Chapter 5

The monophonic numbers of Cartesian and strong product graphs

In this chapter, we investigate the monophonic numbers of Cartesian and strong product of graphs [35, 36]. It is proved that $mn(G \Box H) \leq mn(G)$ for connected graphs G and H such that G is non-complete. Also, it is shown that for integers $m, n \geq 2$, $mn(K_m \Box K_n) = 2$ and $mn(G \Box H) = 2$ for G and H non-complete connected graphs. It is shown that $max\{2, k\} \leq mn(G \Box K_n) \leq mn(G)$ for any non-complete connected graph G with k end vertices. Also, it is shown that for each pair a, b of integers with $2 \leq a \leq b$, there exists a nontrivial connected graph G with $mn(G \Box K_2) = a$ and $g(G \Box K_2) = b$.

If S and T are monophonic sets of G and H respectively, then $S \times T$ is a monophonic set of $G \boxtimes H$ and it is shown that $max\{2, e(G)e(H)\} \leq mn(G \boxtimes H) \leq mn(G)mn(H)$. It is proved that $mn(G \boxtimes H) \leq |S||T| - min\{|S|, |T^o|\}$; and $mn(G \boxtimes K_n) = min\{n|S| - (n-1)|S^o| : S \text{ is a monophonic set of } G\}$, where T^o and S^o respectively denote the monophonic interiors of T and S. It is shown that for integers $m \geq 3$ and $n \geq 2$,

$$mn(C_m \boxtimes K_n) = \begin{cases} 3n & \text{if } m = 3 \\ 4 & \text{if } m = 4, 5 \\ 3 & \text{if } m \geq 6, \end{cases}$$

and for integers $2 \leq r \leq s$ and $n \geq 2$, $mn(K_{r,s} \boxtimes K_n) = 4$.

The monophonic number of Cartesian product graphs

Theorem 5.1 *Let G and H be connected graphs such that G is non-complete. Then $mn(G\Box H) \leq mn(G)$.*

Proof. Let S be a minimum monophonic set of G and v a vertex of H. Then $mn(G) = |S|$. Let $T = S \times \{v\}$. We show that T is a monophonic set of $G\Box H$. Let (x, y) be a vertex in $G\Box H$ such that $(x, y) \notin T$. Let $P : v = v_0, v_1, \ldots, v_m = y$ be a $v - y$ m-path in H. We consider the following two cases.

Case 1. $x \notin S$. Since S is a monophonic set of G, x lies on a $u - u'$ m-path $Q : u = u_0, u_1, \ldots, u_i = x, u_{i+1}, \ldots, u_n = u'$ $(1 \leq i \leq n-1)$, where $u, u' \in S$. Then it is clear that the paths $P_u : (u, v) = (u, v_0), (u, v_1), \ldots, (u, v_m) = (u, y)$; $P_{u'} : (u', v) = (u', v_0), (u', v_1), \ldots, (u', v_m) = (u', y)$ and $Q_y : (u, y) = (u_0, y), (u_1, y), \ldots, (u_{i-1}, y), (u_i, y) = (x, y), (u_{i+1}, y), \ldots, (u_n, y) = (u', y)$ are m-paths in $G\Box H$. Let $R = P_u \cup Q_y \cup P_{u'}^{-1}$. Then R is a $(u, v) - (u', v)$ path in $G\Box H$ containing the vertex (x, y) with $(u, v), (u', v) \in T$. Since $u \neq u_i$ for $i = 1, 2, \ldots, n$ and $y \neq v_j$ for $j = 0, 1, \ldots, m-1$, there is no chord between the vertices of P_u and Q_y. Similarly, since $u' \neq u_i$ for $i = 0, 1, \ldots, n-1$ and $y \neq v_j$ for $j = 0, 1, \ldots, m-1$, there is no chord between the vertices of Q_y and $P_{u'}^{-1}$. Also, since u and u' are distinct and non-adjacent, there is no chord between the vertices of P_u and $P_{u'}^{-1}$. Hence R is a m-path in $G\Box H$ and so T is a monophonic set of $G\Box H$.

Case 2. $x \in S$. We consider two subcases.

Subcase 2.1. x is adjacent to each vertex in S. Since G is non-complete, the subgraph induced by S is not complete and so there exist vertices $u', u'' \in S$ such that u' and u'' are not adjacent in S. Then it is clear that $R : (u', v) = (u', v_0), (u', v_1), \ldots, (u', v_m) = (u', y), (x, y), (u'', y) = (u'', v_m), (u'', v_{m-1}), \ldots, (u'', v_0) = (u'', v)$ is a monophonic path containing the vertex (x, y) with $(u', v), (u'', v) \in T$. Hence T is a monophonic set of $G\Box H$.

Subcase 2.2. There exists a vertex u in S such that u is not adjacent to x. Let $Q' : x = u_0, u_1, \ldots, u_n = u$ be a m-path in G. Then as in case 1, $Q'_y : (x, y) = (u_0, y), (u_1, y), \ldots, (u_n, y) = (u, y)$, $P_x : (x, v) = (x, v_0), (x, v_1), \ldots, (x, v_m) = (x, y)$ and $P_u : (u, v) = (u, v_0), (u, v_1), \ldots, (u, v_m) = (u, y)$ are m-paths in $G\Box H$. Let $R = P_x \cup Q'_y \cup P_u^{-1}$. Then R is a $(x, v) - (u, v)$ m-path in $G\Box H$ containing the vertex (x, y) with $(x, v), (u, v) \in T$. Hence T is a monophonic set of $G\Box H$. Thus $mn(G\Box H) \leq |T| = |S \times \{v\}| = |S| = mn(G)$. ∎

Corollary 5.2 *For non-complete connected graphs G and H, $2 \leq mn(G\Box H) \leq min\{mn(G), mn(H)\}$.*

Corollary 5.3 *Let G and H be connected graphs such that G is non-complete and $mn(G) = 2$. Then $mn(G\Box H) = 2$.*

Theorem 5.4 *For integers $m, n \geq 2$, $mn(K_m\Box K_n) = 2$.*

Proof. Let u, u' be two distinct vertices in K_m and let v, v' be two distinct vertices in K_n. Let $S = \{(u, v), (u', v')\}$. We claim that S is a monophonic set of $K_m \Box K_n$. Let $(x, y) \in V(K_m \Box K_n) - S$. We consider the following two cases.

