thus $eg_2(G) = c$.

Subcase 2.2 Suppose that $k \geq 3$. Let $l = c - b$. Let H be the graph obtained from the path $P : v_0, v_1, \ldots, v_{lk+1}$ by adding $a - 2$ new vertices $w_1, w_2, \ldots, w_{a-2}$ and joining each $w_i (1 \leq i \leq a - 2)$ to v_1. Let K_{b-a} be the complete graph with vertex set $V(K_{b-a}) = \{u_1, u_2, \ldots, u_{b-a}\}$ such that it is vertex disjoint with H. Let G be the graph obtained from H and K_{b-a} by joining each $u_i (1 \leq i \leq b - a)$ to the vertices v_0, v_1 and v_2. The graph G is Shown in Figure 2.11.

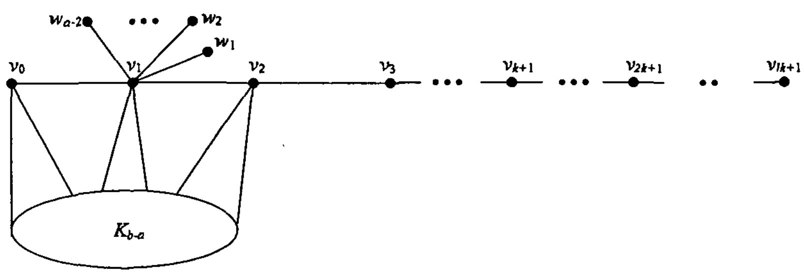

Figure 2.11: G

It is clear that the set $S = \{v_0, v_{lk+1}, w_1, w_2, \ldots, w_{a-2}\}$ of all extreme vertices of G is a geodetic set and so by Theorem 1.19, $g(G) = a$. Since the edges $v_1u_i (1 \leq i \leq b-a)$

and $u_iu_j(i \neq j)$ do not lie on any geodesic of vertices in S, we have S is not an edge geodetic set of G. Now, as in Subcase 2.1, all the vertices $u_1, u_2, \ldots, u_{b-a}$ belong to every edge geodetic set of G. Let $S_1 = S \cup \{u_1, u_2, \ldots, u_{b-a}\}$. Then it is clear that S_1 is an edge geodetic set of G and so by Theorem 1.19, $eg(G) = |S_1| = b - a + a = b$.

The edges $u_iv_1(1 \leq i \leq b-a)$ are the only k-extreme edges of G and so by Theorem 2.10, the vertices $v_1, u_1, u_2, \ldots, u_{b-a}$ belong to every k-edge geodetic set of G. Also S is the set of all k-extreme vertices of G. Let $S_2 = S \cup \{v_1, u_1, u_2, \ldots, u_{b-a}\}$. Now, if $c = b + 1$, then $l = 1$ and it is clear that S_2 is a k-edge geodetic set of G. Hence, by Theorem 2.10, $eg_k(G) = |S_2| = a + b - a + 1 = b + 1 = c$. Next, if $c \geq b + 2$, then it is clear that S_2 is not a k-edge geodetic set of G. Now, let $S_3 = S_2 \cup \{v_{k+1}, v_{2k+1}, \ldots, v_{(l-1)k+1}\}$. Then $|S_3| = c$. That S_3 is a k-edge geodetic set is clear. We show that S_3 is a minimum k-edge geodetic set of G. Suppose that there exists a k-edge geodetic set T of G such that $|T| < |S_3|$. Since S_2 is contained in every k-edge geodetic set of G, we have $S_2 \subseteq T$. Let i be the least positive integer such that $v_{ik+1} \notin T$. Then it is clear that $i \geq 1$. Now, by the choice of i, the vertices $v_{k+1}, v_{2k+1}, \ldots, v_{(i-1)k+1}$ belong to T. Since T is a k-edge geodetic set of G, it follows that there exists at least one j_i with $(i-1)k+1 < j_i < ik+1$ such that $v_{j_i} \in T$ and there exists at least one j_{i+1} with $ik+1 < j_{i+1} < (i+1)k+1$ such that $v_{j_{i+1}} \in T$. Now, suppose that $v_{sk+1} \notin T$ for all $s = i+1, i+2, \ldots, l-1$. Then for each $t = 2, 3, \ldots, l-i$, there exists al least one j_{i+t} with $(i+t-1)k+1 < j_{i+t} < (i+t)k+1$ such that $v_{j_{i+t}} \in T$. It follows that $|T| \geq |S_3| + 1 = c + 1$, which is a contradiction. Otherwise, let p be the least positive integer such that $p \geq i + 1$ and $v_{pk+1} \in T$. Then for each $s = 2, 3, \ldots, p-i$, there exists at least one j_{i+s} with $(i+s-1)k+1 < j_{i+s} < (i+s)k+1$

