# Preface

This thesis embodies the work done by the author under the guidance of Dr. A. P. Santhakumaran. One concept that pervades all of graph theory is that of distance and distance is used in isomorphism testing, graph operations, hamiltonicity problems, extremal problems on connectivity and diameter, and convexity in graphs. The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is defined to be the length of a shortest $u$-$v$ path in $G$. This gives rise to the concepts of convex set, geodetic set, geodetic number, hull set and hull number of a graph [1, 2, 3, 5, 6, 7, 8, 14, 16, 19, 22, 25]. A monophonic $u - v$ path is a $u - v$ chordless path and this gives rise to the concepts of monophonic convex set, monophonic set, monophonic number, monophonic hull set and monophonic hull number of a graph [20, 26, 39]. The detour distance $D(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is defined to be the length of a longest $u$-$v$ path in $G$ and this gives rise to the concepts of detour set and detour number of a graph [4, 9, 10, 11, 27]. Graph theory, as such, has tremendous applications in theoretical Chemistry. Distance concepts in graphs, in particular, the graph invariants arising from various distance are widely applied in Chemical graph theory. Geodetic and monophonic concepts have many applications in location theory and convexity theory. There are interesting applications of these concepts to the problem of designing the routes for shuttles in a city network [30]. The detour concepts and colorings are widely used in the Channel Assignment Problems in radio technologies and also in special situations of molecular problems in theoretical Chemistry [12, 21, 23].

The thesis consists of six chapters.

1. Preliminaries
2. The $k$-edge geodetic number of a graph
3. The edge geodetic number and graph operations
4. The geodetic and hull numbers of strong product graphs
5. The monophonic numbers of Cartesian and strong product graphs
6. The detour hull number of a graph

In this thesis we define and develop various concepts like the $k$-edge geodetic number, the detour hull number and the vertex detour hull number of a graph. Further, we investigate the geodetic number, the edge geodetic number, the $k$-edge geodetic number and the monophonic number of join, Cartesian product and strong product of graphs.

In Chapter 1, we collect the basic definitions and theorems, which are needed for the subsequent chapters. By a graph, we mean a finite undirected connected graph without loops or multiple edges. For basic definitions and graph theoretic terminologies, we refer to [2, 13, 18]. Let $G = (V, E)$ be a connected graph with at least two vertices. The order and size of $G$ are denoted by $n$ and $m$ respectively. The *distance* $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u$ - $v$ path in $G$. An $u$ - $v$ path of length $d(u, v)$ is called an $u$ - $v$ *geodesic*.

The *interval* $I[u, v]$ consists of $u, v$, and all vertices lying on some $u$ - $v$ geodesic of $G$, while for $S \subseteq V$, $I[S] = \bigcup_{u,v \in S} I[u, v]$. The set $S$ is *convex* if $I[S] = S$. The *convex hull* $[S]$ of $S$ is the smallest convex set containing $S$. A set $S \subseteq V$ is a *hull set* of $G$ if $[S] = V$, and a hull set of minimum cardinality is a *minimum hull set* of $G$. The cardinality of a minimum hull set of $G$ is the *hull number* $h(G)$ of $G$. A set $S \subseteq V$ is a *geodetic set* if $I[S] = V$, and a geodetic set of minimum cardinality is a *minimum geodetic set* of $G$. The cardinality of a minimum geodetic set of $G$ is the *geodetic number* $g(G)$ of $G$. For any vertex $v$ in $G$, $N(v)$ denotes the set of all vertices which are adjacent to $v$. A vertex $v$ in $G$ is an *extreme vertex* if the subgraph induced by $N(v)$ is complete. The set of all extreme vertices of $G$ is denoted by $Ext(G)$ and $e(G) = |Ext(G)|$. These concepts were studied in [2, 3, 5, 6, 7, 8, 14, 16, 19, 22, 25]. The geodetic number and the hull number of Cartesian product of two graphs were discussed in [1, 22]. A set $S$ of vertices is an *edge geodetic set* of a graph $G$ if each edge of $G$ lies on a geodesic of vertices in $S$, and the minimum cardinality of an edge geodetic set is the *edge geodetic number* $eg(G)$ of $G$. An edge geodetic set of cardinality $eg(G)$ is called an *eg-set* of $G$. The edge geodetic number of a graph was introduced and studied in [28]. Although the edge geodetic number is greater than or equal to the geodetic number for an arbitrary graph, the properties of the edge geodetic sets and results regarding edge geodetic number are quite different from that of geodetic concepts. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design. In the case of designing the route for a shuttle, although all the vertices are covered by the shuttle when considering geodetic sets, some of the edges may be left out. This drawback is rectified in the case of edge geodetic sets and hence considering edge geodetic sets is more advantageous to the real life application of routing problem. In particular, the edge geodetic sets are more useful than geodetic sets in the case of regulating and routing the goods vehicles to transport the commodities to important places [30]. This leads us to investigate further results on the edge geodetic number of a graph.