Case 1. $x \neq u, u'$ and $y \neq v, v'$. Then it is clear that $P_1 : (u, v), (x, v), (x, y), (u', y), (u', v')$ is a m-path in $K_m \Box K_n$ containing the vertex (x, y). Hence $(x, y) \in J_{K_m \Box K_n}[S]$.

Case 2. $x = u$ or $x = u'$, say $x = u$. Then $x \neq u'$ and $y \neq v$.

Subcase 2.1. $y \neq v'$. The it is clear that $P_2 : (u, v), (x, y), (u', y), (u', v')$ is a m-path in $K_m \Box K_n$ containing the vertex (x, y). Hence $(x, y) \in J_{K_m \Box K_n}[S]$.

Subcase 2.2. $y = v'$. Then $P_3 : (u, v), (x, y), (u', v')$ is a m-path containing the vertex (x, y). Hence $(x, y) \in J_{K_m \Box K_n}[S]$. The other cases are similar and the result follows. ∎

Theorem 5.5 *Let G and H be non-complete connected graphs. Then $mn(G \Box H) = 2$.*

Proof. Let u, u' be the ends of a longest m-path in G, and let v, v' be the ends of a longest m-path in H. Since G is non-complete, u and u' are non-adjacent in G. Similarly, v and v' are non-adjacent in H. Let $T = \{(u, v), (u', v')\}$. We show that T is a monophonic set in $G \Box H$. Let (x, y) be a vertex in $G \Box H$.

Case 1. $x \in J_G[u, u']$ and $y \in J_H[v, v']$. Let $P : u = u_0, u_1, \ldots, u_i = x, \ldots, u_n = u'$ be a $u - u'$ m-path in G and $Q : v = v_0, v_1, \ldots, v_j = y, \ldots, v_m = v'$ a $v - v'$ m-path in H. Let $R = P_v[(u, v), (x, v)] \cup Q_x \cup P_{v'}[(x, v'), (u', v')]$. Then it is clear that R is a $(u, v) - (u', v')$ path in $G \Box H$ containing the vertex (x, y). We show that R is a m-path. Since P and Q are m-paths in G and H respectively, $P_v[(u, v), (x, v)], Q_x$

and $P_{v'}[(x,v'),(u',v')]$ are m-paths in $G\Box H$. Since $x \neq u_k$ for $k = 0,1,\ldots,i-1$ and $v \neq v_l$ for $l = 1,3,\ldots,m$, it follows that there is no chord between the vertices of $P_v[(u,v),(x,v)]$ and Q_x. Similarly, since $x \neq u_k$ for $k = i+1, i+2,\ldots,n$ and $v' \neq v_l$ for $l = 0,1,2,\ldots,m-1$, it is clear that there is no chord between the vertices of Q_x and $P_{v'}[(x,v'),(u',v')]$. Also, since v and v' are non adjacent, there is no chord between the vertices of $P_v[(u,v),(x,v)]$ and $P_{v'}[(x,v'),(u',v')]$. Hence R is a m-path in $G\Box H$ and so T is a monophonic set of $G\Box H$.

Case 2. $x \notin J_G[u,u']$ and $y \in J_H[v,v']$. Let $P' : u = x_0, x_1, \ldots, x_k = x$ be a $u-x$ m-path and $P'' : x = z_0, z_1, \ldots, z_l = u'$ a $x-u'$ m-path in G. Let $Q : v = v_0, v_1, \ldots, v_i = y, \ldots, v_m = v'$ be a $v-v'$ m-path in H containing the vertex y. Let $R = P'_v \cup Q_x \cup P''_{v'}$. The it is clear that R is a $(u,v)-(u',v')$ path containing the vertex (x,y). Since P', P'' and Q are m-paths, it is clear that $P'_v, P''_{v'}$ and Q_x are m-paths in $G\Box H$. Since $x \neq x_i (0 \leq i \leq k-1)$ and $v \neq v_j (1 \leq j \leq n)$, there is no chord between the vertices of P'_v and Q_x. Also, since v and v' are non-adjacent, there is no chord between the vertices of P'_v and $P''_{v'}$. Since $x \neq z_i (1 \leq i \leq l)$ and $v' \neq v_l (0 \leq l \leq m-1)$, there is chord between the vertices of Q_x and $P''_{v'}$. Hence it follows that R is a m-path in $G\Box H$ and so T is a monophonic set of $G\Box H$.

Case 3. $x \in J_G[u,u']$ and $y \notin J_H[v,v']$. This is similar to Case 2.

Case 4. $x \notin J_G[u,u']$ and $y \notin J_H[v,v']$. Let $P' : u = x_0, x_1, \ldots, x_k = x$ and $P'' : x = z_0, z_1, \ldots, z_l = u'$ be $u-x$ and $x-u'$ m-paths in G and let $Q' : v = y_0, y_1, \ldots, y_r = y$ and $Q'' : y = w_0, w_1, \ldots, w_s = v'$ be $v-y$ and $y-v'$ m-paths in H. Since $x \notin J_G[u,u']$ and u and u' are non-adjacent, we see that x is adjacent to at most one of u, u'. Similarly, y is adjacent to at most one of v, v'. We consider the following subcases.

Subase 4.1 x is non-adjacent to u, u'; and y is non-adjacent to v, v'.

Let $R_1 = Q'_u \cup P'_y \cup Q''_x \cup P''_{v'}$. Then R_1 is a $(u,v) - (u',v')$ path in $G\Box H$ containing the vertex (x,y). We show that R_1 is a m-path in $G\Box H$. Since $u \neq x_i(1 \leq i \leq k)$ and $y \neq y_j(0 \leq j \leq r-1)$, there is no chord between the vertices of Q'_u and P'_y. Also, since u is non-adjacent to x, there is no chord between the vertices of Q'_u and Q''_x. Now, since u and u' are ends fo a longest m-path in G, we see that $u \notin V(P'')$. Hence $u \neq z_i(0 \leq i \leq l)$. Similarly, $v' \notin V(Q')$ and so $v' \neq y_j(0 \leq j \leq r)$. Hence there is no chord between the vertices of Q'_u and $P''_{v'}$. Also, it is clear that $x \neq x_i(0 \leq i \leq k-1)$ and $y \neq w_j(1 \leq j \leq s)$ and so there is no chord between the vertices of P'_y and Q''_x. Since y and v' are non-adjacent, there is no chord between the vertices of $P_{y'}$ and $P''_{v'}$. Since $x \neq z_i(1 \leq i \leq l)$ and $v' \neq w_j(0 \leq j \leq s-1)$, there is no chord between Q''_x and $P''_{v'}$. Hence it follows R_1 is m-path in $G\Box H$ and so T is a monophonic set of $G\Box H$.