such that $v_{j_{i+s}} \in T$ and for each $t = 1, 2, \ldots, l - p$, there exists at least one j_{p+t} with $(p + t - 1)k + 1 < j_{p+t} \leq (p + t)k + 1$ such that $v_{j_{p+t}} \in T$. Thus, it follows that $|T| \geq |S_3| + 1 = c + 1$, which is a contradiction. Hence S_3 is an eg_k-set of G so that $eg_k(G) = c$.

Case 3. $a = b < c$. Let G be the graph obtained from the path $P : v_0, v_1, \ldots, v_{(c-a+1)k}$ by adding $a - 2$ new vertices $w_1, w_2, \ldots, w_{a-2}$ and joining each $w_i (1 \leq i \leq a - 2)$ to v_1. The graph G is shown in Figure 2.12.

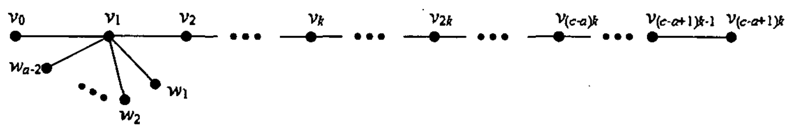

Figure 2.12: G

It is clear that the set $S = \{v_0, v_{(c-a+1)k}, w_1, w_2, \ldots, w_{a-2}\}$ of end vertices is a geodetic set as well as an edge geodetic set and so by Theorem 1.19, $g(G) = a$ and $eg(G) = b$. The edge $v_{(c-a+1)k-1}v_{(c-a+1)k}$ does not lie on any k-geodesic of vertices in S so that S is not a k-edge geodetic set of G. Let $S_1 = S \cup \{v_k, v_{2k}, \ldots, v_{(c-a)k}\}$. Then, exactly similar to that of Subcase 2.2, we can prove that S_1 is a minimum k-edge geodetic set of G and so $eg_k(G) = |S_1| = c$.

Case 4. $a < b = c$. We consider two subcases.

Subcase 4.1. $k = 2$. Let H be the graph obtained from the path $P : v_0, v_1, v_2$ by adding $a - 2$ new vertices $w_1, w_2, \ldots, w_{a-2}$ and joining each $w_i (1 \leq i \leq a - 2)$ to v_1. Let K_{b-a} be the complete graph with vertex set $V(K_{b-a}) = \{u_1, u_2, \ldots, u_{b-a}\}$ such that it is vertex disjoint with H. Let G be the graph obtained from H and K_{b-a} by joining each $u_i (1 \leq i \leq b-a)$ to the vertices v_0, v_1 and v_2. The graph G is shown in Figure 2.13.

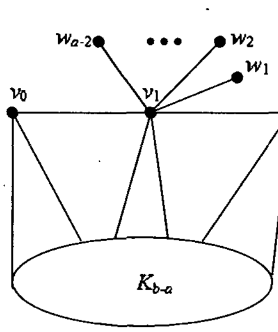

Figure 2.13: G

It is clear that the set $S = \{v_0, v_2, w_1, w_2, \ldots, w_{a-2}\}$ of all extreme vertices is a geodetic set of G and so by Theorem 1.19 $g(G) = |S| = a$. Now, since the edges $u_i u_j (i \neq j)$ are the only extreme edges, by Theorem 2.17, the vertices $u_1, u_2, \ldots, u_{b-a}$ belong to every edge geodetic set of G. Let $S_1 = S \cup \{u_1, u_2, \ldots, u_{b-a}\}$. Then S_1 is an edge geodetic set of G and it follows from Theorem 2.17 that $eg(G) = |S_1| = b$. Now, since $diam(G) = 2$, it follows from Theorem 2.20 that $eg_2(G) = eg(G) = b$.

