Chapter 4

The geodetic and hull numbers of strong product graphs * †

In this chapter, we investigate properties of geodetic and hull numbers of strong product graphs [32, 33]. It is proved that, for connected graphs G and H, $max\{2, e(G)e(H)\} \leq h(G \boxtimes H) \leq h(G)h(H)$ and $h(G \boxtimes H) \leq min\{h(H) + e(H)(h(G) - 1), h(G) + e(G)(h(H) - 1)\}$. Also, we obtain some improved upper bounds of hull numbers of some classes of strong product graphs. It is shown that $h(G\boxtimes H) \leq h(H)$ for any connected graph H with no extreme vertices. If G has no extreme vertices, then we prove that (i) $h(G\boxtimes H) = 2$ if the girth of H is even; and (ii) $h(G\boxtimes H) \leq 3$ if the girth of H is odd and at least 5. Also, it is proved that $h(G\boxtimes K_m) = h(G)+e(G)(m-1)$. A graph G is an *extreme hull graph* if the set of extreme vertices of G is a hull set of G. It is proved that $h(G\boxtimes H) = h(G)h(H)$ if and only if both G and H are extreme hull graphs. It is shown that $g(G \boxtimes H) \geq 4$ and $min\{g(G), g(H)\} \leq g(G \boxtimes H) \leq g(G)g(H)$. Also it is shown that G and H are extreme geodesic graphs if and only if $G \boxtimes H$ is an extreme geodesic graph. A vertex x in a set S of vertices of G is a *geodetic interior vertex* of S

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if $x \in I_G[S-\{x\}]$. The set of all geodetic interior vertices of S is denoted by S^o, called the *geodetic interior of* S. We prove that $g(G \boxtimes H) \leq min\{|S||T| - (|T|-1)|S^o| : S \text{ and } T \text{ are geodetic sets of } G \text{ and } H \text{ respectively}\}$ for any connected graph H having a full degree vertex. Moreover, if H is an extreme geodesic graph, then $g(G \boxtimes H) = min\{e(H)|S| - (e(H)-1)|S^o| : S \text{ is a geodetic set of } G\}$. It is proved that $e(G)(g(H)-1) + g(G) \leq g(G \boxtimes H) \leq e(G)(g(H)-1) + og(G)$ for any extreme geodesic graph H with a full degree vertex. Also, it is proved that for integers $2 \leq r \leq s$ and $n \geq 2$, $g(K_{r,s} \boxtimes K_n) = 4$. The open geodetic number $og(G)$ and the double domination number $\gamma_{\times 2}(G)$ of a graph G were introduced and studied in [25] and [17] respectively. We determine upper bounds for the geodetic number of strong product graphs in terms of open geodetic number and double domination number of the factor graphs.

Hull number of strong product graphs

In this section we determine possible bounds for the hull number of the strong product of two connected graphs. And improved upper bounds are obtained for some classes strong product graphs.

Proposition 4.1 *Let G and H be connected graphs and P a $(u,v)-(u',v')$ geodesic in $G \boxtimes H$ of length n. If $d_G(u,u') \geq d_H(v,v')$, then $\pi_G(P)$ is a $u-u'$ geodesic in G of length n, and if $d_G(u,u') \leq d_H(v,v')$, then $\pi_H(P)$ is a $v-v'$ geodesic in H of length n.*

Proof. Let $P : (u, v) = (u_0, v_0), (u_1, v_1), \ldots, (u_n, v_n) = (u', v')$ be a $(u, v) - (u', v')$ geodesic of length n in $G \boxtimes H$. If $d_G(u, u') \geq d_H(v, v')$, then by Theorem 1.34, $d_G(u, u') = \max\{d_G(u, u'), d_H(v, v')\} = d_{G\boxtimes H}((u, v), (u', v')) = n$. Hence it follows that $\pi_G(P) : u = u_0, u_1, \ldots, u_n = u'$ must be an $u - u'$ geodesic in G. The other case follows similarly. ∎

Remark 4.2 If P is a geodesic in $G \boxtimes H$, then both $\pi_G(P)$ and $\pi_H(P)$ need not be geodesics in the factor graphs G and H respectively. For the graph $G = K_{2,2}$ with partite sets $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $H = P_4$ with $V(H) = \{v_1, v_2, v_3, v_4\}$, it is clear from Theorem 1.34 that $P : (x_1, v_1), (y_1, v_2), (x_2, v_3), (y_2, v_4)$ is a $(x_1, v_1) - (y_2, v_4)$ geodesic in $G \boxtimes H$. However, $\pi_G(P) : x_1, y_1, x_2, y_2$ is a $x_1 - y_2$ path in G, which is not a geodesic and $\pi_H(P) : v_1, v_2, v_3, v_4$ is a geodesic in H.

Theorem 4.3 *Let G and H be connected graphs. Then $Ext(G \boxtimes H) = Ext(G) \times Ext(H)$.*

Proof. Let $(g, h) \in Ext(G \boxtimes H)$. If g is a pendant vertex of G, then $g \in Ext(G)$. So, let $x_1, x_2 \in N_G(g)$ be such that $x_1 \neq x_2$. Then $(x_1, h), (x_2, h) \in N_{G\boxtimes H}((g, h))$. Since the induced subgraph of $N_{G\boxtimes H}(g, h)$ is complete, it follows that $x_1x_2 \in E(G)$ and so the induced subgraph of $N_G(g)$ is complete. Similarly, we can prove that $< N_H(h) >$ is complete. Thus, $(g, h) \in Ext(G) \times Ext(H)$. Conversely, let $(g, h) \in Ext(G) \times Ext(H)$. Let $(x_1, y_1), (x_2, y_2)$ be distinct vertices in $N_{G\boxtimes H}(g, h)$. Then $(x_1, y_1)(g, h) \in E(G \boxtimes H)$ and exactly one of the following three conditions holds.

(1) $x_1 = g$ and $y_1 \in N_H(h)$, or

(2)$x_1 \in N_G(g)$ and $y_1 = h$, or

(3) $x_1 \in N_G(g)$ and $y_1 \in N_H(h)$.

Similarly, $(x_2, y_2)(g, h) \in E(G \boxtimes H)$ and exactly one of the following three conditions holds.

(a) $x_2 = g$ and $y_2 \in N_H(h)$, or

(b)$x_2 \in N_G(g)$ and $y_2 = h$, or

(c) $x_2 \in N_G(g)$ and $y_2 \in N_H(h)$.

Now, there are nine cases.

Case 1. Both (a) and (1) hold. Then $y_1 \neq y_2$. Since $< N_H(h) >$ is complete, we have $y_1y_2 \in E(H)$ so that $(x_1, y_1)(x_2, y_2) \in E(G \boxtimes H)$.

Case 2. Both (c) and (3) hold. Since $< N_G(g) >$ is complete, either $x_1 = x_2$ or $x_1x_2 \in E(G)$. Similarly, we have either $y_1 = y_2$ or $y_1y_2 \in E(H)$. Since $(x_1, y_1) \neq (x_2, y_2)$, it follows that $(x_1, y_1)(x_2, y_2) \in E(G \boxtimes H)$. The other cases are similar. ∎

Theorem 4.4 *Let G and H be connected graphs and S and T hull sets of G and H respectively. Then $S \times T$ is a hull set of $G \boxtimes H$.*

Proof. Let $W = S \times T$. We show that $[W]_{G\boxtimes H} = V(G \boxtimes H)$. Let $(x, y) \in V(G \boxtimes H)$. Now, since $[S]_G = V(G)$, it follows that there exists an integer $m \geq 0$ such that $x \in I_G^m[S]$. We prove that $(x, y) \in [W]_{G\boxtimes H}$. The proof is by induction on m. Let $m = 0$. Then $x \in S$. Now, since $[T]_H = V(H)$, it follows that there exists an integer $n \geq 0$ such that $y \in I_G^n[T]$. We prove that $(x, y) \in [W]_{G\boxtimes H}$. The proof is by induction on n. If $n = 0$ then $y \in T$ and so $(x, y) \in S \times T \subseteq [W]_{G\boxtimes H}$. Assume that $(x, y) \in [W]_{G\boxtimes H}$ for all $y \in I_H^k[T]$. Let $y \in I_H^{k+1}[T]$ be such that $y \notin I_H^k[T]$. Then there exist

y', y'' in $I_H^k[T]$ such that y lies on a $y' - y''$ geodesic $P : y' = y_0, y_1, \ldots, y_t = y''$ with $y \neq y', y''$. Now, by induction hypothesis, $(x, y'), (x, y'') \in [W]_{G \boxtimes H}$. Now, it follows from Theorem 1.34 that the walk $Q : (x, y') = (x, y_0), (x, y_1), \ldots, (x, y_t) = (x, y'')$ is a geodesic in $G \boxtimes H$ which contains the vertex (x, y). Hence $(x, y) \in [W]_{G \boxtimes H}$. Thus, by induction, $(x, y) \in [W]_{G \boxtimes H}$ for all $y \in V(H)$.