Subcase 4.2. x is adjacent to u and y is non-adjacent to v, v'. Then u is non-adjacent to u'. Let $R_2 = P'_v \cup Q'_x \cup P''_y \cup Q''_{u'} = ((u,v),(x,v)) \cup Q'_x \cup P''_y \cup Q''_{u'}$. Then as in Subcase 4.1, we can prove that R_2 is $(u,v) - (u',v')$ m-path containing the vertex (x,y). Hence T is monophonic set of $G\Box H$.

Subcase 4.3. x is adjacent to u and y is adjacent to v. Then x is non-adjacent to u' and y is non-adjacent to v'. Then as above, we can prove that $R_3 = P'_v \cup Q'_x \cup P''_y \cup Q''_{u'} = ((u,v),(x,v),(x,y)) \cup P''_y \cup Q''_{u'}$ is a $(u,v) - (u',v')$ m-path in $G\Box H$ containing the vertex (x,y). Hence T is a monophonic set of $G\Box H$. The other cases are similar. ∎

Theorem 5.6 *If G is a non-complete connected graph with k end vertices, then* $max\{2,k\} \leq mn(G\Box K_n) \leq mn(G)$.

Proof. Let $S = \{u_1, u_2, \ldots, u_k\}$ be the set of end vertices of G. For each $u_i (1 \leq i \leq k)$, let H_{u_i} be the copy of K_n corresponding to u_i in $G\Box K_n$. Let W be a monophonic set of $G\Box K_n$. we show that $W \cap V(H_{u_i}) \neq \phi$ for $i = 1, 2, \ldots, k$. Suppose that $W \cap V(H_{u_i}) = \phi$ for some i with $1 \leq i \leq k$. Let (u_i, y) be a vertex in H_{u_i}. Then $(u_i, y) \notin W$. Since W is a monophonic set of $G\Box K_n$, there exist $(x, y), (x', y') \in W$ such that (u_i, y) lies on a m-path $P : (x, y) = (x_0, y_0), (x_1, y_1), \ldots, (x_j, y_j) = (u_i, y), \ldots, (x_l, y_l) = (x', y')$ with $1 \leq j \leq l-1$. Let α be the greatest integer such that $0 \leq \alpha \leq j-1$ and $(x_\alpha, y_\alpha) \notin H_{u_i}$. Let β be the least integer such that $j+1 \leq \beta \leq l$ and $(x_\beta, y_\beta) \notin H_{u_i}$. Then $(x_{\alpha+1}, y_{\alpha+1}), (x_{\alpha+2}, y_{\alpha+2}), \ldots, (x_j, y_j), \ldots, (x_{\beta-1}, y_{\beta-1}) \in H_{u_i}$. Hence $x_{\alpha+1} = x_{\alpha+2} = \cdots = x_j \cdots = x_{\beta-1} = u_i$ and $x_\alpha x_{\alpha+1}, x_\beta x_{\beta-1} \in E(G)$. Thus $x_\alpha u_i, x_\beta u_i \in E(G)$ and since u_i is an end vertex of G, we have $x_\beta = x_\alpha$. Since P is a path, we have $y_\beta \neq y_\alpha$. Hence $(x_\alpha, y_\alpha), (x_\beta, y_\beta) \in E(G\Box K_n)$, which is a contadiction to the fact that P is a m-path in $G\Box K_n$. Hence $W \cap V(H_{u_i}) \neq \phi$ for $i = 1, 2, \ldots, k$. Thus $mn(G\Box K_n) = |W| \geq max\{2, k\}$. The other inequality follows from Theorem 5.1. ∎

Corollary 5.7 *If the graph G is a tree, then $mn(G\Box K_n) = mn(G)$.*

Proof. Since the set of end vertices of G is the unique monophonic set of G, the result follows. ∎

Remark 5.8 The inequalities in Theorem 5.6 are strict. Let G be the graph in Figure 5.1(a) and the graph $G\Box K_2$ is shown in Figure 5.1(b). Then it follows from Theorem 1.36 that $mn(G) = 4$. Now, as in the proof of Theorem 5.6, we have either

(x_1, y_1) or (x_1, y_2) belongs to every monophonic set of $G \Box K_2$. Similarly, either (x_2, y_1) or (x_2, y_2) belongs to every monophonic set of $G \Box K_2$. It can be easily checked that any pair of the above vertices is not a monophonic set of $G \Box K_2$ and so $mn(G \Box K_2) \geq 3$. Als, it can be easily seen that $S = \{(x_1, y_1), (x_2, y_2), (u, v)\}$ is a monophonic set of $G \Box K_2$ and so $mn(G \Box K_2) = 3$.

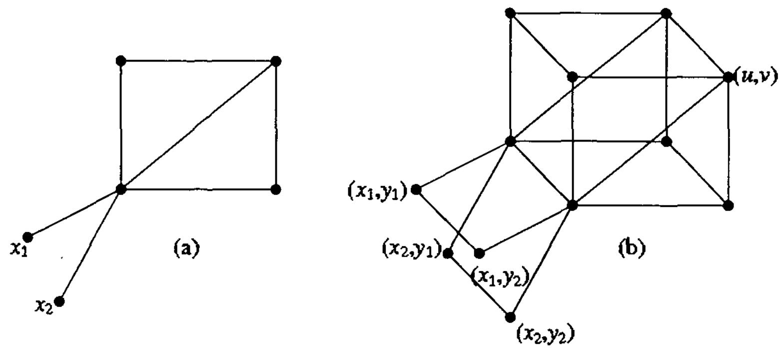

Figure 5.1: Graphs G and $G \Box K_2$ in Remark 5.8

Remark 5.9 The inequalities in Theorem 5.6 are sharp. Let G be the graph in Figure 5.2(a) and the graph $G \Box K_2$ is shown in Figure 5.2(b). It follows from Theorem 1.36 that $mn(G) = 4$. Since $S = \{(u, v), (u', v')\}$ is a monophonic set of $G \Box K_2$, $mn(G \Box K_2) = 2$ and so the left inequality is sharp. By Corollary 5.7, the right inequality is sharp for any tree T.

In view of Theorem 5.1, we have the following realization result.