Subcase 4.2. $k \geq 3$.

First assume that $b = a + 1$. Let G be the graph obtained from the path P : $v_0, v_1, \ldots, v_{2k-2}$ by adding $a-1$ new vertices $w_1, w_2, \ldots, w_{a-3}, x, y$ and joining each w_i with v_1; x with y, v_{k-1}, v_k; and y with v_{k-1}. The graph G is shown in Figure 2.14. It is clear that the set $S = \{v_0, y, v_{2k-2}, w_1, w_2, \ldots, w_{a-3}\}$ of extreme vertices is a geodetic set of G and so by Theorem 1.19, $g(G) = |S| = a$. The vertex v_{k-1} is such that $v_{k-1} \in N(x)$ and is adjacent to all other vertices in $N(x)$. Hence, by

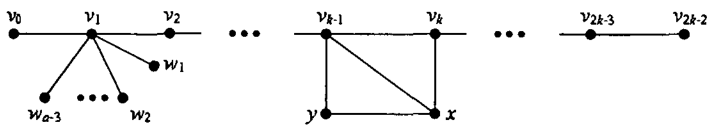

Figure 2.14: G

Proposition 2.14, x belongs to every edge geodetic set of G. Now, since $S \cup \{x\}$ is an edge geodetic set, it follows from Theorem 1.19 that $eg(G) = |S| + 1 = a + 1 = b$.

Now, G has no k-extreme edges and S is the set of all k-extreme vertices. Just as above, it follows from Corollary 2.15 that x belongs to every k-edge geodetic set of G. Also, $S \cup \{x\}$ is a k-edge geodetic set of G and so it follows from Theorem 2.10 that $eg_k(G) = |S| + 1 = a + 1 = b$.

Next, assume that $b \geq a + 2$. Let H be the graph obtained from the path $P : v_0, v_1, \ldots, v_k$ by adding $a - 2$ new vertices $w_1, w_2, \ldots, w_{a-2}$ and joining each $w_i (1 \leq i \leq a - 2)$ to v_1. Let K_{b-a} be the complete graph with vertex set $V(K_{b-a}) = \{u_1, u_2, \ldots, u_{b-a}\}$ such that it is vertex disjoint with H. Let G be the graph obtained from H and K_{b-a} by joining each $u_i (1 \leq i \leq b - a)$ to the vertices v_1 and v_3. The graph G is shown in Figure 2.15.

Suppose that $k = 3$. Then $S = \{v_0, w_1, w_2, \ldots, w_{a-2}\}$ is the set of all extreme vertices of G. Since S is not a geodetic set and $S \cup \{v_3\}$ is a geodetic set of G, it follows from Theorem 1.19 that $g(G) = |S| + 1 = a$. Now, the edges $u_i u_j (i \neq j)$ are the only extreme edges and so by Theorem 2.17, the vertices $u_1, u_2, \ldots, u_{b-a}$ belong to every edge geodetic set of G. Let $S_1 = S \cup \{u_1, u_2, \ldots, u_{b-a}\}$. Now, no edge incident

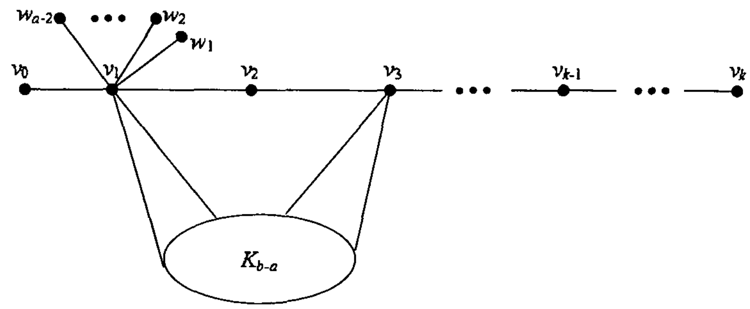

Figure 2.15: G

at v_3 lies on a geodesic joining a pair of vertices in S_1 and so S_1 is not an edge geodetic set of G. It is clear that $S_1 \cup \{v_3\}$ is an edge geoddetic set and so by Theorem 2.17, $eg(G) = |S_1| + 1 = b$. Also, just as above, S_1 is the set of all k-extreme vertices together with the ends of all k-extreme edges. As earlier, S_1 is not a k-edge geodetic set and $S_1 \cup \{v_3\}$ is a k-edge geodetic set of G. Hence it follows from Theorem 2.10 that $eg_k(G) = |S_1| + 1 = b$.