Assume that the result is true for $m = l$. Then $(x, y) \in [W]_{G \boxtimes H}$ for all $x \in I_G^l[S]$ and $y \in V(H)$. Let $x \in V(G)$ be such that $x \in I_G^{l+1}[S]$ and $x \notin I_G^l[S]$. Then there exist $x', x'' \in I_G^l[S]$ such that x lies on a $x' - x''$ geodesic $P' : x' = x_0, x_1, \ldots, x_j = x, \ldots, x_s = x'$ with $1 \leq j \leq s - 1$. Now, by induction hypothesis, $(x', y), (x'', y) \in [W]_{G \boxtimes H}$. By Theorem 1.34, it follows that the walk $Q' : (x', y) = (x_0, y), (x_1, y) \ldots, (x_j, y) = (x, y), \ldots, (x_s, y) = (x'', y)$ is a geodesic. Hence $(x, y) \in [W]_{G \boxtimes H}$. Thus, by induction $(x, y) \in [W]_{G \boxtimes H}$ for all $x \in V(G)$ and $y \in V(H)$ so that $[W]_{G \boxtimes H} = V(G \boxtimes H)$. ∎

Remark 4.5 The converse of Theorem 4.4 need not be true. Let G be the cycle $C_4 : u_1, u_2, u_3, u_4, u_1$ and let H be the complete graph K_2, with vertex set $\{v_1, v_2\}$. Let $S = \{u_1, u_3\}$ and $T = \{v_1\}$. Then, it is clear that $I_{G \boxtimes H}^2[S \times T] = V(G \boxtimes H)$ and so $S \times T$ is a hull set of $G \boxtimes H$. However, T is not a hull set of K_2.

Corollary 4.6 *Let G and H be connected graphs. Then $max\{2, e(G)e(H)\} \leq h(G \boxtimes H) \leq h(G)h(H)$.*

Proof. Let S and T be minimum hull sets of G and H respectively. By Theorem 4.4, $W = S \times T$ is a hull set of G so that $h(G \boxtimes H) \leq h(G)h(H)$. The other inequality follows from Theorems 1.17 and 4.3. ∎

Lemma 4.7 *Let G and H be connected graphs. Then, for any $x \in V(G)$ and $T \subseteq V(H)$, $x \times I_H^k[T] \subseteq I_{G\boxtimes H}^k[x \times T]$ for all $k \geq 0$.*

Proof. For $k = 0$, it is obvious. We first show that $x \times I_H[T] \subseteq I_{G\boxtimes H}[x \times T]$. Let $(x,y) \in x \times I_H[T]$. If $y \in T$, then $(x,y) \in I_{G\boxtimes H}[x \times T]$. If $y \notin T$, then y lies on a $y' - y''$ geodesic $P : y' = y_0, y_1, \ldots, y_i = y, \ldots, y_n = y''$ with $y', y'' \in T$. It follows from Theorem 1.34 that $Q : (x,y') = (x,y_0), (x,y_1), \ldots, (x,y_i) = (x,y), \ldots, (x,y_n) = (x,y'')$ is a geodesic in $G \boxtimes H$ with $(x,y'), (x,y'') \in x \times T$ and so $(x,y) \in I_{G\boxtimes H}[x \times T]$. Hence $x \times I_H[T] \subseteq I_{G\boxtimes H}[x \times T]$.

Now, $x \times I_H^2[T] = x \times I_H[I_H[T]] \subseteq I_{G\boxtimes H}[x \times I_H[T]] \subseteq I_{G\boxtimes H}^2[x \times T]$. Proceeding like this , we see that $x \times I_H^k[T] \subseteq I_{G\boxtimes H}^k[x \times T]$ for all $k \geq 0$. ∎

Theorem 4.8 *Let G and H be connected graphs. Then $h(G \boxtimes H)$ $\leq min\{h(H) + e(H)(h(G) - 1), h(G) + e(G)(h(H) - 1)\}$.*

Proof. Let S and T be minimum hull sets of G and H respectively. Let $W = (Ext(G) \times Ext(H)) \cup ((S - Ext(G)) \times Ext(H)) \cup (u \times (T - Ext(H)))$, where $u \in S$. Then $|W| \leq e(G)e(H) + (h(G) - e(G))e(H) + (h(H) - e(H)) = h(H) + e(H)(h(G) - 1)$. We prove that W is a hull set of $G \boxtimes H$.

Step 1. $u \times V(H) \subseteq [W]_{G\boxtimes H}$.

Let $y \in V(H)$. Since T is a hull set of H, it follows that $y \in I_H^k[T]$ for some $k \geq 0$ and so $(u,y) \in u \times I_H^k[T]$. Hence by Lemma 4.7, $(u,y) \in I_{G\boxtimes H}^k[u \times T]$. Since $u \in S$, it is clear from the definition of W that $u \times T \subseteq W$ and so $I_{G\boxtimes H}^k[u \times T] \subseteq I_{G\boxtimes H}^k[W] \subseteq$

$[W]_{G\boxtimes H}$. Thus $(u, y) \in [W]_{G\boxtimes H}$ and so $u \times V(H) \subseteq [W]_{G\boxtimes H}$.

Step 2. If $x \in V(G)$ and $x \times V(H) \subseteq [W]_{G\boxtimes H}$, then $x' \times V(H) \subseteq [W]_{G\boxtimes H}$ for $x' \in N_G(x)$.

Let $y \in V(H)$. If $y \notin Ext(H)$, then there exist vertices $y', y'' \in N_H(y)$ such that y' and y'' are non-adjacent. It is clear that $Q : (x, y'), (x', y), (x, y'')$ is a geodesic in $G \boxtimes H$ with $(x, y'), (x, y'') \in [W]_{G\boxtimes H}$ and so $(x', y) \in [W]_{G\boxtimes H}$. Now, assume that $y \in Ext(H)$. Since S is a hull set of G, it follows that $x' \in I^l_G[S]$ for some $l \geq 0$. Thus $(x', y) \in I^l_G[S] \times y$. By Lemma 4.7, $(x', y) \in I^l_{G\boxtimes H}[S \times y]$. Since $y \in Ext(H)$, it is clear from the definition of W that $S \times y \subseteq [W]_{G\boxtimes H}$ and so $(x', y) \in [W]_{G\boxtimes H}$. Hence $x' \times V(H) \subseteq [W]_{G\boxtimes H}$.

Now, since G and H are connected, it follows from Step 1 and Step 2 that $V(G) \times V(H) \subseteq [W]_{G\boxtimes H}$ and so W is a hull set of $G \boxtimes H$. Hence $h(G \boxtimes H) \leq |W| \leq h(H) + e(H) + (h(G) - 1)$. Similarly, we can prove that $h(G \boxtimes H) \leq h(G) + e(G)(h(H) - 1)$. Thus the result follows. ∎

Corollary 4.9 *Let G and H be connected graphs such that H has no extreme vertices. Then $h(G \boxtimes H) \leq h(H)$.*

Corollary 4.10 *Let G and H be connected graphs having no extreme vertices. Then $2 \leq h(G \boxtimes H) \leq min\{h(G), h(H)\}$.*

Corollary 4.11 *For any connected graph G, $h(G \boxtimes K_{r_1, r_2, \ldots, r_n}) = 2$, where $n \geq 2$ and $r_i \geq 2$ for $i = 1, 2, \ldots n$.*

Proof. Let $n \geq 2$ and $r_i \geq 2$ for $i = 1, 2, \ldots, n$. Any two vertices of a partite set of

$K_{r_1,r_2,\ldots,r_n}$ form a hull set and so $h(K_{r_1,r_2,\ldots,r_n}) = 2$. Also $K_{r_1,r_2,\ldots,r_n}$ has no extreme vertices. Hence the result follows from Corollary 4.9. ∎

Corollary 4.12 *For any connected graph G, $h(G \boxtimes C_{2n}) = 2$ for all $n \geq 2$.*

Proof. This follows from Corollary 4.9. ∎

In the following we introduce a class of graphs for which the upper bound of hull number is further improved.