Theorem 5.10 *For integers a, b and n with $2 \leq a \leq b$ and $n \geq 2$, there exists a connected graph G with $mn(G) = b$ and $mn(G \Box K_n) = a$.*

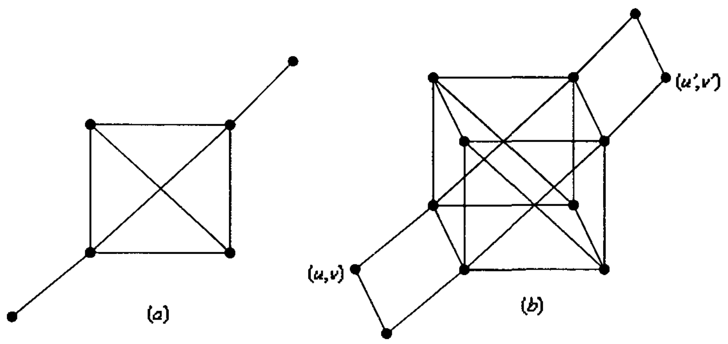

Figure 5.2: Graphs G and $G\Box K_2$ in Remark 5.9

Proof. Le $a = b$. By Corollary 5.7, the graph $G = K_{1,a}$ has the desired properties. So, assume that $a < b$. Let H be the graph obtained from the path $P_3 : u_1, u_2, u_3$ by adding $a-1$ new vertices $x_1, x_2, \ldots, x_{a-1}$ and joining each $x_i (1 \leq i \leq a-1)$ to u_1. Let G be the graph obtained from H by adding $b-a$ new vertices $z_1, z_2, \ldots, z_{b-a}$ and joining each $z_i (1 \leq i \leq b-a)$ to both u_1 and u_2. We show that the graph G has the desired properties. Let $S = \{x_1, x_2, \ldots, x_{a-1}, u_3, z_1, z_2, \ldots, z_{b-a}\}$ be the set of extreme vertices of G. Since S is a monophonic set of G, by Theorem 1.36, $mn(G) = |S| = b$. Since $S' = \{x_1, x_2, \ldots, x_{a-1}, u_3\}$ is the set of end vertices of G, by Theorem 5.6, $mn(G\Box K_n) \geq |S'| = a$. For any vertex v in K_n, let $T = S' \times \{v\}$. Then $|T| = |S'| = a$. We show that T is a monophonic set of $G\Box K_n$. Let $(x, y) \in V(G\Box K_n)$. First, assume that $x \neq z_i (1 \leq i \leq b-a)$. Hence x lies on the m-path $P : x_i, u_1, u_2, u_3$ for some i. If $y = v$, then P_v : $(x_i, v), (u_1, v), (u_2, v), (u_3, v)$ is a m-path in $G\Box K_n$ containing the vertex (x, y) with $(x_i, v), (u_3, v) \in T$. If $y \neq v$, then $Q : (x_i, v), (x_i, y), (u_1, y), (u_2, y), (u_3, y), (u_3, v)$ is a $(x_i, v) - (u_3, v)$ m-path in $G\Box K_n$ containing the vertex (x, y) with $(x_i, v), (u_3, v) \in T$. Next, assume that $x = z_i (1 \leq i \leq b-a)$. Now, if $y = v$, then it is clear that

$Q : (x_1, v), (u_1, v), (z_i, v), (z_i, y'), (u_2, y'), (u_3, y'), (u_3, v)$ is a $(x_1, v) - (u_3, v)$ m-path in $G \square K_n$ containing the vertex (x, y), where $y' \in V(K_n)$ such that $y' \neq v$. If $y \neq v$, then $Q : (x_1, v), (x_1, y), (u_1, y), (z_i, y), (z_i, v), (u_2, v), (u_3, v)$ is a $(x_1, v) - (u_3, v)$ m-path in $G \square K_n$ containing the vertex (x, y). Hence T is a monophonic set of $G \square K_n$ and so $mn(G \square K_n) \leq |T| = |S'| = a$. Thus $mn(G \square K_n) = a$. ∎

We use the following lemma to give another realization result.

Lemma 5.11 [3] *Let G be a connected graph of order at least 3. If G contains a minimum geodetic set S with a vertex x such that every vertex of G lies on some $x - w$ geodesic in G for some $w \in S$, then $g(G) = g(G \square K_2)$.*

Proposition 5.12 *For each pair a, b of integers with $2 \leq a \leq b$, there exists a nontrivial connected graph G with $mn(G \square K_2) = a$ and $g(G \square K_2) = b$.*

Proof. For $a = b$, it follows from Lemma 5.11 and Corollary 5.7 that $mn(K_{1,a} \square K_2) = g(K_{1,a} \square K_2) = a$. So, assume that $a < b$. Let G be the graph constructed in Theorem 5.10. Then the set $S = \{x_1, x_2, \ldots, x_{a-1}, z_1, z_2, \ldots, z_{b-a}, u_3\}$ is a minimum geodetic set of G which satisfies the hypothesis of Lemma 5.11 and so $g(G \square K_2) = g(G) = b$. It is proved in Theorem 5.10 that $mn(G \square K_2) = a$. ∎

Now, we leave the following problem as an open question.

Problem 5.13 Characterize the class of all graphs G for which $mn(G \square K_n) = mn(G)$.

Monophonic number of strong product graphs

The following lemma is used to prove an upper bound for the monophonic number of strong product graphs.

Lemma 5.14 *Let G and H be connected graphs. Then for $u, u' \in V(G)$ and $v, v' \in V(H)$, $J_G[u,u'] \times J_H(v,v') \subseteq J_{G\boxtimes H}((u,v),(u',v'))$.*

Proof. Let $P : u = u_0, u_1, \ldots, u_i = x, \ldots, u_n = u'$ be a $u - u'$ m-path in G containing the vertex x with $0 \le i \le n$ and let $Q : v = v_0, v_1, \ldots, v_j = y, \ldots, v_m = v'$ be a $v - v'$ m-path in H containing the vertex y with $1 \le j \le m-1$. We consider two cases