For an integer $k \geq 1$, a geodesic in a connected graph $G$ of length $k$ is called a *k-geodesic.* A set $S \subseteq V$ is called a *k-geodetic set* of $G$ if each vertex $v$ in $V - S$ lies on a $k$-geodesic of vertices in $S$. The minimum cardinality of a $k$-geodetic set of $G$ is its *k-geodetic number* $g_k(G)$. A $k$-geodetic set of cardinality $g_k(G)$ is called a *$g_k$-set.* The $k$-geodetic number of a graph was refered to as $k$-geodomination number and studied in [15, 24, 25].

A *chord* of a path $P : u_0, u_1, \ldots, u_n$ is an edge $u_iu_j$, with $j \geq i+2$. Any chordless path connecting $u$ and $v$ is a *u-v monophonic path* or *m-path.* The *monophonic interval* $J[u, v]$ consists of all vertices lying on some $u - v$ monophonic path of $G$. For $S \subseteq V$, the *monophonic closure* $J[S]$ is the set formed by the union of all monophonic intervals $J[u, v]$ with $u, v \in S$. A set $S$ of vertices of $G$ is a *monophonic set* if $J[S] = V$. The *monophonic number*, $mn(G)$ of $G$ is the minimum cardinality of a monophonic set in $G$. The monophonic number of a graph was studied in [20, 26, 39]. The monophonic numbers of join and composition of graphs were discussed in [26].

For vertices $u$ and $v$ in a nontrivial connected graph $G$ of order $n$, the *detour distance* $D(u, v)$ is the length of a longest $u - v$ path in $G$. An $u - v$ path of length $D(u, v)$ is an $u - v$ detour. It is known that the detour distance is a metric on the vertex set $V$ of $G$. The detour distance of a graph was studied in [4]. The detour interval $I_D[u, v]$ consists of $u$, $v$, and all vertices lying in some $u-v$ detour of $G$, while for $S \subseteq V$, $I_D[S] = \bigcup_{u,v \in S} I_D[u, v]$. A set $S \subseteq V$ is called a *detour set* if $I_D[S] = V$. The *detour number* $dn(G)$ of $G$ is the minimum cardinality of a detour set and any detour set of cardinality $dn(G)$ is called a *minimum detour set* of $G$. The detour number of a graph was introduced in [10] and further studied in [9, 11, 27].

In Chapter 2, we introduce and investigate the concept of $k$-edge geodetic number of a graph [29, 31]. For an integer $k \geq 1$, a set $S$ of vertices in a connected graph $G$ is called a *$k$-edge geodetic set* of $G$ if each edge $e$ in $E - E(< S >)$ lies on a $k$-geodesic of vertices in $S$. The minimum cardinality of a $k$- edge geodetic set of $G$ is its *$k$-edge geodetic number* $eg_k(G)$. A $k$-edge geodetic set of cardinality $eg_k(G)$ is called a *$eg_k$-set* of $G$. Certain general properties of this concept are studied. The $k$-edge geodetic numbers of certain classes of graphs are determined. Graphs of order $n$ having edge geodetic number $n$ are characterized. Also, we characterize trees of diameter $d$ for which the $d$-edge geodetic number and the edge geodetic number are equal. It is shown that for each triple $a, b, k$ of integers with $2 \leq a \leq b$ and $k \geq 2$, there is a connected graph $G$ with $g_k(G) = a$ and $eg_k(G) = b$. Also, it is shown that for integers $a, b, c$ and $k \geq 2$ with $3 \leq a \leq b \leq c$, there exists a connected graph $G$ such that $g(G) = a$, $eg(G) = b$ and $eg_k(G) = c$. We also investigate how the edge geodetic number and the $k$-edge geodetic number of a graph are affected by adding a pendant edge to the graph. It is proved that if $G'$ is a graph obtained from $G$ by adding a pendant edge, then $eg(G) \leq eg(G') \leq eg(G) + 1$ and $eg_2(G) \leq eg_2(G') \leq eg_2(G) + 1$. For any integer $k \geq 2$, it is proved that $eg_k(G') \leq eg_k(G) + 2$. It is shown that for any integer $k \geq 4$ and for every pair $a, b$ of integers with $4 \leq a \leq b + 2$, there is a connected graph $G$ such that $eg_k(G) = b$ and $eg_k(G') = a$.