Let $\Im$ denote the class of connected graphs G such that every non-extreme vertex of G has two non-adjacent neighbors which are not extreme. The graph G in Figure 4.1 belongs to the class $\Im$. Obviously, complete graphs and graphs having no extreme vertices belong to $\Im$.

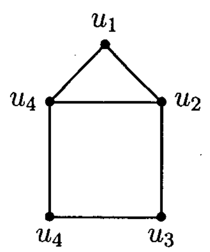

Figure 4.1: G

Theorem 4.13 *If G and H are connected graphs having extreme vertices and belong to $\Im$, then $h(G \boxtimes H) \leq e(G)e(H) + h(G) + h(H) - e(G) - e(H)$.*

Proof. Let S and T be minimum hull sets of G and H respectively. Let $u \in Ext(G)$ and $v \in Ext(H)$. We show that $W = (Ext(G) \times Ext(H)) \cup ((S - Ext(G)) \times v) \cup (u \times (T - Ext(H)))$ is a hull set of $G \boxtimes H$. It is clear that $S \times v, u \times T \subseteq W \subseteq [W]_{G \boxtimes H}$.

Step 1. $V(G) \times v \subseteq [W]_{G \boxtimes H}$.

Let $x \in V(G)$. Since S is a hull set of G, we have $x \in I_G^k[S]$ for some $k \geq 0$ and so $(x, v) \in I_G^k[S] \times v$. By Lemma 4.7, $(x, v) \in I_{G \boxtimes H}^k[S \times v] \subseteq [W]_{G \boxtimes H}$. Hence $V(G) \times v \subseteq [W]_{G \boxtimes H}$.

Step 2. If $y \in V(H)$ and $(V(G) - Ext(G)) \times y \subseteq [W]_{G \boxtimes H}$, then $(V(G) - Ext(G)) \times y' \subseteq [W]_{G \boxtimes H}$ for $y' \in N_H(y)$.

Let $x \in V(G) - Ext(G)$. Since $G \in \Im$, there exist $x', x'' \in N_G(x)$ such that x', x'' are non-extreme and non-adjacent. Now, it is clear that $Q : (x', y), (x, y'), (x'', y)$ is a geodesic in $G \boxtimes H$ with $(x', y), (x'', y) \in (V(G) - Ext(G)) \times y \subseteq [W]_{G \boxtimes H}$ and so $(x, y') \in [W]_{G \boxtimes H}$. Thus $(V(G) - Ext(G)) \times y' \subseteq [W]_{G \boxtimes H}$.

Now, since G and H are connected, it follows from Step 1 and Step 2 that $(V(G) - Ext(G)) \times V(H) \subseteq [W]_{G \boxtimes H}$. Similarly, we can prove that $V(G) \times (V(H) - Ext(H)) \subseteq [W]_{G \boxtimes H}$. Also, by the definition of W, we have $Ext(G) \times Ext(H) \subseteq W \subseteq [W]_{G \boxtimes H}$. Hence $[W]_{G \boxtimes H} = V(G \boxtimes H)$ and so $h(G \boxtimes H) \leq |W| \leq e(G)e(H) + h(G) + h(H) - e(G) - e(H)$. ∎

Exact hull numbers

In this section we determine the exact values of the hull numbers of the strong product for several classes of graphs. We also give several classes of graphs G and H with $h(G \boxtimes H) = 2$. It is to be noted that the graphs given in Corollaries 4.11 and 4.12 belong to this class. We also characterize graphs G and H for which $h(G \boxtimes H) = h(G)h(H)$.

Theorem 4.14 *Let G and H be connected graphs such that G has no extreme vertices. Then*

(i) $h(G \boxtimes H) = 2$ if the girth of H is even,

(ii) $h(G \boxtimes H) \leq 3$ if the girth of H is odd and at least 5.

Proof. (*i*) Let the girth of H be $2n (n \geq 2)$ and let $C : y_0, y_1, \ldots, y_{2n-1}, y_0$ be a cycle of length $2n$. For any $x \in V(G)$, we show that the set $W = \{(x, y_0), (x, y_n)\}$ is a hull set of $G \boxtimes H$. We first prove the following two steps.

Step 1. $V(G) \times \{y_0, y_n\} \subseteq [W]_{G \boxtimes H}$. Let $u \in V(G)$. We use induction on $d_G(x, u)$ to prove that $(u, y_0), (u, y_n) \in [W]_{G \boxtimes H}$. Let $d_G(x, u) = 0$ or 1. Since C is a shortest cycle in H, it follows that the path $P : y_0, y_1, \ldots, y_n$ and $P_1 : y_n, y_{n+1}, \ldots, y_{2n-1}, y_0$ are geodesics in H. Then it follows from Theorem 1.34 that $Q : (x, y_0), (x, y_1), \ldots, (x, y_{n-1}), (x, y_n)$ and $Q_1 : (x, y_n), (x, y_{n+1}), \ldots, (x, y_{2n-1}), (x, y_0)$ are geodesics in $G \boxtimes H$ and so $(x, y_1), (x, y_{2n-1}), (x, y_{n-1}), (x, y_{n+1}) \in [W]_{G \boxtimes H}$. It is clear that $Q_2 : (x, y_1), (u, y_0), (x, y_{2n-1})$ and $Q_3 : (x, y_{n-1}), (u, y_n), (x, y_{n+1})$ are geodesics in $G \boxtimes H$. Hence $(u, y_0), (u, y_n) \in [W]_{G \boxtimes H}$.

Assume that the result is true for $d_G(x, u) = k$. Let u be a vertex such that $d_G(x, u) = k + 1$. Let $x = x_0, x_1, \ldots, x_k, x_{k+1} = u$ be a $x - u$ geodesic in G. By induction hypothesis, $(x_k, y_0), (x_k, y_n) \in [W]_{G \boxtimes H}$. As above, we see that $Q_3 : (x_k, y_0), (x_k, y_1), \ldots, (x_k, y_n)$ and $Q_4 : (x_k, y_n), (x_k, y_{n+1}), \ldots, (x_k, y_{2n-1}), (x_k, y_0)$ are geodesics in $G \boxtimes H$ so that $(x_k, y_1), (x_k, y_{2n-1}), (x_k, y_{n-1}), (x_k, y_{n+1}) \in [W]_{G \boxtimes H}$. It is clear that $Q_5 : (x_k, y_1), (u, y_0), (x_k, y_{2n-1})$ and $Q_6 : (x_k, y_{n-1}), (u, y_n), (x_k, y_{n+1})$ are geodesics in $G \boxtimes H$. Hence $(u, y_0), (u, y_n) \in [W]_{G \boxtimes H}$ and so $V(G) \times \{y_0, y_n\} \subseteq [W]_{G \boxtimes H}$.

Step 2. If $y \in V(H)$ and $V(G) \times y \in [W]_{G\boxtimes H}$, then $V(G) \times y' \subseteq [W]_{G\boxtimes H}$ for $y' \in N_H(y)$.

Let $u \in V(G)$. Since G has no extreme vertices, there exist vertices $u', u'' \in N_G(u)$ such that u' and u'' are non-adjacent. Now, it is clear that $Q_5 : (u', y), (u, y'), (u'', y)$ is a geodesic in $G \boxtimes H$ with $(u', y), (u'', y) \in [W]_{G\boxtimes H}$ and so $(u, y') \in [W]_{G\boxtimes H}$. Hence $V(G) \times y' \subseteq [W]_{G\boxtimes H}$. Now, since G and H are connected, it follows from Step 1 and Step 2 that $V(G) \times V(H) \subseteq [W]_{G\boxtimes H}$ and so W is a hull set of $G \boxtimes H$. Hence $h(G \boxtimes H) = 2$.