Case 1. $1 \le i \le n-1$. Since $(u_{i-1}, v)(x, v_1), (x, v_{m-1})(u_{i+1}, v') \in E(G \boxtimes H)$, it follows that $R = P_v[(u_0,v),(u_{i-1},v)] \cup \{(u_{i-1},v),(x,v_1)\} \cup Q_x[(x,v_1),(x,v_{m-1})] \cup \{(x,v_{m-1}),(u_{i+1},v')\} \cup P_{v'}[(u_{i+1},v'),(u_n,v')]$ is a $(u,v)-(u',v')$ path in $G\boxtimes H$ containing the vertex (x,y). We show that R is a $(u,v)-(u',v')$ m-path in $G\boxtimes H$ containing the vertex (x,y). Since P and Q are m-paths, we see that $P_v[(u_0,v),(u_{i-1},v)]$, $Q_x[(x,v_1),(x,v_{m-1})]$ and $P_{v'}[(u_{i+1},v'),(u_n,v')]$ are m-paths in $G \boxtimes H$. Now, since $x \neq u_k$ are non-adjacent for $k = 0, 1, \ldots, i-2 (i \ge 2)$; and $v \neq v_l$ are non-adjacent for $l = 2, 3, \ldots, m-1$, there is no chord between the vertices of $P_v[(u_0,v),(u_{i-1},v)]$ and $Q_x[(x,v_1),(x,v_{m-1})]$. Note that if $i = 1$, then $P_v[(u_0,v),(u_{i-1},v)]$ is the single point (u_0,v) and so there is no chord between the vertices of $P_v[(u_0,v),(u_{i-1},v)]$ and $Q_x[(x,v_1),(x,v_{m-1})]$. Similarly, we can show that there is no chord between the vertices of $Q_x[(x,v_1),(x,v_{m-1})]$ and $P_{v'}[(u_{i+1},v'),(u_n,v')]$. Also, since v and v' are distinct and non-adjacent, there is no chord between the vertices of $P_v[(u_0,v),(u_{i-1},v)]$

and $P_{v'}[(u_{i+1}, v'), (u_n, v')]$. Hence it follows that R is a $(u, v) - (u', v')$ m-path in $G \boxtimes H$ containing the vertex (x, y). Since $(x, y) \neq (u, v), (u', v')$, we have $(x, y) \in J_{G\boxtimes H}((u, v), (u', v'))$.

Case 2. $i = 0$ or $i = n$. Without loss of generality assume that $i = 0$ and so $x = u = u_0$. Since (u, v_{m-1}) and (u_1, v') are adjacent, $R = Q_u[(u, v_0), (u, v_{m-1})] \cup \{(u, v_{m-1}), (u_1, v')\} \cup P_{v'}[(u_1, v'), (u_n, v')]$ is a $(u, v) - (u', v')$ path in $G \boxtimes H$ containing the vertex (x, y). Now, since $u \neq u_k$ are non-adjacent for $k = 2, 3, \ldots, n$; and $v' \neq v_l$ are non-adjacent for $l = 0, 1, \ldots, m-2$, there is no chord between the vertices of $Q_u[(u, v_0), (u, v_{m-1})]$ and $P_{v'}[(u_1, v'), (u_n, v')]$. Hence R is a $(u, v) - (u', v')$ m-path in $G \boxtimes H$ containing the vertex (x, y). Since $y \neq v, v'$, we have $(x, y) \neq (u, v), (u', v')$. Thus $(x, y) \in J_{G\boxtimes H}((u, v), (u', v'))$. ∎

Proposition 5.15 *Let G and H be connected graphs. Let S and T be monophonic sets of G and H respectively. Then $S \times T$ is a monophonic set of $G \boxtimes H$.*

Proof. Let $(x, y) \in V(G \boxtimes H)$. Since S and T are monophonic sets of G and H respectively, we have $x \in J_G[u, u']$ and $y \in J_H[v, v']$ for some $u, u' \in S$ and $v, v' \in T$. If $x \in J_G(u, u')$ or $y \in J_H(v, v')$, then by Lemma 5.14, $(x, y) \in J_{G\boxtimes H}((u, v), (u', v'))$. Otherwise, $(x, y) \in S \times T$. Thus $S \times T$ is a monophonic set of $G \boxtimes H$. ∎

Proposition 5.16 *Let G and H be connected graphs. Then $max\{2, e(G)e(H)\} \leq mn(G \boxtimes H) \leq mn(G)mn(H)$.*

Proof. Let S and T be minimum monophonic sets of G and H respectively. By Proposition 5.15, $S \times T$ is a monophonic set of $G \boxtimes H$ and so $mn(G \boxtimes H) \leq |S \times T| =$

$mn(G)mn(H)$. The other inequality follows from Theorems 1.36 and 4.3. ∎

Corollary 5.17 *Let G and H be extreme monophonic graphs. Then $mn(G \boxtimes H) = mn(G)mn(H) = e(G)e(H)$.*

Problem 5.18 Let G and H be connected graphs such that $G \boxtimes H$ is an extreme monophohic graph. Is it true that G and H are extreme monophonic graphs ?

In the following we introduce the concepts of open monophonic sets and open monophonic number of a graph and obtain an upper bound of $mn(G \boxtimes H)$ in terms of the open monophonic numbers of the factor graphs. Also, we find the exact value of the monophonic number of several classes of strong product graphs.

A set $S \subseteq V(G)$ is an *open monophonic set* if for each vertex v, either (1) v is an extreme vertex of G and $v \in S$, or (2) v lies as an internal vertex of an x-y m-path for some $x, y \in S$. An open monophonic set of minimum cardinality is a *minimum open monophonic set* of G and this cardinality is the *open monophonic number* $og(G)$. A vertex x in a set S of vertices of G is a *monophonic interior vertex* of S if $x \in J_G[S - \{x\}]$. The set of all monophonic interior vertices of S is denoted by S^o, called the *monophonic interior of* S. Through the section, for a set S of vertices, S^o denotes the monophonic interior of S. It is clear that a monophonic set S of G is an open monophonic set if and only if $S^o = S - Ext(G)$.

Theorem 5.19 *Let G and H be connected graphs. Let S and T be monophonic sets of G and H respectively. Then $mn(G \boxtimes H) \leq |S||T| - min\{|S|, |T^o|\}$.*

Proof. Let $S = \{g_1, g_2, \ldots, g_p\}$ and $T = \{h_1, h_2, \ldots, h_q\}$ be monophonic sets of G and H respectively. If $T^o = \phi$, then the result follows from Proposition 5.15. So, assume that $T^o \neq \phi$. Let $T^o = \{h_1, h_2, \ldots, h_m\}$, where $1 \leq m \leq q$.

Case 1. $p \geq m$. Let $W = S \times T - \bigcup_{i=1}^{m}\{(g_i, h_i)\}$. Then $|W| = pq - m$. We show that W is a monophonic set of $G \boxtimes H$. Let (x, y) be a vertex of $G \boxtimes H$. Since S and T are monophonic sets of G and H respectively, we have $x \in J_G[g_i, g_j]$ and $y \in J_H[h_k, h_l]$ for $0 \leq i < j \leq p$ and $0 \leq k < l \leq q$. We consider two subcases.