Finally, suppose that $k \geq 4$. Then the set $T = \{v_0, v_k, w_1, w_2, \ldots, w_{a-2}\}$ of all extreme vertices is a geodetic set of G and it follows from Theorem 1.19 that $g(G) = |T| = a$. Since the edges $u_iu_j (i \neq j)$ are the only extreme edges, we have by Theorem 2.17, the vertices $u_1, u_2, \ldots, u_{b-a}$ belong to every edge geodetic set of G. It is clear that $T_1 = T \cup \{u_1, u_2, \ldots, u_{b-a}\}$ is an edge geodetic set and so by Theorem 2.17, $eg(G) = |T_1| = b$. Also, just as above, T_1 is the set of all k-extreme vertices together with the ends of all k-extreme edges. Since T_1 is a k-edge geodetic set, it follows from Theorem 2.10 that $eg_k(G) = |T_1| = b$. ∎

The edge geodetic number and addition of a pendant edge

In this section we discuss how the edge geodetic number of a connected graph G is affected by adding a pendant edge to G. Let G' be a graph obtained from a connected graph G by adding a pendant edge uv, where u is not a vertex of G and v is a vertex of G.

Theorem 2.28 *If G' is a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G, then $eg(G) \leq eg(G') \leq eg(G) + 1$.*

Proof. Let S be any eg-set of G and let $S' = S \cup \{u\}$. We claim that S' is an edge geodetic set of G'. Let e be an edge of G'. If $e \in E(G)$, then e lies on a geodesic of vertices in S. If $e = uv$, then, since every edge geodetic set of G is a geodetic set of G, it follows that the vertex v lies on a $x - y$ geodesic P with $x, y \in S$. Then, it is clear that the portion $P[x, v]$ of the $x - v$ path on P together with the edge uv is a $x - u$ geodesic of G', which contains the edge e with $x, u \in S'$. Hence S' is an edge geodetic set of G' and so $eg(G') \leq eg(G) + 1$. Let S' be an eg-set of G'. By Theorems 1.19 and 1.24, $u \in S'$ and $v \notin S'$. Also, it is clear that $S = (S' - \{u\}) \cup \{v\}$ is an edge geodetic set of G so that $eg(G) \leq |S'| - 1 + 1 = |S'| = eg(G')$. Hence the result. ∎

Remark 2.29 The bounds for $eg(G')$ in Theorem 2.28 are sharp. If the graph G is the path $P_n (n \geq 3)$ on n vertices, then, by Theorem 1.21, $eg(P_n) = 2$. Let G' be the path obtained from P_n by adding a pendant edge at one of its end vertices. Then, by Theorem 1.21, $eg(G') = 2 = eg(G)$. If G' is the tree obtained from P_n by adding a pendant edge at a cut vertex of P_n, then by Theorem 1.21, $eg(G') = 3 = eg(G) + 1$.

Theorem 2.30 *Let G' be a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G. Then $eg(G) = eg(G')$ if and only if v is a vertex of some eg-set of G.*

Proof. First, assume that there is an eg-set S of G such that $v \in S$. Let $S' = (S - \{v\}) \cup \{u\}$. We show that S' is an eg-set of G'. If $e = uv$, then it is clear that e lies on every $w - u$ geodesic of G, where $w \in S'(w \neq u)$. Let e be any edge of G. Since S is an eg-set of G, e lies on a $x - y$ geodesic in G with $x, y \in S$. If both $x, y \in S - \{v\}$, then e also lies on a $x - y$ geodesic in G' with $x, y \in S'$. If e lies on a $x - v$ geodesic in G with $x \in S - \{v\}$, then e also lies on $x - u$ geodesic in G'. Thus S' is an edge geodetic set of G' so that $eg(G') \leq |S'| = |S| = eg(G)$. Now, the result follows from Theorem 2.28.