(ii) Let the girth of H be $2n + 1 (n \geq 2)$ and let $C : y_0, y_1, \ldots, y_{2n}, y_0$ be a cycle of length $2n + 1$. For any $x \in V(G)$, let $W = \{(x, y_0), (x, y_n), (x, y_{n+1}\}$. Then, as in (i), we can prove that W is a hull set of $G \boxtimes H$ and so $h(G \boxtimes H) \leq |W| = 3$. ∎

In the following we give a class of strong product graphs for which the bound in Theorem 4.8 is attained.

Theorem 4.15 *Let G and H be connected graphs and $S \subseteq V(G \boxtimes H)$. Then $I^k_G[\pi_G(S)] \subseteq \pi_G(I^k_{G\boxtimes H}[S])$.*

Proof. For $k = 0$, it is obvious. We first show that $I_G[\pi_G(S)] \subseteq \pi_G(I_{G\boxtimes H}[S])$. Let $x \in I_G[\pi_G(S)]$. If $x \in \pi_G(S)$, then there exists $y \in V(H)$ such that $(x, y) \in S \subseteq I_{G\boxtimes H}[S]$ and so $x \in \pi_G(I_{G\boxtimes H}[S])$. If $x \notin \pi_G(S)$, then there exist $g, g' \in \pi_G(S)$ such that x lies on a $g - g'$ geodesic $P : g = g_0, g_1, \ldots, g_i = x, g_{i+1}, \ldots, g_n = g'$ with $1 \leq i \leq n - 1$ so that $d_G(g, g') = n$. Since $g, g' \in \pi_G(S)$, there exist $h, h' \in V(H)$

such that $(g,h),(g',h') \in S$. Let $d_H(h,h') = m$. and let $Q : h = h_0, h_1, \ldots, h_m = h'$ be a $h - h'$ geodesic in H. We consider the following two cases.

Case 1. $m \geq n$. Then it follows from Theorem 1.34 that $Q' : (g,h) = (g_0,h_0),(g_1,h_1), \ldots,(g_i,h_i) = (x,h_i),(g_{i+1},h_{i+1}),\ldots,(g_n,h_n),(g_n,h_{n+1}),\ldots,(g_n,h_m) = (g',h')$ is a $(g,h)-(g',h')$ geodesic in $G \boxtimes H$ containing the vertex (x,h_i). Hence $(x,h_i) \in I_{G\boxtimes H}[S]$ and so $x \in \pi_G(I_{G\boxtimes H}[S])$.

Case 2. $m < n$. Then it follows from Theorem 1.34 that the walk $Q'' : (g,h) = (g_0,h_0),(g_1,h_1),\ldots,(g_m,h_m),(g_{m+1},h_m),\ldots,(g_n,h_m) = (g',h')$ is a $(g,h)-(g',h')$ geodesic in $G \boxtimes H$ containing the vertex (x,h_i) for some i with $1 \leq i \leq m$. Hence $(x,h_i) \in I_{G\boxtimes H}[S]$ and so $x \in \pi_G(I_{G\boxtimes H}[S])$. Thus $I_G[\pi_G(S)] \subseteq \pi_G(I_{G\boxtimes H}[S])$.

Now, $I_G^2[\pi_G(S)] = I_G[I_G[\pi_G(S)]] \subseteq I_G[\pi_G(I_{G\boxtimes H}[S])] \subseteq \pi_G(I^2_{G\boxtimes H}[S])$.

Proceeding like this, we get $I_G^k[\pi_G(S)] \subsetneq \pi_G(I^k_{G\boxtimes H}[S])$. ∎

Remark 4.16 Strict inclusion can hold in Theorem 4.15. Let G and H be the paths $P_4 : u_1, u_2, u_3, u_4$ and $P_5 : v_1, v_2, v_3, v_4, v_5$ respectively. Let $S = \{(u_1,v_1),(u_2,v_2),(u_2,v_4)\}$. Then it is easily checked that $I_{G\boxtimes H}[S] = \{(u_1,v_1),(u_1,v_3),(u_1,v_2),(u_3,v_3),(u_2,v_3),(u_2,v_4),(u_2,v_2)\}$ and so $\pi_G(I_{G\boxtimes H}[S]) = \{u_1,u_2,u_3\}$. But $I_G[\pi_G(S)] = I_G[\{u_1,u_2\}] = \{u_1,u_2\} \subsetneq \pi_G(I_{G\boxtimes H}[S])$.

The following theorem shows that equality holds for the graph $H = K_m$ in the inclusion in Theorem 4.15.

Theorem 4.17 *Let G be a connected graph and $S \subseteq V(G \boxtimes K_m)$, where $m \geq 2$. Then $I_G^k[\pi_G(S)] = \pi_G(I^k_{G\boxtimes H}[S])$ for all $k \geq 0$.*

Proof. For $k = 0$, it is obvious. By Theorem 4.15, it is enough to show that $\pi_G(I^k_{G\boxtimes H}[S]) \subseteq I^k_G[\pi_G(S)]$. We first show that $\pi_G(I_{G\boxtimes H}[S]) \subseteq I_G[\pi_G(S)]$. Let $x \in \pi_G(I_{G\boxtimes H}[S])$. Then $(x,y) \in I_{G\boxtimes H}[S]$ for some $y \in V(K_m)$. If $(x,y) \in S$, then the result is trivial. If $(x,y) \notin S$, then there exist $(g,h),(g',h') \in S$ such that (x,y) lies on a $(g,h) - (g',h')$ geodesic $P : (g,h) = (g_0,h_0),(g_1,h_1),\ldots,(g_i,h_i) = (x,y),\ldots,(g_n,h_n) = (g',h')$ of length $n \geq 2$ with $1 \leq i \leq n-1$. Since $d_H(h,h') = 1$, it follows that $d_G(g,g') > d_H(h,h')$. By Proposition 4.1, $\pi_G(P)$ is a $g - g'$ geodesic in G containing the vertex x, where $g, g' \in \pi_G(S)$. Hence $x \in I_G[\pi_G(S)]$. Thus $\pi_G(I_{G\boxtimes H}[S]) \subseteq I_G[\pi_G(S)]$. Now, $\pi_G(I^2_{G\boxtimes H}[S]) \subseteq I_G[\pi_G(I_{G\boxtimes H}[S])] \subseteq I^2_G[\pi_G(S)]$. Proceeding like this, we get $\pi_G(I^k_{G\boxtimes H}[S]) = I^k_G[\pi_G(S)]$ for all $k \geq 0$. ∎

Theorem 4.18 *For a connected graph G, $h(G \boxtimes K_m) = h(G) + e(G)(m-1)$.*

Proof. Let S be a minimum hull set of G and let $W = Ext(G) \times V(K_m) \cup ((S - Ext(G)) \times v)$, where $v \in V(K_m)$. Then, as in the proof of Theorem 4.8, we can prove that W is a hull set of $G \boxtimes K_m$. Hence $h(G \boxtimes K_m) \leq |W| = h(G) + e(G)(m-1)$. On the other hand, if there exists a hull set W' of $G \boxtimes K_m$ such that $|W'| < |W|$, then it follows from Theorems 1.17 and 4.3 that $W' = Ext(G) \times V(K_m) \cup T$, where $T \cap (Ext(G) \times V(K_m)) = \phi$. This implies that $|T| < |(S - Ext(G)) \times v| = |S - Ext(G)| = h(G) - e(G)$. Now, since W' is a hull set of $G \boxtimes K_m$, there exists an integer $k \geq 0$ such that $[W']_{G\boxtimes K_m} = I^k_{G\boxtimes K_m}[W'] = V(G \boxtimes K_m)$. By Theorem 4.17, $I^k_G[\pi_G(W')] = \pi_G(I^k_{G\boxtimes K_m}[W']) = V(G)$ and so $\pi_G(W')$ is a hull set of G. It is easily seen that $\pi_G(W') = Ext(G) \cup \pi_G(T)$ and so $|\pi_G(W')| \leq |Ext(G)| + |\pi_G(T)| \leq |Ext(G)| + |T| < e(G) + h(G) - e(G) = h(G)$. Thus $\pi_G(W')$ is a hull set of G such that $|\pi_G(W')| <$

$h(G)$, which is a contradiction. Hence W is a minimum hull set of $G \boxtimes H$ so that $h(G \boxtimes K_m) = |W| = h(G) + e(G)(m-1)$. ∎

It is proved in [2] that $h(G \circ K_m) = h(G) + e(G)(m-1)$. The proof is much involved. We observe that $G \circ K_m = G \boxtimes K_m$ and so the following corollary gives a very simple and alternate proof of the above result proved in [2].