Subcase 1.1. $x \in J_G(g_i, g_j)$ or $y \in J_H(h_k, h_l)$. Suppose that $i = k$. Then $i \neq l$ and $j \neq k$ so that $(g_i, h_l), (g_j, h_k) \in W$. It follows from Lemma 5.14 that $(x, y) \in J_{G\boxtimes H}((g_i, h_l), (g_j, h_k)) \subseteq J_{G\boxtimes H}[W]$. Now, suppose that $i \neq k$. If $j = l$, then $i \neq l$ and $j \neq k$. Hence this is similar to the above case. If $j \neq l$, then $(g_j, h_l) \in W$. Since $i \neq k$, we have $(g_i, h_k) \in W$. Now, it follows from Lemma 5.14 that $(x, y) \in J_{G\boxtimes H}((g_i, h_k), (g_j, h_l))$. Hence $(x, y) \in J_{G\boxtimes H}[W]$.

Subcase 1.2 $x \in \{g_i, g_j\}$ and $y \in \{h_k, h_l\}$. Let $y = h_k$. If $k \geq m + 1$, then it is clear that $(x, y) \in W \subseteq J_{G\boxtimes H}[W]$. If $k \leq m$, then $h_k \in T^o$ and so $h_k \in J_H(h_r, h_s)$ for $0 \leq r < s \leq q$. Hence we have $x \in J_G[g_i, g_j]$ and $y = h_k \in J_H(h_r, h_s)$.Then as in Subcase 1.1 we can prove that $(x, y) \in J_{G\boxtimes H}[W]$. If $y = h_l$, then we can prove similarly that $(x, y) \in J_{G\boxtimes H}[W]$. Hence W is a monophonic set of $G \boxtimes H$.

Case 2. $p < m$. Let $W = S \times T - \bigcup_{i=1}^{p}\{(g_i, h_i)\}$. Then, as in case 1, we can prove that W is a monophonic set of $G \boxtimes H$. Hence the result follows. ∎

Corollary 5.20 *Let G and H be connected graphs. Then*

$$mn(G \boxtimes H) \leq mn(G)omn(H) - min\{mn(G), omn(H) - e(H)\}.$$

Proof. Let S be a minimum monophonic set of G and T a minimum open monophonic set of H. Then $mn(G) = |S|$ and $omn(H) = |T|$ and $T^o = T - Ext(H)$. Hence the result follows from Theorem 5.19. ∎

Proposition 5.21 *Let G be a connected graph. If S is a monophonic set of $G \boxtimes K_n$, then $\pi_G(S)$ is a monophonic set of G.*

Proof. Let x be a vertex of G such that $x \notin \pi_G(S)$. Then $(x, y) \notin S$ for any $y \in V(K_n)$. Since S is a monophonic set of $G \boxtimes K_n$, there exist $(u, v), (u', v') \in S$ such that (x, y) lies on an m-path $P : (u, v) = (u_0, v_0), (u_1, v_1), \ldots, (u_k, v_k) = (x, y), \ldots, (u_m, v_m) = (u', v')$ with $1 \leq k \leq m-1$. We first claim that all the u_i's ($1 \leq i \leq m$) are distinct. Suppose that $u_i = u_j$ for some $i < j$. If $j = i+1$, then it follows that either (u_{i-1}, v_{i-1}) is adjacent to (u_j, v_j) or (u_{j+1}, v_{j+1}) is adjacent to (u_i, v_i), which is a contradiction to the fact that P is an m-path in $G \boxtimes K_n$. If $j \neq i+1$, then the edge $(u_i, v_i)(u_j, v_j)$ is a chord of the path P, which is also a contradiction. Thus all the u_i's are distinct and it follows that $\pi_G(P) : u = u_0, u_1, \ldots, u_k = x, \ldots, u_m = u'$ is an m-path in G with $u, u' \in \pi_G(S)$. Hence $\pi_G(S)$ is a monophonic set of G. ∎

Corollary 5.22 *Let G be a connected graph. Then $mn(G) \leq mn(G \boxtimes K_n)$.*

Proof. Let S be a minimum monophonic set of $G \boxtimes K_n$. Then by Proposition 5.21, $\pi_G(S)$ is a monophonic set of G. Hence $mn(G) \leq |\pi_G(S)| \leq |S| = mn(G \boxtimes K_n)$. ∎

Theorem 5.23 *Let G be a connected graph. Then*

$$mn(G \boxtimes K_n) = min\{n|S| - (n-1)|S^o| : S \text{ is a monophonic set of } G\}.$$

Proof. Let S be a monophonic set of G and let $W = ((S - S^o) \times V(K_n)) \cup (S^o \times \{v\})$, where v is any vertex in K_n. Then $|W| = n|S| - (n-1)|S^o|$. We first show that W is a monophonic set of $G \boxtimes K_n$. Let (x, y) be a vertex in $G \boxtimes K_n$.

Case 1. $x \notin S$. Then there exist $u, u' \in S$ such that x lies on an m-path $P : u = u_0, u_1, \ldots, u_i = x, \ldots, u_m = u'$ with $1 \leq i \leq m - 1$. Now, it is not diffcult to prove that $P' : (u, v) = (u_0, v), (u_1, v), \ldots, (u_{i-1}, v), (u_i, y), (u_{i+1}, v), \ldots, (u_m, v) = (u', v)$ is an m-path in $G \boxtimes K_n$ containing the vertex (x, y) with $(u, v), (u', v) \in W$. Hence W is a monophonic set of $G \boxtimes K_n$.