In Chapter 3, we discuss the edge geodetic number and the 2-edge geodetic number of join and Cartesian product of graphs [30, 34]. The Cartesian product of two graphs $G$ and $H$, denoted by $G \Box H$, has the vertex set $V(G) \times V(H)$, where two distinct vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if either $x_1 = x_2$ and $y_1 y_2 \in E(H)$, or $y_1 = y_2$ and $x_1 x_2 \in E(G)$. For $S \subseteq V(G)$ and $T \subseteq V(H)$, it is shown that $S$ and $T$ are edge geodetic sets of $G$ and $H$ respectively if and only if $S \times T$ is an edge geodetic set of $G \Box H$. It is proved that for any connected graphs $G$ and $H$, $\max\{eg(G), eg(H)\} \leq eg(G \Box H) \leq eg(G)eg(H) - \min\{eg(G), eg(H)\}$. An edge geodetic set $S = \{x_1, x_2, \ldots, x_k\}$ of a graph $G$ is called a *linear edge geodetic set* if for any edge $e$ of $G$, there exists an index $i, 1 \leq i < k$ such that the edge $e$ lies on some $x_i$ - $x_{i+1}$ geodesic of $G$. Let $G$ and $H$ be connected graphs with $eg(G) = p$ and $eg(H) = q$. If both $G$ and $H$ contain linear minimum edge geodetic sets, then it is proved that $eg(G \Box H) \leq \lfloor \frac{pq}{2} \rfloor$. For integers $m \geq n \geq 2$, it is shown that $eg(K_m \Box K_n) = mn - n$. If $G$ has a minimum edge geodetic set $S$, which can be partitioned into pairwise disjoint non-empty subsets $S_1, S_2, \ldots, S_n (n \geq 2)$ such that every edge of $G$ lies on an $S_i$ - $S_j$ geodesic for every $i, j$ with $i \neq j$, then it is shown that $eg(G \Box H) = eg(G)$ for every connected graph $H$ with $eg(H) = n$. An edge geodetic set $S$ of a graph $G$ is called a *perfect edge geodetic set* if for every edge $e$ of $G$, there exists a vertex $x \in S$ such that the edge $e$ lies on a $x$ - $w$ geodesic of $G$ for every $w \in S$, where $w \neq x$. It is proved that for connected graphs $G$ and $H$, each having a perfect minimum edge geodetic set, $eg(G \Box H) = \max\{eg(G), eg(H)\}$. For a connected graph $G$, a set $S \subseteq V(G)$ is called a *(edge, vertex)-geodetic set* if for every pair of an edge $e$ and a vertex $v$ of $G$, there exist $x$ and $y$ in $S$ such that $e$ and $v$ lie on geodesics between $x$ and $y$. If a connected graph $G$ has an (edge, vertex)-geodetic set $S$ of cardinality $eg(G)$, then it is shown that $eg(G \Box G) = eg(G)$. Also, we prove that $eg(T_1 \Box T_2) = \max\{eg(T_1), eg(T_2)\}$ for any trees $T_1$ and $T_2$; and we obtain a necessary condition of $G$ for which $eg(G \Box K_2) = eg(G)$.

If $G$ and $H$ are connected graphs of order $m$ and $n$ respectively with $2 \leq m \leq n$, then it is shown that $n \leq eg_2(G \Box H) \leq mn - m$. It is also proved that $eg_2(G \Box K_m) = mn - n$ for $n \leq m$; and $mn - n \leq eg_2(G \Box K_m) \leq mn - m$ for $m < n$. Also, for connected graphs $G$ and $H$ of order $m$ and $n$ with $2 \leq m \leq n$ such that $H$ has an independent set of $n - m + 1$ vertices, it is shown that $eg_2(G \Box H) \leq mn - n$. The *join* $G + H$ of two vertex disjoint graphs $G$ and $H$ consists of $G \cup H$ and all the edges joining a vertex of $G$ and a vertex of $H$. If $G$ and $H$ are connected graphs of order $n$ and $m$ respectively such that neither $G$ nor $H$ contains a full degree vertex, then $eg_2(G + H) = min\{n + eg_2(H), m + eg_2(G)\}$.