Corollary 4.19 *For a connected graph G, $h(G \circ K_m) = h(G) + e(G)(m-1)$.*

Extreme hull graphs

In this section we characterize the class of graphs for which the upper bound in Corollary 4.6 is attained.

Definition 4.20 A graph G is an *extreme hull graph* if the set of extreme vertices of G is a hull set of G.

Example 4.21 For the graph G in Figure 4.2, the set $S = \{u_1, u_5\}$ of extreme vertices is a hull set of G so that G is an extreme hull graph.

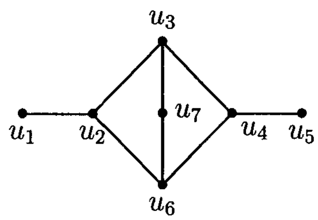

Figure 4.2: G

Remark 4.22 Every extreme geodesic graph is an extreme hull graph. The graph G given in Figure 4.2 is an extreme hull graph, which it is not an extreme geodesic graph.

By Theorem 1.17, $0 \leq e(G) \leq h(G)$ for every graph G. The following theorem is a realization of this result.

Theorem 4.23 *For every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $e(G) = a$ and $h(G) = b$.*

Proof. If $a = b$, then $a \geq 2$ and $G = K_a$ has the desired properties. Thus we assume that $a < b$. Let $G_i (1 \leq i \leq b - a)$ be the graphs given in Figure 4.3. Let H be the graph obtained from $\bigcup_{i=1}^{b-a} G_i$ by adding a new vertex w and joining w to x_i and z_i $(1 \leq i \leq b - a)$. Now, let G be the graph obtained from H by adding the new vertices $s_1, s_2, \ldots, s_a$ and joining these to w. The graph G is shown in Figure 4.4. Then

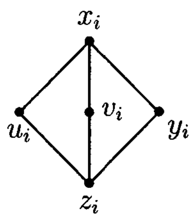

Figure 4.3: G_i

$S = \{s_1, s_2, \ldots, s_a\}$ is the set of extreme vertices of G and so $e(G) = a$. We prove that $h(G) = b$. By Theorem 1.17, the vertices $s_1, s_2, \ldots, s_a$ belong to every hull set of G. Since $V(G) - V(G_i)$ is a convex set for each $i = 1, 2, \ldots, b - a$, it follows that every hull set of G contains at least one vertex from each G_i. Hence $h(G) \geq a + b - a = b$.

Now, since the set $S' = S \cup \{v_1, v_2, \ldots, v_{b-a}\}$ is a hull set of G, we have $h(G) = b$. ∎

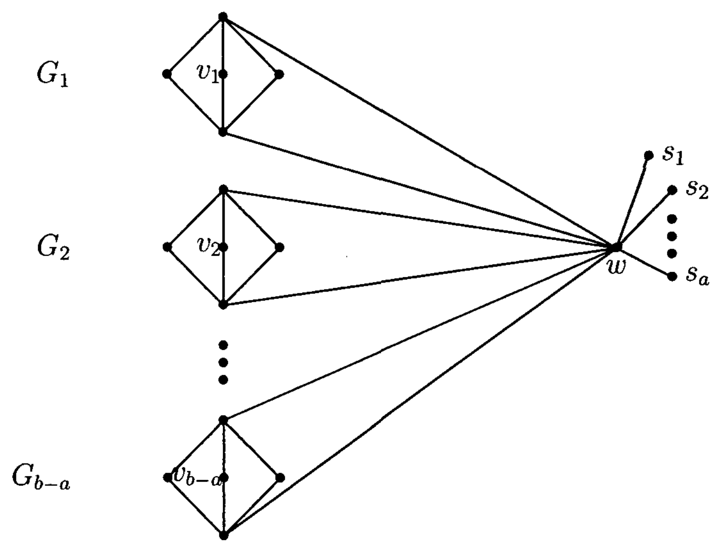

Figure 4.4: G

Theorem 4.24 *For every pair a, b of integers with $2 \leq a \leq b$, there exists an extreme hull graph G with $h(G) = a$ and $g(G) = b$.*

Proof. If $a = b$, then $G = K_a$ has the desired properties. Thus we assume that $a < b$. We construct a graph G with the required geodetic number a and hull number b. Let $G_i (1 \leq i \leq b - a)$ be the graphs given in Figure 4.3. Let G be the graph obtained from $\bigcup_{i=1}^{b-a} G_i$ by adding the new vertices $w_i (1 \leq i \leq a)$ and the edges (1) $w_i u_1 (1 \leq i \leq a - 1)$, $w_a y_{b-a}$ and (2) $y_i u_{i+1} (1 \leq i \leq b - a - 1)$. The graph G is shown in Figure 4.5. Let $S = \{w_1, w_2, \ldots, w_a\}$ be the set of extreme vertices of G. Then it is clear that $I[S] = V(G) - \{v_1, v_2, \ldots, v_{b-a}\}$ and $I^2[S] = V(G)$. Hence by Theorem 1.17, S is the unique minimum hull set of G and so $h(G) = a$.

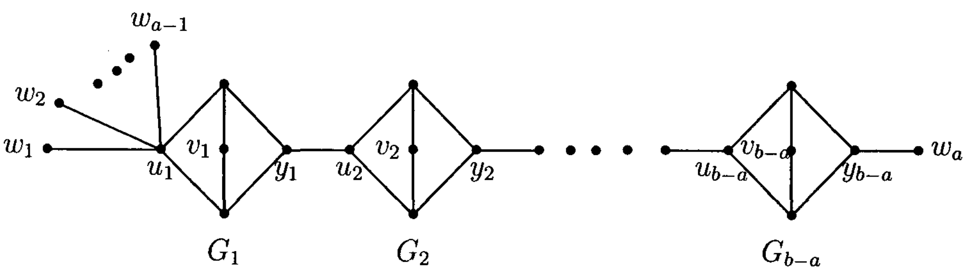

Figure 4.5: G

Next, we show that $g(G) = b$. It is clear that $I[S] = V(G) - \{v_1, v_2, \ldots, v_{b-a}\}$ and each v_i must belong to every minimum geodetic set of G. Since $W = S \cup \{v_1, v_2, \ldots, v_{b-a}\}$ is a geodetic set of G, it follows from Theorem 1.19 that $g(G) = b$. ∎

Theorem 4.25 *Let G and H be connected graphs. Then $h(G \boxtimes H) = h(G)h(H)$ if and only if both G and H are extreme hull graphs.*

Proof. Let G and H be extreme hull graphs. Then $Ext(G)$ and $Ext(H)$ are minimum hull sets of G and H respectively. Therefore, $h(G) = e(G)$ and $h(H) = e(H)$. Now, it follows from Theorems 4.3 and 4.4 that $Ext(G \boxtimes H) = Ext(G) \times Ext(H)$ is a hull set of $G \boxtimes H$. Hence by Theorem 1.17, we have $h(G \boxtimes H) = e(G)e(H) = h(G)h(H)$.

Conversely, assume that $h(G \boxtimes H) = h(G)h(H)$. Let S and T be minimum hull sets of G and H respectively. If $Ext(G) = \phi$, then, by Corollary 4.9, $h(G \boxtimes H) \leq h(G) < h(G)h(H)$, which is a contradiction. Hence $Ext(G) \neq \phi$. Similarly, we can prove that $Ext(H) \neq \phi$. Now, by Theorem 4.8, $h(G \boxtimes H) \leq h(H) + e(H)(h(G) - 1)$. Hence $h(G)h(H) \leq h(H) + e(H)(h(G) - 1)$. This implies that $h(H)(h(G) - 1) \leq e(H)(h(G) - 1)$. Since $h(G) \geq 2$, we have $h(H) \leq e(H)$. Hence it follows from Theorem 1.17 that $h(H) = e(H)$ and so H is an extreme hull graph. Similarly, G is also an extreme hull graph. ∎

Corollary 4.26 *Let G and H be connected graphs. If G and H are extreme hull graphs, then $G \boxtimes H$ is an extreme hull graph.*

Proof. This follows from Theorems 4.3 and 4.25. ∎

The converse of the above corollary seems to be a diffcult problem and we leave it open.

