Chapter 1

Preliminaries

In this chapter we collect the basic definitions and theorems which are needed for the subsequent chapters. For graph theoretic terminology, we refer to [2, 13, 18].

Definition 1.1 A *graph* G is a finite nonempty set of objects called *vertices* together with a set of unordered pairs of distinct vertices of G, called *edges*. The vertex set and the edge set of G are denoted by $V(G)$ or simply V and $E(G)$ or simply E respectively.

The number of vertices in G, denoted by n, is called the *order* of G, while the number of edges in G, denoted by m, is called the *size* of G. A graph of order n and size m is called a (n, m)-*graph*.

If $e = \{u, v\}$ is an edge of a graph G, written $e = uv$, we say that e *joins* the vertices u and v; u and v are *adjacent* vertices; u and v are *incident* with e. If two vertices are not joined, then we say that they are *non-adjacent*. If two distinct edges e and f are incident with a common vertex v, then e and f are said to be *adjacent* to each other. A set of vertices in a graph is *independent* if no two vertices in the set are adjacent. Similarly, a set of edges in a graph is *independent* if no two edges in the set are adjacent.

Definition 1.2 The *edge independence number* or the *matching number* $\beta_1(G)$ of a graph G is the maximum cardinality of an independent set of edges.

Theorem 1.3 [13] *For any graph G of order $n \geq 2$, $\beta_1(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Definition 1.4 The *degree* of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $deg_G(v)$ or $deg(v)$. A vertex of degree 0 in G is called an *isolated vertex* and a vertex of degree 1 is called a *pendent vertex* or an *end vertex* of G. A graph is said to be *k-regular* if every vertex of G has degree k.

Definition 1.5 A graph H is called a *subgraph* of G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a *proper subgraph* of G. A *spanning subgraph* of G is a subgraph H with $V(H) = V(G)$. For any set S of vertices of G, the *induced subgraph* $< S >$ is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in $< S >$ if and only if they are adjacent in G. Let v be a vertex of a graph G. The induced subgraph $< V(G) - \{v\} >$ is denoted by $G - v$; it is the subgraph of G obtained by the removal of v and edges incident with v. Similarly, if e is an edge of a graph G, then $G - e$ is the subgraph of G having the same vertex set as G and whose edge set consists of all edges of G except e.

Definition 1.6 Two graphs G and H are *equal* if $V(G) = V(H)$ and $E(G) = E(H)$. A graph G is said to be *isomorphic* to a graph H, if there exists a one-to-one correspondence ϕ from $V(G)$ to $V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$.

Definition 1.7 A graph G is *complete* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n.

Definition 1.8 A *bipartite graph* G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2; (V_1, V_2) is called a *bipartition* of G. If G contains every edge joining V_1 and V_2, then G is called a *complete bipartite graph*. The complete bipartite graph with bipartition (V_1, V_2) such that $|V_1| = r$ and $|V_2| = s$ is denoted by $K_{r,s}$. A *star* is a complete bipartite graph $K_{1,s}$.

Definition 1.9 Let u and v be vertices of a graph G. A u–v *walk* of G is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, e_n, u_n = v$ of vertices and edges in G beginning with vertex u and ending with vertex v such that $e_i = u_{i-1}u_i$, $i = 1, 2, \ldots, n$. The number n is called the *length* of the walk. The walk is said to be *open* if u and v are distinct vertices; it is *closed* otherwise. A walk $u_0, e_1, u_1, e_2, u_2, \ldots, e_n, u_n$ is determined by the sequence $u_0, u_1, u_2, \ldots, u_n$ of its vertices and hence we specify this walk by $W: u_0, u_1, u_2, \ldots, u_n$. A walk in which all the vertices are distinct is called a *path*. A closed walk $u_0, u_1, u_2, \ldots, u_n$ in which $u_0, u_1, u_2, \ldots, u_{n-1}$ are distinct is called a *cycle*. A path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. Given a path P in a graph G and two vertices x, y on P, we use $P[x, y]$ to denote the portion of P between x and y, inclusive of x and y.