Conversely, suppose that $eg(G) = eg(G')$. Suppose that v does not belong to any eg-set of G. Let S' be an eg-set of G'. Since u is an end vertex of G' and v is a cut vertex of G', by Theorems 1.19 and 1.24, $u \in S'$ and $v \notin S'$. Let $S = (S' - \{u\}) \cup \{v\}$. Then $S \subseteq V(G)$ and $|S| = |S'| = eg(G') = eg(G)$. Let e be any edge of G. Then e is also an edge of G' and so e lies on a geodesic P in G' joining a pair of vertices $x, y \in S'$. If $x \neq u$ and $y \neq u$, then $x \in S$ and $y \in S$ so that e lies on a geodesic joining a pair of vertices in S. Otherwise, let $x \neq u$ and $y = u$. Then it follows that e lies on a geodesic in G joining x and v in S. Thus, S is an edge geodetic set of G and since $|S| = eg(G)$, it follows that S is an eg-set of G. Since $v \in S$, this is contradiction to our assumption. This completes the proof. ∎

Remark 2.31 If a vertex v is added to a connected graph G such that more than one edge is incident with v, then the edge geodetic number of the resulting graph can stay the same, increase significantly or decrease significantly. For example, for the complete bipartite graph $K_{m,n}$ we have, by Theorem 1.25, $eg(K_{m,n}) = m$ for all $2 \leq m \leq n$. However, if we add a new vertex to $K_{m,n}$ and join this vertex to all the vertices of the minimum partite set containing m vertices, the resulting graph is $K_{m,n+1}$ and again by Theorem 1.25, the edge geodetic number is m. Hence a new vertex may be added to a graph along with a large number of edges such that it does not affect the edge geodetic number. On the other hand, it is clear that $eg(C_n) = 2$ for all even $n \geq 4$. If we add a vertex v to this C_n and join v to all the vertices of C_n, the resulting graph is the wheel $K_1 + C_n$. Now, it follows from Theorem 1.26 that $eg(K_1 + C_n) = n$ and so the edge geodetic number of the resulting graph increases significantly. Also, it is clear that $eg(K_{1,n}) = n$ for all $n \geq 2$. If we add a vertex v and join it to all the end vertices of $K_{1,n}$ then we obtain the graph $K_{2,n}$. By Theorem 1.25, $eg(K_{2,n}) = 2$, and so the edge geodetic number of the resulting graph decreases significantly for large n.

The k-edge geodetic number and addition of a pendant edge

We now consider how the k-edge geodetic number of a connected graph G is affected by the addition of a pendant edge.

Proposition 2.32 *Let G' be a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G. Then $eg_k(G') \leq eg_k(G) + 2$.*

Proof. Let S be an eg_k-set of G. Then $S \cup \{u, v\}$ is a k-edge geodetic set of G' and so $eg_k(G') \leq |S \cup \{u, v\}| \leq eg_k(G) + 2$. ∎

Proposition 2.33 *There is no connected graph G with $diam(G) \geq k$ such that $eg_k(G') = 2$, where G' is a graph obtained from G by adding a pendant edge at a vertex of G.*

Proof. Suppose that there exists a connected graph G with $diam(G) \geq k$ such that $eg_k(G') = 2$. Let G' be a graph obtained from G by adding a pendant edge uv at a vertex v of G. By Theorem 1.19, u belongs to every edge geodetic set of G'. Let $S' = \{u, y\}$ be an eg_k-set of G'. Then $y \neq v$ and it is clear that $S = \{v, y\}$ is a eg_{k-1}-set of G. Hence S is an eg-set of G and $d(v, y) = k - 1$. Now, by Theorem 1.23, v and y are antipodal vertices and so $diam(G) = k - 1$, which is a contradiction. Hence the result follows. ∎

Observation 2.34 In a connected graph G, each edge in G has at least one end in every 2-edge geodetic set of G.