Problem 4.27 Let G and H be graphs such that $G \boxtimes H$ is an extreme hull graph. Is is true that G and H are extreme hull graphs ?

The geodetic number of strong product graphs

In this section, we determine bounds for the geodetic number of strong product graphs and also obtain exact values of geodetic numbers of certain classes of strong product graphs.

Theorem 4.28 *Let G and H be nontrivial connected graphs. Then $g(G \boxtimes H) \geq 4$.*

Proof. Suppose that there is a geodetic set of $G \boxtimes H$ of cardinality 3, say $W = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$. We consider three cases.

Case 1. $x_1 = x_2 = x_3 = x$(say). Then y_1, y_2 and y_3 are distinct. Let $x' \in V(G)$ be such that $x' \neq x$. Then it follows from Proposition 4.1 that $(x', y_1) \in I_{G \boxtimes H}[(x, y_2), (x, y_3)]$ and so $y_1 \in I_H[y_2, y_3]$. Similarly, we have $y_2 \in I_H[y_1, y_3]$ and

$y_3 \in I_H[y_1, y_2]$. Thus we get a contradiction.

Case 2. $x_1 = x_2 \neq x_3$. Then $y_1 \neq y_2$. Hence $y_3 \neq y_1$ or $y_3 \neq y_2$. Assume that $y_3 \neq y_1$. Hence it follows from Proposition 4.1 that $(x_3, y_1) \in I_{G\boxtimes H}[(x_1, y_2), (x_3, y_3)]$ and so $y_1 \in I_H[y_2, y_3]$. Thus $y_2 \neq y_3$. Hence it follows similarly from Proposition 4.1 that $(x_1, y_3) \in I_{G\boxtimes H}[(x_1, y_1), (x_1, y_2)]$ and $(x_3, y_2) \in I_{G\boxtimes H}[(x_1, y_1), (x_3, y_3)]$. Again Proposition 4.1 shows that $y_3 \in I_H[y_1, y_2]$ and $y_2 \in I_H[y_1, y_3]$, which is a contradiction.

Case 3. $x_1 \neq x_2 \neq x_3$. We consider only the case $y_1 \neq y_2 \neq y_3$, since the other cases are similar to the above cases. As in the previous case, we have $(x_2, y_1) \in I_{G\boxtimes H}[(x_1, y_1), (x_3, y_3)]$ or $(x_2, y_1) \in I_{G\boxtimes H}[(x_2, y_2), (x_3, y_3)]$.

Subcase 3.1 Assume that $(x_2, y_1) \in I_{G\boxtimes H}[(x_1, y_1), (x_3, y_3)]$. Then, by Proposition 4.1, $d_G(x_1, x_3) > d_H(y_1, y_3)$ and $x_2 \in I_G[x_1, x_3]$. Again, it follows from Proposition 4.1 that $(x_1, y_2) \in I_{G\boxtimes H}[(x_2, y_2), (x_3, y_3)]$. Hence $d_G(x_2, x_3) > d_H(y_2, y_3)$ and $x_1 \in I_G[x_2, x_3]$. Now, it is clear from Proposition 4.1 that $(x_3, y_1) \in I_{G\boxtimes H}[(x_1, y_1), (x_2, y_2)]$ and $x_3 \in I_G[x_1, x_2]$, which is a contradiction.

Subcase 3.2. This is similar to Subcase 3.1. Thus the proof is complete. ∎

Theorem 4.29 *Let G and H be connected graphs and S a geodetic set of $G \boxtimes H$. Then, $\pi_G(S)$ is a geodetic set of G or $\pi_H(S)$ is a geodetic set of H.*

Proof. Suppose that both $\pi_G(S)$ and $\pi_H(S)$ are not geodetic sets of G and H respectively. Then there exist vertices x in G and y in H such that $x \notin I_G[\pi_G(S)]$ and $y \notin I_H[\pi_H(S)]$. Since S is a geodetic set of $G \boxtimes H$, there exist $(g, h), (g', h') \in S$ such that (x, y) lies on a $(g, h) - (g', h')$ geodesic P in $G \boxtimes H$. Now, it follows from

Proposition 4.1 that $x \in I_G[\pi_G(S)]$ or $y \in I_H[\pi_H(S)]$, which is a contradiction. Hence $\pi_G(S)$ is a geodetic set of G or $\pi_H(S)$ is a geodetic set of H. ∎

Corollary 4.30 *Let G and H be connected graphs. Then $min\{g(G), g(H)\} \leq g(G \boxtimes H)$.*

The following theorem is useful in giving an improved lower bound of $g(G \boxtimes H)$ for a class of graphs.

Theorem 4.31 *Let G and H be connected graphs and S a geodetic set of $G \boxtimes H$. If $Ext(G) \neq \phi$, then $\pi_H(S)$ is a geodetic set of H.*

Proof. Let $S_1 = \pi_H(S)$. We show that S_1 is a geodetic set of of H. Let $x \in Ext(G)$ and $y \in V(H)$. Since S is a g-set of $G \boxtimes H$, the vertex (x, y) lies on a geodesic P : $(g_0, h_0), (g_1, h_1), \ldots, (g_i, h_i) = (x, y), \ldots, (g_n, h_n)$ of length n with $(g_0, h_0), (g_n, h_n) \in S$. First, suppose that $d_G(g_0, g_n) \leq d_H(h_0, h_n)$. Then it follows from Proposition 4.1 that $\pi_H(P)$ is a $h_0 - h_n$ geodesic in H containing the vertex y, with $h_0, h_n \in S_1$. Next, suppose that $d_G(g_0, g_n) > d_H(h_0, h_n)$. Then, as above, by Proposition 4.1, $\pi_G(P)$ is a $g_0 - g_n$ geodesic in G containing the vertex x. Now, since the vertex x is extreme, either $x = g_0$ or $x = g_n$ and it follows that either $y = h_0$ or $y = h_n$. Hence S_1 is a geodetic set of H. ∎

Corollary 4.32 *Let G and H be connected graphs such that $Ext(G) \neq \phi$. Then $g(H) \leq g(G \boxtimes H)$.*

Corollary 4.33 *Let G and H be connected graphs such that $Ext(G) \neq \phi$ and $Ext(H) \neq \phi$. Then $max\{g(G), g(H)\} \leq g(G \boxtimes H)$.*

Corollary 4.34 *Let G be a connected graph and $m \geq 2$ an integer. Then*

(i) $g(G) \leq g(G \boxtimes K_m)$.

(ii) $g(G) \leq g(G \boxtimes K_{1,m})$.

The following lemma is useful in proving an upper bound for the geodetic number of $G \boxtimes H$.

Lemma 4.35 *Let G and H be connected graphs. If $g \in I_G[g', g'']$ and $h \in I_H[h', h'']$, then $(g, h) \in I_{G \boxtimes H}[S]$, where $S = \{g', g''\} \times \{h', h''\}$.*

Proof. Let g be a vertex of the geodesic $P : g' = g_0, g_1, \ldots, g_i = g, \ldots, g_n = g''$ in G and h a vertex of the geodesic $Q : h' = h_0, h_1, \ldots, h_j = h, \ldots, h_m = h''$ in H. Then $d_G(g_0, g_i) = i$ and $d_G(g_i, g_n) = n - i$ for all $0 \leq i \leq n$. Similarly, $d_H(h_0, h_j) = j$ and $d_H(h_j, h_m) = m - j$ for all $0 \leq j \leq m$. Without loss of generality, we may assume that $m \leq n$. Suppose that $(g, h) \notin I_{G \boxtimes H}[S]$. We consider two cases.