In Chapter 4, we investigate properties of geodetic and hull numbers of strong product graphs [32, 33]. The *strong product* of graphs $G$ and $H$, denoted by $G \boxtimes H$, has vertex set $V(G) \times V(H)$, where two distinct vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent with respect to the strong product if, (a) $x_1 = x_2$ and $y_1y_2 \in E(H)$, or (b) $y_1 = y_2$ and $x_1x_2 \in E(G)$, or (c) $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. It is proved that, for connected graphs $G$ and $H$, $max\{2, e(G)e(H)\} \leq h(G \boxtimes H) \leq h(G)h(H)$ and $h(G \boxtimes H) \leq min\{h(H) + e(H)(h(G) - 1), h(G) + e(G)(h(H) - 1)\}$. It is shown that $h(G \boxtimes H) \leq h(H)$ for any connected graph $H$ with no extreme vertices. If $G$ has no extreme vertices, then we prove that $(i)$ $h(G \boxtimes H) = 2$ if the girth of $H$ is even; and $(ii)$ $h(G \boxtimes H) \leq 3$ if the girth of $H$ is odd and at least 5.

Also, it is proved that $h(G \boxtimes K_m) = h(G) + e(G)(m-1)$. A graph $G$ is an *extreme hull graph* if the set of extreme vertices of $G$ is a hull set of $G$. It is proved that $h(G \boxtimes H) = h(G)h(H)$ if and only if both $G$ and $H$ are extreme hull graphs.

It is shown that $g(G \boxtimes H) \geq 4$ and $min\{g(G), g(H)\} \leq g(G \boxtimes H) \leq g(G)g(H)$. Also it is shown that $G$ and $H$ are extreme geodesic graphs if and only if $G \boxtimes H$ is an extreme geodesic graph. A vertex $x$ in a set $S$ of vertices of $G$ is a *geodetic interior vertex* of $S$ if $x \in I[S - \{x\}]$. The set of all geodetic interior vertices of $S$ is denoted by $S^o$, called the *geodetic interior of S*. We prove that $g(G \boxtimes H) \leq min\{|S||T| - (|T|-1)|S^o| : S \; and \; T \; are \; geodetic \; sets \; of \; G \; and \; H \; respectively\}$ for any connected graph $H$ having a full degree vertex. Moreover, if $H$ is an extreme geodesic graph, then $g(G \boxtimes H) = min\{e(H)|S| - (e(H)-1)|S^o| : S \; is \; a \; geodetic \; set \; of \; G\}$. It is proved that $e(G)(g(H)-1) + g(G) \leq g(G \boxtimes H) \leq e(G)(g(H)-1) + og(G)$ for any extreme geodesic graph $H$ with a full degree vertex. Also, it is proved that for integers $2 \leq r \leq s$ and $n \geq 2$, $g(K_{r,s} \boxtimes K_n) = 4$.

In Chapter 5, we investigate the monophonic numbers of Cartesian and strong product of graphs [35, 36]. It is proved that $mn(G \Box H) \leq mn(G)$ for connected graphs $G$ and $H$ such that $G$ is non-complete. Also, it is shown that for integers $m, n \geq 2$, $mn(K_m \Box K_n) = 2$ and $mn(G \Box H) = 2$ for $G$ and $H$ non-complete connected graphs. It is shown that $max\{2, k\} \leq mn(G \Box K_n) \leq mn(G)$ for any non-complete connected graph $G$ with $k$ end vertices. Also, it is shown that for each pair $a, b$ of integers with $2 \leq a \leq b$, there exists a nontrivial connected graph $G$ with $mn(G \Box K_2) = a$