Theorem 2.35 *If G' is a graph obtained from a connected graph G by adding a pendant edge at a vertex of G, then $eg_2(G) \leq eg_2(G') \leq eg_2(G) + 1$.*

Proof. Let G' be the graph obtained from G by adding a pendant edge uv at a vertex v of G. Let S be an eg_2-set of G. Let $S' = S \cup \{u\}$. We claim that S' is a 2-edge geodetic set of G'. Let e be an edge of G' be such that $e \notin E(< S' >)$. If $e \in E(G)$, then e lies on a 2-geodesic of vertices in S. If $e \notin E(G)$, then $e = uv$ and $v \notin S$. Let vw be an edge of G. Then, by Observation 2.34, we have $w \in S$. Now, it is clear that the edge uv lies on the 2-geodesic $P : w, v, u$ of G' with $w, u \in S'$. Hence S' is a 2-edge geodetic set of G' and so $eg_2(G') \leq |S'| = eg_2(G) + 1$.

Now, let T' be any eg_2-set of G'. Then by Theorem 2.10, $u \in T'$. Let $T = (T' - \{u\}) \cup \{v\}$. Then $|T| \leq |T'|$. We show that T is a 2-edge geodetic set of G. Let $e = xy$ be any edge of G such that $e \notin E(< T >)$. Then it is clear that $e \notin E(< T' >)$. Now, since $e \in E(G')$ and T' is a eg_2-set of G', we see that $e = xy$ lies on a 2-geodesic P of vertices in T'. By Observation 2.34, we may assume that $x \in T'$. Assume that the geodesic P is $P : x, y, z$ with $x, z \in T'$. Since $xy \in E(G)$, we have $x \neq u$ and so $x \in T$. Now, if $z = u$, then $y = v$ and so $xy \in E(< T >)$, which is a contradiction. Hence $z \neq u$ and so $z \in T$. Thus T is a 2-edge geodetic set of G so that $eg_2(G) \leq |T| \leq |T'| = eg_2(G')$. ∎

Theorem 2.36 *Let G' be a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G. If v belongs to some eg_2-set of G', then $eg_2(G') = eg_2(G) + 1$.*

Proof. Let T' be an eg_2-set of G' such that $v \in T'$. By Theorem 2.10, $u \in T'$. Now, let $T = T' - \{u\}$. Then $|T| = |T'| - 1 = eg_2(G') - 1$ and as in the proof of Theorem 2.35, T is a 2-edge geodetic set of G so that $eg_2(G) \leq |T| = eg_2(G') - 1$ Now, the result follows from Theorem 2.35. ∎

Remark 2.37 The converse of Theorem 2.36 is not true. For the graph $G = K_{1,n}(n \geq 2)$, we have that $eg_2(G) = n$. However, if we add a pendant edge to the cut vertex of $K_{1,n}$, then the resulting graph G' is $K_{1,n+1}$ and so $eg_2(G') = n + 1 = eg_2(G)+1$. However, the cut vertex of $K_{1,n+1}$ does not belong to any eg_2-set of $K_{1,n+1}$.

Problem 2.38 Charaterize graphs G for which $eg_2(G') = eg_2(G)$, where G' is a graph obtained from G by adding a pendant edge.

In view of Proposition 2.32, we have the following realization theorem.

Theorem 2.39 *Let $k \geq 4$ be an integer. For each pair a, b of integers with $4 \leq a \leq b + 2$, there is a connected graph G with $eg_k(G) = b$ and $eg_k(G') = a$, where G' is a graph obtained from G by adding a pendant edge.*

Proof. We prove the theorem by considering five cases.

Case 1. Let $a = b$. Let G be the graph obtained from the path $P : v_0, v_1, \ldots, v_k$ by adding $b - 3$ new vertices $u_1, u_2, \ldots, u_{b-3}$ and joining them to v_2. The graph G is shown in Figure 2.16. It is clear that the edges $u_iv_2(1 \leq i \leq b - 3)$ are the only k-extreme edges of G. Hence $S = \{v_0, v_k, u_1, u_2, \ldots, u_{b-3}, v_2\}$ is the set of all k-extreme vertices and the ends of all k-extreme edges of G. Since S is a k-edge geodetic set of G, it follows from Theorem 2.10 that $eg_k(G) = |S| = b$.

Now, let G' be the graph obtained from G by adding a pendant edge v_kx. It is clear that G' has no k-extreme edges. Let $S' = \{v_0, u_1, u_2, \ldots, u_{b-3}, x\}$ be the set of all k-extreme vertices of G'. Since the edges v_0v_1 and v_1v_2 do not lie on any k-geodesic joining a pair of vertices in S', we have S' is not a k-edge geodetic set of

G'. Since $S' \cup \{v_k\}$ is a k-edge geodetic set of G', it follows from Theorem 2.10 that $eg_k(G') = |S'| + 1 = b = a$.