Case 1. $j \leq i$. First we show that $m - j > n - i$. Assume the contrary. Let P_1 be a $(g_0, h_0) - (g_i, h_j)$ geodesic and P_2 a $(g_i, h_j) - (g_n, h_m)$ geodesic in $G \boxtimes H$. Then it follows from Theorem 1.34 that $l(P_1) = i$ and $l(P_2) = n - i$. Now, $P_3 = P_1 \cup P_2$ is a $(g_0, h_0) - (g_n, h_m)$ walk in $G \boxtimes H$, which contains (g, h). Since $l(P_3) = n$, it follows from Theorem 1.34 that P_3 is a $(g_0, h_0) - (g_n, h_m)$ geodesic in $G \boxtimes H$ containing the vertex (g, h), which is a contradiction to our assumption that $(g, h) \notin I_{G \boxtimes H}[S]$. Hence $m - j > n - i$. Similarly, we can show that $j > n - i$.

Now, let P' be a $(g_n, h_0) - (g_i, h_j)$ geodesic and P'' a $(g_i, h_j) - (g_n, h_m)$ geodesic in $G \boxtimes H$. Since $m-j > n-i$ and $j > n-i$, it follows from Theorem 1.34 that $l(P') = j$ and $l(P'') = m - j$. Now, $P' \cup P''$ is a $(g_n, h_0) - (g_n, h_m)$ walk in $G \boxtimes H$, which contains (g, h). Since $l(P' \cup P'') = m$, it follows from Theorem 1.34 that $P' \cup P''$ is a $(g_n, h_0) - (g_n, h_m)$ geodesic, which contains (g, h). Thus $(g, h) \in I_{G \boxtimes H}[S]$, which is a contradiction.

Case 2. $i < j$. As in Case 1, we can prove that $n - i > m - j$ and $i > m - j$. Let Q' be a $(g_0, h_m) - (g_i, h_j)$ geodesic and Q'' a $(g_i, h_j) - (g_n, h_m)$ geodesic in $G \boxtimes H$. Then, as in Case 1, we can show that $Q' \cup Q''$ is a $(g_0, h_m) - (g_n, h_m)$ geodesic, which contains (g, h). Thus $(g, h) \in I_{G \boxtimes H}[S]$, which is a contradiction. Hence the result follows. ∎

Theorem 4.36 *Let G and H be connected graphs. If S and T are geodetic sets of G and H respectively, then $S \times T$ is a geodetic set of $G \boxtimes H$.*

Proof. Let $U = S \times T$. Let $(g, h) \in V(G \boxtimes H)$. Since S and T are geodetic sets of G and H respectively, there exist $g', g'' \in S$ and $h', h'' \in T$ such that $g \in I_G[g', g'']$ and $h \in I_H[h', h'']$. Then, by Lemma 4.35, $(g, h) \in I_{G \boxtimes H}[W] \subseteq I_{G \boxtimes H}[U]$, where $W = \{g', g''\} \times \{h', h''\}$. Hence U is a geodetic set of $G \boxtimes H$. ∎

Corollary 4.37 *Let G and H be connected graphs. Then $g(G \boxtimes H) \leq g(G)g(H)$.*

Theorem 4.38 *Let G and H be connected graphs. Then $min\{g(G), g(H)\} \leq g(G \boxtimes H) \leq g(G)g(H)$.*

Proof. This follows from Corollaries 4.30 and 4.37. ∎

Now, we proceed to characterize graphs G and H for which $g(G \boxtimes H) = e(G)e(H)$.

Theorem 4.39 *Let G and H be connected graphs. Then G and H are extreme geodesic graphs if and only if $G \boxtimes H$ is an extreme geodesic graph.*

Proof. Let G and H be extreme geodesic graphs. Then $Ext(G)$ and $Ext(H)$ are geodetic sets of G and H respectively. Then it follows from Theorems 4.3 and 4.36 that $Ext(G \boxtimes H) = Ext(G) \times Ext(H)$ is a geodetic set of $G \boxtimes H$. Hence $G \boxtimes H$ is an extreme geodesic graph.

Conversely, let $G \boxtimes H$ be an extreme geodesic graph. Then $Ext(G \boxtimes H)$ is a geodetic set of $G \boxtimes H$. Then it follows from Theorems 4.3 and 4.31 that $Ext(G)$ and $Ext(H)$ are geodetic sets of G and H respectively. Thus G and H are extreme geodesic graphs. ∎

Corollary 4.40 *Let G and H be connected graphs. Then G and H are extreme geodesic graphs if and only if $g(G \boxtimes H) = e(G)e(H)$.*

Proof. This follows from Theorems 4.3 and 4.39. ∎

A vertex x in a set S of vertices of G is a *geodetic interior vertex* of S if $x \in I_G[S - \{x\}]$. The set of all geodetic interior vertices of S is denoted by S^o. For a geodetic set S, we have (i) $S^o \subseteq S - Ext(G)$ and (ii) $S^o = S - Ext(G)$ if and only if S is an open geodetic set of G.

Theorem 4.41 *Let G and H be connected graphs such that H has a full degree vertex v_0. Then*

$g(G \boxtimes H) \leq min\{|S||T|-(|T|-1)|S^o| : S \text{ and } T \text{ are geodetic sets of } G \text{ and } H \text{ respectively}\}$

Moreover, if H is an extreme geodesic graph, then

$g(G \boxtimes H) = min\{e(H)|S| - (e(H) - 1)|S^o| : S \text{ is a geodetic set of } G\}$.

Proof. Let S and T be geodetic sets of G and H respectively and let $W = ((S - S^o) \times T) \cup (S^o \times \{v_0\})$. Then $|W| = |S||T| - (|T| - 1)|S^o|$. We show that W is a geodetic set of $G \boxtimes H$. Let $(x, y) \in V(G \boxtimes H)$. Since T is a geodetic set of H, y lies on a $h - h'$ geodesic $P : h = h_0, h_1, \ldots, h_j = y, \ldots, h_m = h'$ in H with $h, h' \in T$. Now, we consider the following two cases.

Case 1. $x \in S - S^o$. Then, it follows from Theorem 1.34 that $P' : (x, h) = (x, h_0), (x, h_1), \ldots, (x, h_j) = (x, y), \ldots, (x, h_m) = (x, h')$ is a geodesic in $G \boxtimes H$ with $(x, h), (x, h') \in (S - S^o) \times T$. Hence $(x, y) \in I_{G\boxtimes H}[(S - S^o) \times T] \subseteq I_{G\boxtimes H}[W]$.

Case 2. $x \notin S - S^o$. Then x lies on a $g - g'$ geodesic $Q : g = g_0, g_1, \ldots, g_i = x, g_{i+1}, \ldots, g_n = g'$, where $1 \leq i \leq n - 1$ and $g, g' \in S$. We consider the following three subcases.

Subcase 2.1. Both $g, g' \in S - S^o$. Let $X = \{g, g'\} \times \{h, h'\}$. Then, by Lemma 4.35, $(x, y) \in I_{G\boxtimes H}[X] \subseteq I_{G\boxtimes H}[(S - S^o) \times T] \subseteq I_{G\boxtimes H}[W]$.

Subcase 2.2. Both $g, g' \notin S - S^o$. Then $g, g' \in S^o$. Since v_0 is a full degree vertex of H, it follows from Theorem 1.34 that $Q_1 : (g, v_0) = (g_0, v_0), (g_1, v_0), \ldots, (g_{i-1}, v_0), (g_i, y) = (x, y), (g_{i+1}, v_0), \ldots, (g_n, v_0) = (g', v_0)$ is a $(g, v_0) - (g', v_0)$ geodesic that contains the vertex (x, y), where $(g, v_0), (g', v_0) \in S^o \times \{v_0\} \subseteq W$.