Definition 1.10 If a graph G has a spanning cycle C, then G is called a *Hamiltonian graph* and C a *Hamiltonian cycle.* A graph G is said to be *hypohamiltonian* if G does not itself have a Hamiltonian cycle but every graph formed by removing a single vertex from G is Hamiltonian.

Definition 1.11 A graph G is said to be *connected* if any two distinct vertices of G are joined by a path. A maximal connected subgraph of G is called a *component* of G.

Definition 1.12 A *cut-vertex* (*cut-edge*) of a graph G is a vertex (edge) whose removal increases the number of components. A *non separable graph* is connected, nontrivial and has no cut vertices. A *block* of a graph is a maximal non separable subgraph. A graph in which each block is complete is called a *block graph.* For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ in G is called a *branch* of G at v. An end-block of G is a block containing exactly one cut-vertex of G. Thus every end-block is a branch of G.

Definition 1.13 For a vertex v in a connected graph G, $N(v)$ denotes the set of all neighbors of v, and $N[v] = N(v) \cup \{v\}$. A vertex v in G is an *extreme vertex* if the subgraph induced by $N(v)$ is complete. The set of all extreme vertices of G is denoted by $Ext(G)$ and $e(G) = |Ext(G)|$.

Definition 1.14 A graph G is called *acyclic* if it has no cycles. A connected acyclic graph is called a *tree.* A nontrivial path is a tree with exactly two end vertices. A graph G with exactly one cycle is called a *unicyclic* graph. For a connected graph G with cycles, the length of a shortest cycle in G is the *girth* of G and the length of a longest cycle in G is the *circumference* of G. A *caterpillar* is a tree of order 3 or more, for which the removal of all end-vertices leaves a path.

Definition 1.15 For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u, v)$ is called a *$u-v$ geodesic*. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, $rad(G)$ and the maximum eccentricity is its *diameter*, $diam(G)$ of G. Two vertices u and v of G are *antipodal* if $d(u, v) = diam(G)$. A *double star* is a tree of diameter 3.

Definition 1.16 For vertices u and v in a connected graph G, the *closed interval* $I[u, v]$ consists of all vertices lying on some u - v geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{u,v \in S} I[u, v]$. The set S is *convex* if $I[S] = S$. The *convex hull* $[S]$ of S in G is the smallest convex set containing S. The convex hull $[S]$ can also be formed from the sequence $\{I^k[S]\}$, $k \geq 0$, where $I^0[S] = S$, $I^1[S] = I[S]$ and $I^k[S] = I[I^{k-1}[S]]$ for $k \geq 2$. From some term on, this sequence must be constant. Let p be the smallest number such that $I^p[S] = I^{p+1}[S]$. Then $I^p[S]$ is the convex hull $[S]$. A set S of vertices of G is a *hull set* of G if $[S] = V$, and a hull set of minimum cardinality is a *minimum hull set* or a *h-set* of G. The cardinality of a minimum hull set of G is the *hull number* $h(G)$ of G. A set S of vertices of G is a *geodetic set* if $I[S] = V$, and a geodetic set of minimum cardinality is a *minimum geodetic set* or a *g-set* of G. The cardinality of a minimum geodetic set of G is the *geodetic number* $g(G)$ of G. A graph G is an *extreme geodesic graph* if $Ext(G)$ forms a geodetic set of G.

Theorem 1.17 [5] *Each extreme vertex of a connected graph G belongs to every hull set of G. In particular, if the set of all extreme vertices W is a hull set of G then W is the unique h-set of G.*

Definition 1.18 An *edge geodetic set* of G is a set $S \subseteq V$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The *edge geodetic number* $eg(G)$ of G is the minimum cardinality of its edge geodetic sets. An edge geodetic set of cardinality $eg(G)$ is called a *minimum edge geodetic set* or an *eg-set* of G.