Case 2. $x \in S$. If $x \notin S^o$, then $(x, y) \in (S - S^o) \times V(K_n) \subseteq W$. So, we assume that $x \in S^o$. Hence there exist $z, z' \in S$ such that x lies on a $z - z'$ m-path $Q : z = z_0, z_1, \ldots, z_i = x, \ldots, z_m = z'$ with $1 \leq i \leq m - 1$. Then, as in Case 1, we have W is a monophonic set of $G \boxtimes K_n$. Thus $mn(G \boxtimes K_n) \leq n|S| - (n-1)|S^o|$ for any monophonic set S of $G \boxtimes K_n$. Now, let W be a minimum monophonic set of $G \boxtimes K_n$. Then $mn(G \boxtimes K_n) = |W|$. By Proposition 5.21, $S_1 = \pi_G(W)$ is a monophonic set of G. We claim that $(S_1 - S_1^o) \times V(K_n) \subseteq W$. Let $(x, y) \in (S_1 - S_1^o) \times V(K_n)$. Then $x \notin S_1^o$. If $(x, y) \notin W$, then there exist $(u, v), (u', v') \in W$ such that (x, y) lies on an m-path $P : (u, v) = (u_0, v_0), (u_1, v_1), \ldots, (u_i, v_i) = (x, y), \ldots, (u_m, v_m) = (u', v')$ with $1 \leq i \leq m - 1$. Then, as in the proof of Proposition 5.21, we can show that all the u_i's are distinct and so $\pi_G(P) : u = u_0, u_1, \ldots, u_i = x, \ldots, u_m = u'$ is an m-path in G with $x \neq u, u'$. Thus $x \in J_G(u, u')$ with $u, u' \in S_1$ and so $x \in S_1^o$, which is a contradiction. Hence $(x, y) \in W$ and so $(S_1 - S_1^o) \times V(K_n) \subseteq W$. Let $X = W - ((S_1 - S_1^o) \times V(K_n))$. Now, we claim that $\pi_G(X) = S_1^o$. Let $x \in \pi_G(X)$. Then $(x, y) \in X$ for some $y \in V(K_n)$ and so $x \notin S_1 - S_1^o$. Since $x \in \pi_G(W) = S_1$, we have $x \in S_1^o$. Thus $\pi_G(X) \subseteq S_1^o$. Let $x \in S_1^o$. Then $x \in S_1$. Since $S_1 = \pi_G(W)$, there exists $y \in V(K_n)$ such that $(x, y) \in W$. Since $x \notin S_1 - S_1^o$, we have $(x, y) \in X$ and so $x \in \pi_G(X)$. Thus $\pi_G(X) = S_1^o$. Let $T = ((S_1 - S_1^o) \times V(K_n)) \cup (S_1^o \times \{v\})$, where $v \in V(G)$. Then, as in the first part of the proof of this theorem, T is a monophonic set of $G \boxtimes K_n$. Now, if $|X| > |S_1^o|$, then $|T| = |(S_1 - S_1^o) \times V(K_n)| + |S_1^o| < |(S_1 - S_1^o) \times V(K_n)| + |X| = |W|$, which is a contradiction to the fact that W is a minimum monophonic set of $G \boxtimes K_n$. Hence we have $|X| = |S_1^o|$. Thus $|W| = |(S_1 - S_1^o) \times V(K_n)| + |X| = |(S_1 - S_1^o) \times V(K_n)| + |S_1^o|$. Hence it follows that $mn(G \boxtimes K_n) = min\{n|S| - (n-1)|S^o| : S \textit{ is a monophonic set of } G\}$. ■

Corollary 5.24 *Let G be a connected graph. Then*
$e(G)(n-1) + mn(G) \leq mn(G \boxtimes K_n) \leq e(G)(n-1) + omn(G)$.

Proof. Suppose that $mn(G \boxtimes K_n) < e(G)(n-1) + mn(G)$. Then, by Theorem 5.23, there exists a monophonic set S of G such that $n|S| - (n-1)|S^o| < e(G)(n-1) + mn(G)$. Thus, $n|S| < e(G)(n-1) + mn(G) + (n-1)|S^o|$. Since $S^o \subseteq S - Ext(G)$, we have $n|S| < e(G)(n-1) + mn(G) + (n-1)(|S| - e(G))$. This implies that $n|S| < mn(G) + (n-1)|S|$. Hence $|S| < mn(G)$, which is a contradiction. Thus $e(G)(n-1) + mn(G) \leq mn(G \boxtimes K_n)$. For the other inequality, let S be a minimum open monophonic set of G. Then $omn(G) = |S|$. By Theorem 5.23, we have $mn(G \boxtimes K_n) \leq n|S| - (n-1)|S^o| = n|S| - (n-1)(|S| - e(G)) = e(G)(n-1) + omn(G)$. ■

Corollary 5.25 *Let G be a connected graph. Then*
$mn(G \boxtimes K_n) = e(G)(n-1) + mn(G)$ *if and only if* $mn(G) = omn(G)$.

Proof. Suppose that $mn(G) = omn(G)$. Then the result follows from Corollary 5.24. Conversely, assume that $mn(G \boxtimes K_n) = e(G)(n-1) + mn(G)$. By Theorem 5.23, there exists a monophonic set S of G such that $mn(G \boxtimes K_n) = n|S| - (n-1)|S^o|$ and so $n|S| - (n-1)|S^o| = e(G)(n-1) + mn(G)$ .....(1). Since $S^o \subseteq S - Ext(G)$, we have $e(G)(n-1) + mn(G) = n|S| - (n-1)|S^o| \geq n|S| - (n-1)(|S| - e(G)) = |S| + (n-1)e(G)$. Thus $|S| \leq mn(G)$ and so S is a minimum monophonic set of G. Hence $mn(G) = |S|$. Now we claim that S is an open monophonic set of G From (1), we have $(n-1)|S^o| = n|S| - e(G)(n-1) - mn(G) = n.mn(G) - e(G)(n-1) - mn(G)$. Hence $(n-1)|S^o| = (n-1)(mn(G) - e(G))$ and so $|S^o| = mn(G) - e(G) = |S| - e(G)$. Therefore, $S^o = S - Ext(G)$ and hence S is an open monophonic set of G. Thus $omn(G) \leq |S| = mn(G)$. Since $mn(G) \leq omn(G)$, the result follows. ∎

Corollary 5.26 If G is a connected graph such that $omn(G) = mn(G) + 1$, then $mn(G \boxtimes K_n) = e(G)(n-1) + mn(G) + 1$.

Proof. This follows from Corollaries 5.24 and 5.25. ∎

Next, we proceed to find a class of graphs for which the upper bound of Corollary 5.24 is attained.