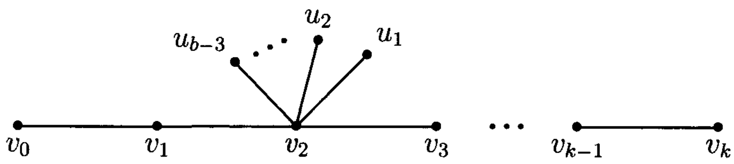

Figure 2.16: G

Case 2. $a = b + 1$. Let G be the graph obtained from the path $P : v_0, v_1, \ldots, v_k$ by adding $b - 2$ new vertices $u_1, u_2, \ldots, u_{b-2}$ and joining each u_i to v_1. Let $S = \{u_1, u_2, \ldots, u_{b-2}, v_0, v_k\}$. Then S is the set of all k-extreme vertices of G. It is clear that G has no k-extreme edges and S is a k-edge geodetic set of G and so by Theorem 2.10, $eg_k(G) = |S| = b$. Now, let G' be the graph obtained from G by adding a new vertex x and joining it to v_1. Then, just as above, the set $S' = \{u_1, u_2, \ldots, u_{b-2}, x, v_0, v_k\}$ of all k-extreme vertices of G' is the eg_k-set of G'. Hence $eg_k(G') = |S'| = b + 1 = a$.

Case 3. $a = b + 2$. Let G be the graph constructed in Case 2. Then, as in Case 2, $eg_k(G) = b$. Now, let G' be the graph obtained from G by adding a new vertex x and joining it to v_2. Then the edge xv_2 is the only k-extreme edge in G'. Since the set $S' = \{u_1, u_2, \ldots, u_{b-2}, v_0, v_k, x, v_2\}$ of all k-extreme vertices together with the ends of the k-extreme edge xv_2 is a k-edge geodetic set, it follows from Theorem 2.10 that $eg_k(G') = |S'| = b + 2 = a$.

Case 4. $a = b - 1$. Let G_1 be the graph obtained from the path $P : v_0, v_1, \ldots, v_k$ by adding a new vertex w and joining it to the vertices v_1, v_2 and v_3. Let Q :

$x_0, x_1, \ldots, x_{k-2}$ be a path such that it is vertex disjoint with G_1. Let G_2 be the graph obtained from G_1 and Q by identifying the vertices v_2 and x_0. Let G be the graph obtained from G_2 by adding $b-5$ new vertices $z_1, z_2, \ldots, z_{b-5}$ and joining each z_i to v_1. The graph is G shown in Figure 2.17. It is clear that the edge v_2w is the only k-extreme edge of G. Since the set $S = \{v_0, z_1, z_2, \ldots, z_{b-5}, v_k, x_{k-2}, v_2, w\}$ of all k-extreme vertices and the ends of the k-extreme edge v_2w of G is a k-edge geodetic set, it follows from Theorem 2.10 that $eg_k(G) = |S| = b$.

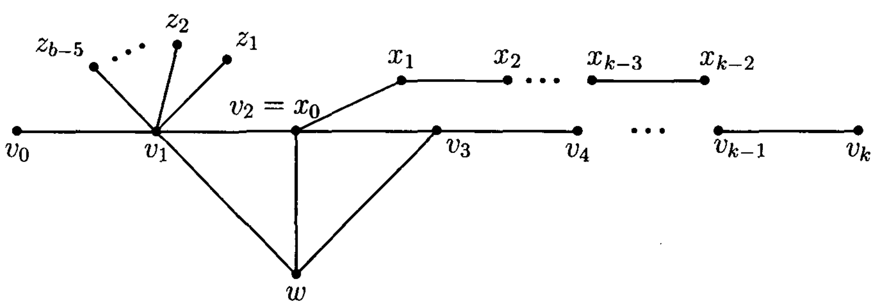

Figure 2.17: G

Now, let G' be the graph obtained from G by adding a pendant edge wx. It is clear that G' has no k-extreme edges. Since the set $S' = \{v_0, z_1, z_2, \ldots, z_{b-5}, x, v_k, x_{k-2}\}$ of all k-extreme vertices of G' is a k-edge geodetic set of G', it follows from Theorem 2.10 that $eg_k(G') = |S'| = b - 1 = a$.