The edge geodetic number of a graph was introduced and studied in [28].

Theorem 1.19 [6, 28] *Each extreme vertex of a connected graph G belongs to every geodetic set (edge geodetic set) of G. In particular, if the set of all extreme vertices W is a geodetic set (edge geodetic set) of G then W is the unique g-set (eg-set) of G.*

Theorem 1.20 [28] *For the complete graph K_n, $eg(K_n) = n$.*

Theorem 1.21 [28] *For any tree T, the edge geodetic number $eg(T)$ equals the number of end vertices in T. In fact, the set of all end vertices of T is the unique minimum edge geodetic set of T.*

Theorem 1.22 [28] *Every edge geodetic set of a connected graph G is a geodetic* *set of G.*

Theorem 1.23 [28] *For a connected graph G, $eg(G) = 2$ if and only if there exist two antipodal vertices u and v such that every edge lies on a $u - v$ geodesic of G.*

Theorem 1.24 [28] *For a connected graph G, no cut vertex belongs to any eg-set of G.*

Theorem 1.25 [28] *For the complete bipartite graph $K_{m,n}(m, n \geq 2)$, $eg(K_{m,n}) = min\{m, n\}$.*

Theorem 1.26 [28] *If a connected graph G of order n has exactly one vertex v of degree $n - 1$, then $eg(G) = n - 1$.*

Definition 1.27 For an integer $k \geq 1$, a geodesic in a connected graph G of length k is called a *k-geodesic*. A vertex v is called a *k-extreme vertex* if v is not the internal vertex of a k-geodesic joining any pair of distinct vertices of G. Obviously, each extreme vertex of a connected graph G is k-extreme vertex of G. In particular, each end vertex of G is a k-extreme vertex of G. A set $S \subseteq V$ is called a *k-geodetic set* of G if each vertex v in $V - S$ lies on a k-geodesic of vertices in S. The minimum cardinality of a k-geodetic set of G is its *k-geodetic number* $g_k(G)$. A k-geodetic set of cardinality $g_k(G)$ is called a *g_k-set*.

Theorem 1.28 [24] *For an integer $k \geq 1$, each k-geodetic set of G contains every k-extreme vertex of G. In particular, if the set W of k-extreme vertices is a k-geodetic set of G, then W is the unique g_k-set of G.*

Theorem 1.29 [24] *If G is a connected graph of diameter 2, then $g_2(G) = g(G)$.*

A set $S \subseteq V$ is an *open geodetic set* of G if for each vertex v, either (1) v is an extreme vertex of G and $v \in S$, or (2) v lies as an internal vertex of an x-y geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a *minimum open geodetic set* or *og-set* of G and this cardinality is the *open geodetic number* $og(G)$. The open geodetic number of a graph was studied in [25]. A set $S \subseteq V$ is a *double dominating set* of G if $|N[v] \cap S| \geq 2$ for all $v \in V$. The *double domination number* $\gamma_{\times 2}(G)$ of G is the minimum cardinality of its double dominating sets. Any double dominating set of cardinality $\gamma_{\times 2}(G)$ is a $\gamma_{\times 2}$-*set* of G. The double domination number of a graph was introduced and studied in [17].

Definition 1.30 The *Cartesian product* of graphs G and H, denoted by $G \square H$, has vertex set $V(G) \times V(H)$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2$ and $y_1 y_2 \in E(H)$, or $y_1 = y_2$ and $x_1 x_2 \in E(G)$. The mappings $\pi_G : (x, y) \mapsto x$ and $\pi_H : (x, y) \mapsto y$ from $V(G \square H)$ onto G and H respectively are called *projections*. For a set $S \subseteq V(G \square H)$, we define the *G-projection* on G as $\pi_G(S) = \{x \in V(G) : (x, y) \in S \text{ for some } y \in V(H)\}$, and the *H-projection* $\pi_H(S) = \{y \in V(H) : (x, y) \in S \text{ for some } x \in V(G)\}$. For any $y \in V(H)$, the subgraph of $G \square H$ induced by $\{(x, y) : x \in V(G)\}$ is isomorphic to G. We denote it by G_y and call it the *copy* of G corresponding to y. Similarly, for any x in $V(G)$ the subgraph of $G \square H$ induced by $\{(x, y) : y \in V(H)\}$ is isomorphic to H, and we denote it by H_x and call it the *copy* of H corresponding to x.