and $g(G\square K_2) = b$. If $S$ and $T$ are monophonic sets of $G$ and $H$ respectively, then $S \times T$ is a monophonic set of $G \boxtimes H$ and it is shown that $max\{2, e(G)e(H)\} \leq mn(G \boxtimes H) \leq mn(G)mn(H)$. It is proved that $mn(G \boxtimes H) \leq |S||T| - min\{|S|, |T^o|\}$; and $mn(G \boxtimes K_n) = min\{n|S| - (n-1)|S^o| : S \textit{ is a monophonic set of } G\}$, where $T^o$ and $S^o$ respectively denote the monophonic interiors of $T$ and $S$. It is shown that for integers $m \geq 3$ and $n \geq 2$,

$$mn(C_m \boxtimes K_n) = \begin{cases} 3n & \text{if } m = 3 \\ 4 & \text{if } m = 4, 5 \\ 3 & \text{if } m \geq 6, \end{cases}$$

and for integers $2 \leq r \leq s$ and $n \geq 2$, $mn(K_{r,s} \boxtimes K_n) = 4$.

In Chapter 6, we introduce and study the detour and vertex detour hull numbers of a graph [37, 38]. A set $S$ of vertices is a *detour convex set* if $I_D[S] = S$. The *detour convex hull* $[S]_D$ is the smallest detour convex set containing $S$. A set $S$ of vertices of $G$ is a *detour hull set* if $[S]_D = V$ and the minimum cardinality of a detour hull set is the *detour hull number* $d_h(G)$ of $G$. Some general properties satisfied by this concept are studied. It is shown that for each pair of positive integers $r$ and $s$, there is a connected graph $G$ with $r$ detour extreme vertices each of degree $s$. It is proved that every two integers $a$ and $b$ with $2 \leq a \leq b$ are realizable as the detour hull number and the detour number respectively. For each triple $D, k$ and $n$ of positive integers with $2 \leq k \leq n - D + 1$ and $D \geq 2$, there is a connected graph of order $n$, detour diameter $D$ and detour hull number $k$. It is proved that for a connected graph $G$ with $diam_D(G) \leq 4$, $dn(G) = d_h(G)$; and for positive integers $a, b$ and $k \geq 2$ with $a < b \leq 2a$, there exists a connected graph $G$ with $rad_D(G) = a, diam_D(G) = b$ and

$d_h(G) = k$, where $rad_D(G)$ and $diam_D(G)$ respectively denotes the detour radius and the detour diameter of $G$. Graphs $G$ for which $d_h(G) = n - 1$; and $d_h(G) = n - 2$ are characterized.

Let $x$ be any vertex in a connected graph $G$. For a vertex $y$ in $G$, we define the set $I_D[y]^x$ consists of all the vertices distinct from $x$ lying on some $x - y$ detour of $G$; while for $S \subseteq V$, $I_D[S]^x = \bigcup_{y \in S} I_D[y]^x$. For a set $S$ of vertices in $G$ with $x \notin S$, we call $S$ an *x-detour convex set* if $I_D[S]^x = S$. The *x-detour convex hull* of $S$, $[S]_D^x$ is the smallest $x$-detour convex set containing $S$. For $x \notin S$, $S$ is an *x-detour hull set* if $[S]_D^x = V - \{x\}$ and an $x$-detour hull set of minimum cardinality is the *x-detour hull number* $dh_x(G)$ of $G$. The $x$-detour number $d_x(G)$ of a graph $G$ was introduced and studied in [27]. Certain general properties of $x$-detour hull number of a graph is studied. It is proved that for any vertex $x$ in a connected graph $G$, $d_h(G) \leq dh_x(G)+1$ and for each pair of positive integers $a, b$ with $2 \leq a \leq b+1$, there is a connected graph $G$ and a vertex $x$ such that $d_h(G) = a$ and $dh_x(G) = b$. Also, it is proved that every two integers $a$ and $b$ with $1 \leq a \leq b$ are realizable as the $x$-detour hull number and the $x$-detour number respectively. We determine bounds for $dh_x(G)$ and characterize graphs $G$ which realize these bounds. Finally, we investigate how the detour and vertex detour hull numbers of a graph are affected by adding a pendant edge. If $G'$ is the graph obtained from $G$ by adding a pendant edge $uv$ at a vertex $v$ of $G$, then it is proved that $d_h(G) \leq d_h(G') \leq d_h(G) + 1$ and $dh_x(G) \leq dh_x(G') \leq dh_x(G) + 1$ for every vertex $x$ distinct from $v$. Also, we characterize graphs for which the bounds are attained.

For publication position of the thesis, refer [29], [30], [31], [32] and [33].