Theorem 5.27 *Let G be a connected graph. Then $max\{2, e(G)\} \leq mn(G) \leq omn(G) \leq 3mn(G) - 2e(G)$.*

Proof. The lower bound follows from Theorem 1.36. To prove the upper bound of $omn(G)$, let $mn(G) = p$. If $p = e(G)$, then $omn(G) = p$ so that $omn(G) =$

$3mn(G) - 2e(G)$. So, assume that $e(G) < p$. Let S be a minimum monophonic set of G. Then $Ext(G) \subseteq S$. Let $S - Ext(G) = \{v_1, v_2, \ldots, v_{p-e(G)}\}$. For each j with $1 \leq j \leq p - e(G)$, let $v_{j,1}$ and $v_{j,2}$ be two non-adjecent neighbors of v_j. Then v_j lies on the m-path $P : v_{j,1}, v_j, v_{j,2}$. Let $T = S \cup \{v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, \ldots, v_{p-e(G),1}, v_{p-e(G),2}\}$. Then $|T| \leq p+2(p-e(G)) = 3mn(G)-2e(G)$. We show that T is an open monophonic set of G. Let $x \in T - Ext(G)$. If $x \notin S$, then $x \in J_G(y,z)$ for some $y, z \in S$ and so $x \in T^o$. If $x \in S$, then $x = v_j$ for some j with $1 \leq j \leq p - e(G)$ and so $x \in J_G(v_{j,1}, v_{j,2})$. Hence $x \in T^o$ and so $T^o = T - Ext(G)$. This shows that T is an open monophonic set of G. Thus $omn(G) \leq |T| = 3mn(G) - 2e(G)$. ∎

Theorem 5.28 *Let G be a connected graph and $n \geq 2(mn(G) - e(G)) + 1$. Then $mn(G \boxtimes K_n) = e(G)(n-1) + omn(G)$.*

Proof. Let S be a monophonic set of G. If S is an open monophonic set, then $S^o = S - Ext(G)$. Hence $n|S| - (n-1)|S^o| = n|S| - (n-1)(|S| - e(G)) = e(G)(n-1) + |S| \geq e(G)(n-1) + omn(G)$. Now, if S is not an open monophonic set of G, then $S^o \subsetneq S - Ext(G)$. Thus, $n|S| - (n-1)|S^o| \geq n|S| - (n-1)(|S| - e(G) - 1) = (n-1)e(G) + |S| + (n-1) \geq (n-1)e(G) + mn(G) + 2(mn(G) - e(G)) = (n-1)e(G) + 3mn(G) - 2e(G)$. By Theorem 5.27, we have $n|S| - (n-1)|S^o| \geq (n-1)e(G) + omn(G)$. Hence the result follows from Corollary 5.24. ∎

Remark 5.29 The converse of Theorem 5.28 is not true. For the graph $G = C_6$, we have that $mn(C_6 \boxtimes K_3) = 3$ (see Theorem 5.31). However, $n = 3 < 5 = 2(mn(G) - e(G)) + 1$.

In view of Theorem 5.28, we leave the following problem as an open question.

Problem 5.30 Characterize the class of graphs G for which $mn(G \boxtimes K_n) = (n-1)e(G) + omn(G)$.

In the following, we obtain the exact values of the monophonic numbers of some standard classes of strong product graphs.

Theorem 5.31 *For integers $m \geq 3$ and $n \geq 2$,*

$$mn(C_m \boxtimes K_n) = \begin{cases} 3n & \text{if } m = 3 \\ 4 & \text{if } m = 4, 5 \\ 3 & \text{if } m \geq 6 \end{cases}$$

Proof. If $m = 3$, then $C_m \boxtimes K_n = K_{3n}$ and so $mn(C_m \boxtimes K_n) = 3n$. So, assume that $m \geq 4$. Since any two non-adjecent vertices of C_m forms a monophonic set of C_m, we have $mn(C_m) = 2$. Let $m = 4$ and let S be a monophonic set of C_4. If $|S| = 2$, then $|S^o| = 0$ and so $n|S| - (n-1)|S^o| = 2n \geq 4$. If $|S| = 3$, then $|S^o| = 1$ and so $n|S| - (n-1)|S^o| = 2n + 1 \geq 5$. If $|S| = 4$, then $S = V(C_4)$ so that $S = S^o$. This implies that $n|S| - (n-1)|S^o| = 4$. Hence it follows from Theorem 5.23 that $mn(C_4 \boxtimes K_n) = 4$. Let $m = 5$ and let S be a monophonic set of C_5. If $|S| = 2$, then $|S^o| = 0$ and so $n|S| - (n-1)|S^o| = 2n \geq 4$. If $|S| = 3$, then we have $|S^o| = 1$ or $|S^o| = 2$. Hence $n|S| - (n-1)|S^o| = 2n + 1$ or $n|S| - (n-1)|S^o| = n + 2$, which is gretater than or equal to 4. Also, if $|S| = 4$ or $|S| = 5$, then $S = S^o$ and so $n|S| - (n-1)|S^o| = |S|$. Hence it follows from Theorem 5.23 that $mn(C_5 \boxtimes K_n) = 4$ $(n \geq 2)$. Finally, let $m \geq 6$. Then $omn(C_m) = 3 = mn(C_m) + 1$ and so by Corollary

5.26, $mn(C_m \boxtimes K_n) = 3$. ∎

Theorem 5.32 *For integers $2 \leq r \leq s$ and $n \geq 2$, $mn(K_{r,s} \boxtimes K_n) = 4$.*

Proof. If $r \geq 4$, then it is easily seen that $mn(K_{r,s}) = omn(K_{r,s}) = 4$ and so by Corollary 5.25, $mn(K_{r,s} \boxtimes K_n) = 4$. If $r = 3$, then $mn(K_{r,s}) = 3$ and $omn(K_{r,s}) = 4$. Hence, by Corollary 5.26, $mn(K_{r,s} \boxtimes K_n) = 4$. Now, let $r = 2$. If $s = 2$, then $K_{r,s} = C_4$ and so by Theorem 5.31, $mn(K_{r,s} \boxtimes K_n) = 4$. If $s \geq 3$, let (X, Y) be the partite sets of $K_{2,s}$ with $|X| = 2$. Now, X and Y are monophonic sets of $K_{2,s}$. Let S be any monophonic set of $K_{2,s}$. If $S = X$ or Y, then $S^o = \phi$ and so $n|S| - (n-1)|S^o| = n|S| \geq 4$. Assume that $S \neq X, Y$. Then $|S| \geq 3$. If $|S| = 3$, then $|S^o| = 1$ and so $n|S| - (n-1)|S^o| = 2n + 1 \geq 5$. If $|S| \geq 4$, then $S^o = S$ or $|S^o| = 1$. If $|S^o| = 1$, then $n|S| - (n-1)|S^o| \geq 3n + 1 \geq 7$. If $S^o = S$, then $n|S| - (n-1)|S^o| = |S|$. Now, let $S = \{x_1, x_2, y_1, y_2\}$, where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then S is a monophonic set of $K_{2,s}$ with $S^o = S$. Hence it follows from Theorem 5.23 that $mn(K_{2,s} \boxtimes K_n) = 4$. ∎