Definition 1.31 The composition of two graphs G and H, written as $G \circ H$, is the graph with vertex set $V(G) \times V(H)$ and with (x_1, y_1) is adjacent to (x_2, y_2) if either x_1 is adjacent to x_2 in G or $x_1 = x_2$ and y_1 is adjacent to y_2 in H. The *join* $G + H$ of two vertex disjoint graphs G and H consists of $G \cup H$ and all the edges joining a vertex of G and a vertex of H.

Definition 1.32 The *strong product* of graphs G and H, denoted by $G \boxtimes H$, has vertex set $V(G) \times V(H)$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent with respect to the strong product if,

(a) $x_1 = x_2$ and $y_1y_2 \in E(H)$, or

(b) $y_1 = y_2$ and $x_1x_2 \in E(G)$, or

(c) $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$.

The mappings $\pi_G : (x, y) \mapsto x$ and $\pi_H : (x, y) \mapsto y$ from $V(G \boxtimes H)$ onto G and H respectively are called *projections.* For a set $S \subseteq V(G \boxtimes H)$, we define the *G-projection* on G as $\pi_G(S) = \{x \in V(G) : (x, y) \in S \text{ for some } y \in V(H)\}$, and the *H-projection* $\pi_H(S) = \{y \in V(H) : (x, y) \in S \text{ for some } x \in V(G)\}$.

Let G and H be connected graphs. For a walk $P : (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in $G \boxtimes H$ ($G \square H$, respectively), we define the *G-projection* $\pi_G(P)$ of P as a sequence that is obtained from $(x_1, x_2, \ldots, x_n)$ by changing each constant subsequence with its unique element. For example, if $P : (x_2, y_3), (x_2, y_4), (x_2, y_5), (x_4, y_5), (x_4, y_2), (x_3, y_2), (x_2, y_2)$, then $\pi_G(P)$ is (x_2, x_4, x_3, x_2) (it is obtained from the sequence $(x_2, x_2, x_2, x_4, x_4, x_3, x_2)$). The H-projection $\pi_H(P)$ is defined similarly. It is clear from the definition that for any walk P in $G \boxtimes H$ ($G \square H$, respectively), both $\pi_G(P)$ and $\pi_H(P)$

are walks in the factor graphs G and H respectively. Also, for a path $P : u = u_0, u_1, \ldots, u_n = u'$ in G and $y \in V(H)$, we use P_y to denote the path $P_y : (u, y) = (u_0, y), (u_1, y), \ldots, (u_n, y) = (u', y)$ in $G \boxtimes H$ ($G \Box H$, respectively). Similarly, we can define Q_x, where Q is a path in H and $x \in V(G)$.

Theorem 1.33 [18] *Let G and H be connected graphs with (u, v) and (x, y) arbitrary vertices of the Cartesian product $G \Box H$ of G and H. Then $d_{G \Box H}((u, v), (x, y)) = d_G(u, x) + d_H(v, y)$. Moreover, if P is a $(u, v) - (x, y)$ geodesic in $G \Box H$, then the G-projection $\pi_G(P)$ is a u - x geodesic in G and the H-projection $\pi_H(P)$ is v - y geodesic in H.*