Case 5. $4 \leq a \leq b-2$. Let G_1 be the graph obtained from the path $P : v_0, v_1, \ldots, v_k$ by adding a new vertex w and joining it to both v_2 and v_4. Let G_2 be the graph obtained from G_1 by adding $b-a-1$ new vertices $u_1, u_2, \ldots, u_{b-a-1}$ and joining each u_i to the vertices v_2, v_3, v_4 and w. Let G_3 be the graph obtained from G_2 by adding

$a-4$ new vertices $z_1, z_2, \ldots, z_{a-4}$ and joining each z_i to v_1. Let $Q : x_0, x_1, \ldots, x_{k-3}$ be a path such that it is vertex disjoint with G_3. Let G be the graph obtained from G_3 and Q by identifying the vertices v_3 and x_0. The graph G is shown in Figure 2.18. It is clear that the edges u_iv_3 and u_iw $(1 \leq i \leq b-a-1)$ are the only k-extreme edges of G and so by Theorem 2.10, the vertices $u_1, u_2, \ldots, u_{b-a-1}, v_3, w$ belong to every k-edge geodetic set of G.

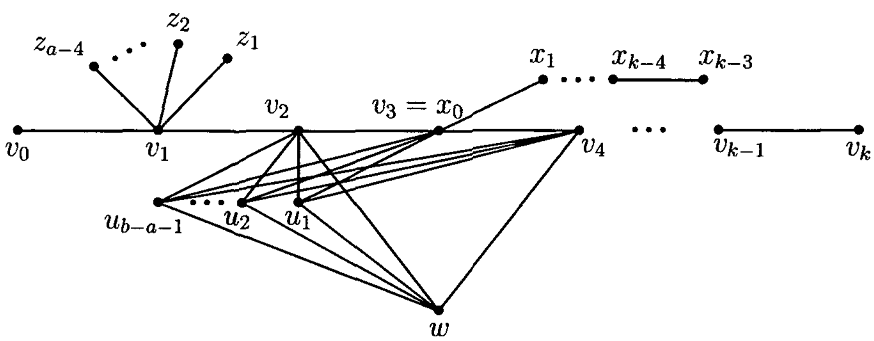

Figure 2.18: G

First, suppose that $k = 4$. Let $S = \{v_0, u_1, u_2, \ldots, u_{b-a-1}, v_3, w, z_1, z_2, \ldots, z_{a-4}, x_1\}$. Then S is the set of all k-extreme vertices and the ends of all k-extreme edges of G. It is clear that S is not a k-edge geodetic set of G and $S \cup \{v_4\}$ is a k-edge geodetic set of G so that by Theorem 2.10, $eg_k(G) = |S| + 1 = b - 1 + 1 = b$. Now, let G' be the graph obtained from G by adding a new vertex x and joining it to w. Then the graph G' has no k-extreme edges. Let $S' = \{v_0, z_1, z_2, \ldots, z_{a-4}, x, x_1\}$. Then S' is the set of all k-extreme vertices of G'. It is clear that S' is not a k-edge geodetic set of G' and $S' \cup \{v_4\}$ is a k-edge geodetic set of G' so that by Theorem 2.10, $eg_k(G') = |S'| + 1 = a - 1 + 1 = a$. Next, suppose that $k \geq 5$. Let $T = \{v_0, u_1, u_2, \ldots, u_{b-a-1}, v_3, w, z_1, z_2, \ldots, z_{a-4}, x_{k-3}, v_k\}$. Then T is the set of all

k-extreme vertices and the ends of all k-extreme edges of G. It is clear that T is a k-edge geodetic set of G and so by Theorem 2.10, $eg_k(G) = |T| = b$. Let G' be the graph obtained from G by adding a new vertex x and joining it to w. Then G' has no k-extreme edges and $T' = \{v_0, z_1, z_2, \ldots, z_{a-4}, x, x_{k-3}, v_k\}$ is the set of all k-extreme vertices of G'. Since T' is a k-edge geodetic set of G', it follows from Theorem 2.10 that $eg_k(G') = |T'| = a$. Thus the proof is complete. ∎