Theorem 1.34 [18] *Let G and H be connected graphs with (u, v) and (x, y) arbitrary vertices of the strong product $G \boxtimes H$ of G and H. Then $d_{G \boxtimes H}((u, v), (x, y)) = max\{d_G(u, x), d_H(v, y)\}$.*

Definition 1.35 A *chord* of a path $P : u_0, u_1, \ldots, u_n$ is an edge $u_i u_j$, with $j \geq i+2$. Any chordless path connecting u and v is a *u-v monophonic path* or *m-path.* The *monophonic closed interval* $J[u, v]$ consists of all the vertices lying on some $u - v$ monophonic path in G. For $S \subseteq V$, the set $J[S]$ is the union of all sets $J[u, v]$ for $u, v \in S$. A set $S \subseteq V$ is a *monophonic set* of G if $J[S] = V$ and a monophonic set of minimum cardinality is the *minimum monophonic set* or an *mn-set* of G. The cardinality of a minimum monophonic set of G is the *monophonic number* of G, denoted by $mn(G)$.

Theorem 1.36 [20] *Each extreme vertex of a connected graph G belongs to every monophonic set of G.*

Definition 1.37 For vertices u and v in a connected graph G, the *detour distance* $D(u,v)$ is the length of a longest $u-v$ path in G. A $u-v$ path of length $D(u,v)$ is called a $u-v$ *detour*. A vertex x is said to *lie on* a $u-v$ detour P if x is a vertex of P including the vertices u and v. The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance from v to a vertex of G. The *detour radius*, $rad_D(G)$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D(G)$ of G is the maximum detour eccentricity among the vertices of G.

Theorem 1.38 [2, 4] *In a connected graph G, the distance $d(u,v)$ and the detour distance $D(u,v)$ are metrics on the vertex set V.*

Definition 1.39 For vertices u and v in a connected graph G, the *detour interval* $I_D[u,v]$ consists of all those vertices lying on a $u-v$ detour in G, while for $S \subseteq V$, $I_D[S] = \bigcup_{u,v \in S} I_D[u,v]$. A set S of vertices of G is called a *detour set* if $I_D[S] = V$. The *detour number* $dn(G)$ of G is the minimum cardinality of a detour set and any detour set of cardinality $dn(G)$ is called a *minimum detour set* of G. A vertex v that belongs to every minimum detour set of G is a *detour vertex* in G.

Theorem 1.40 [10] *Each end vertex of a connected graph G belongs to every detour set of G. Also if the set S of all end-vertices of G is a detour set, then S is the unique minimum detour set of G.*

Theorem 1.41 [10] *If G is a connected graph of order n and detour diameter D, then $dn(G) \leq n - D + 1$.*

Theorem 1.42 [10] *Let G be a connected graph of order $n \geq 4$. Then $dn(G) = n - 1$ if and only if $G = K_{1,n-1}$.*

Theorem 1.43 [10] *Let G be a connected graph of order $n \geq 5$. Then $dn(G) = n - 2$ if and only if G is a double star or $K_{1,n-1} + e$.*

Definition 1.44 Let x be any vertex in a connected graph G. For a vertex y in G, we define the set $I_D[y]^x$ consists of all the vertices distinct from x lying on some $x - y$ detour of G; while for $S \subseteq V$, $I_D[S]^x = \bigcup_{y \in S} I_D[y]^x$. A set S of vertices of G such that $x \notin S$ is a x-detour set of G if $I_D[S] = V - \{x\}$ and a x-detour set of minimum cardinality is the *minimum x-detour set* or *d_x-set* of G. The cardinality of a minimum x-detour set of G is the *x-detour number $d_x(G)$ of G.*

Theorem 1.45 [27] *Each end vertex of G other than x (whether x is an end vertex or not) belongs to every minimum x-detour set of G.*

Theorem 1.46 [27] *For any vertex x in a connected graph G of order n, $d_x(G) \leq n - e_D(x)$.*

Throughout the thesis, G denotes a connected graph with at least two vertices.