Subcase 2.3. $g \in S - S^o$ and $g' \notin S - S^o$. Then $(g,h),(g,h'),(g',v_0) \in W$. Let $y \neq h,h'$. Since $diam(H) \leq 2$ and y lies on the $h - h'$ geodesic P, it follows that y is adjacent to both h,h'. Now, it is clear from Theorem 1.34 that $Q_2 : (g,h) = (g_0,h),(g_1,y),\ldots,(g_i,y) = (x,y),\ldots,(g_{n-1},y),(g_n,v_0) = (g',v_0)$ is a $(g,h)-(g',v_0)$ geodesic in $G \boxtimes H$ containing the vertex (x,y). If $y = h$ or h', say $y = h$, then as above (x,y) lies on a $(g,h)-(g',v_0)$ geodesic $Q_3 : (g,h) = (g_0,h),(g_1,h),\ldots,(g_i,h) = (x,y),\ldots,(g_{n-1},h),(g_n,v_0) = (g',v_0)$. Thus W is a geodetic set of $G \boxtimes H$ and the first part of the theorem follows.

Now, assume that H is an extreme geodesic graph. Then $T = Ext(H)$ is a geodetic set of H. Let W_1 be a g-set of $G\boxtimes H$. Then $g(G\boxtimes H) = |W_1|$. By Theorem 4.31, $S_1 = \pi_G(W_1)$ is a geodetic set of G. We first claim that $(S_1 - S_1^o) \times T \subseteq W_1$. Let $(x,y) \in (S_1 - S_1^o) \times T$. Then $x \notin S_1^o$. If $(x,y) \notin W_1$, then there exists $(u,v),(u',v') \in W_1$ such that (x,y) lies on a $(u,v)-(u',v')$ geodesic $P : (u,v) = (u_0,v_0),(u_1,v_1),\ldots,(u_i,v_i) = (x,y),\ldots,(u_m,v_m) = (u',v')$ with $1 \leq i \leq m-1$. Since y is an extreme vertex of H, it follows from Proposition 4.1 that $\pi_G(P) : u = u_0,u_1,\ldots,u_i = x,\ldots,u_m = u'$ is a $u - u'$ geodesic in G with $x \neq u,u'$. Thus $x \in I_G(u,u')$ with $u,u' \in S_1$ and so $x \in S_1^o$, which is a contradiction. Hence $(x,y) \in W_1$ and so $((S_1 - S_1^o) \times T) \subseteq W_1$. Let $X = W_1 - ((S_1 - S_1^o) \times T)$. Now, we claim that $S_1^o \subseteq \pi_G(X)$. Let $x \in S_1^o$. Then $x \in S_1$. Since $S_1 = \pi_G(W_1)$, there exists y such that $(x,y) \in W_1$. Since $x \notin S_1 - S_1^o$, we have $(x,y) \in X$ and so $x \in \pi_G(X)$. Thus $S_1^o \subseteq \pi_G(X)$ and so $|S_1^o| \leq |\pi_G(X)| \leq |X|$. If $|S_1^o| < |X|$, let $W_2 = ((S_1 - S_1^o) \times T) \cup (S_1^o \times \{v_0\})$. Then, as in the first part of the proof of this theorem, W_2 is a geodetic set of $G \boxtimes H$. Now, $|W_2| = |(S_1 - S_1^o) \times T| + |S_1^o| < |(S_1 - S_1^o) \times T| + |X| = |W_1|$, which is a contradiction to the fact that W is a minimum geodetic set of $G \boxtimes H$. Hence we have $|X| = |S_1^o|$ and so $|W_1| = |(S_1 - S_1^o) \times T| + |X| = |(S_1 - S_1^o) \times T| + |S_1^o| = |S_1||T| - (|T| - 1)|S_1^o|$. This completes the second part of the theorem. ∎

Corollary 4.42 *Let G be a connected graph. Then*

(i) $g(G \boxtimes K_n) = min\{n|S| - (n-1)|S^o| : S \text{ is a geodetic set of } G\}$

(ii) $g(G \boxtimes K_{1,n}) = min\{n|S| - (n-1)|S^o| : S \text{ is a geodetic set of } G\}$.

Corollary 4.43 *Let G and H be connected graphs such that H is an extreme geodesic graph with a full degree vertex. Then*

$e(G)(g(H) - 1) + g(G) \leq g(G \boxtimes H) \leq e(G)(g(H) - 1) + og(G)$.

Proof. Suppose that $g(G \boxtimes H) < e(G)(g(H) - 1) + g(G)$. Then, by Theorem 4.41, there exists a geodetic set S of G such that $e(H)|S| - (e(H)-1)|S^o| < e(G)(g(H)-1) + g(G)$. Thus, $e(H)|S| < e(G)(g(H)-1) + g(G) + (e(H)-1)|S^o|$. Since $S^o \subseteq S - Ext(G)$ and $e(H) = g(H)$, we have $g(H)|S| = e(H)|S| < e(G)(g(H) - 1) + g(G) + (e(H) - 1)(|S| - e(G)) = g(G) + (g(H) - 1)|S|$. Hence $|S| < g(G)$, which is a contradiction. Thus $e(G)(g(H)-1) + g(G) \leq g(G \boxtimes H)$. For the other inequality, let S be a minimum open geodetic set of G. Then $og(G) = |S|$ and $S^o = S - Ext(G)$. By Theorem 4.41, we have $g(G \boxtimes H) \leq e(H)|S| - (e(H) - 1)|S^o| = e(H)|S| - (e(H) - 1)(|S| - e(G)) = e(G)(e(H) - 1) + og(G)$. ∎

Theorem 4.44 *Let G be a connected graph and H an extreme geodesic graph with a full degree vertex. Then $g(G \boxtimes H) = e(G)(g(H) - 1) + g(G)$ if and only if $g(G) = og(G)$.*

Proof. Suppose that $og(G) = g(G)$. Then the result follows from Corollary 4.43. Conversely, assume that $g(G \boxtimes H) = e(G)(g(H)-1)+g(G)$. Let W be a g-set of $G \boxtimes H$. Then $|W| = e(G)(g(H)-1)+g(G) = e(G)(e(H)-1)+g(G)$. By Theorem 1.19 and 4.3, $W = (Ext(G) \times Ext(H)) \cup D$, where $D \subseteq V(G \boxtimes H)$ with $(Ext(G) \times Ext(H)) \cap D = \phi$. Hence $|D| = g(G) - e(G)$ and so $|\pi_G(W)| \leq e(G) + |\pi_G(D)| \leq e(G) + |D| = g(G)$. By Theorem 4.31, $\pi_G(W)$ is a geodetic set of G and so it follows that $|\pi_G(W)| = g(G)$. Now, we show that $\pi_G(W)$ is an open geodetic set of G. Let $x \in V(G)$ be such that $x \notin Ext(G)$. If $x \notin \pi_G(W)$, then, since $\pi_G(W)$ is a geodetic set of G, x lies as an internal vertex of a $g - g'$ geodesic in G with $g, g' \in \pi_G(W)$. Now, assume that $x \in \pi_G(W)$. First we prove that $\{x\} \times Ext(H) \nsubseteq W$. Otherwise, we have $\{x\} \times Ext(H) \subseteq W$. Then, since $Ext(G) \times Ext(H) \subseteq W$ and $\pi_G(W)$ contains $g(G) - e(G) - 1$ non-extreme vertices other than x, it follows that $|W| \geq e(H) + e(G)e(H) + (g(G) - e(G) - 1) = e(G)(e(H) - 1) + g(G) + (e(H) - 1) > e(G)(e(H) - 1) + g(G)$, which is a contradiction. Thus $\{x\} \times Ext(H) \nsubseteq W$. Hence there exists a $y \in Ext(H)$ such that $(x, y) \notin W$. Since W is a geodetic set of $G \boxtimes H$, it is clear that (x, y) lies on a $(g, h) - (g', h')$ geodesic P in $G \boxtimes H$ with $(g, h), (g', h') \in W$ and $(x, y) \neq (g, h), (g', h')$. Now, if $d_H(h, h') \geq d_G(g, g')$, then it follows from Proposition 4.1 that $\pi_H(P)$ is a $h - h'$ geodesic in H of length that of P so that y lies as an internal vertex of $\pi_H(P)$, which is a contradiction to y an extreme vertex of H. Hence, by Proposition 4.1, $\pi_G(P)$ is a geodesic in G that contains the vertex x with $x \neq g, g'$. Thus $\pi_G(W)$ is an open geodetic set of G and $|\pi_G(W)| = g(G)$. Hence $og(G) = g(G)$. ■

Theorem 4.45 *For integers $2 \leq r \leq s$ and $n \geq 2$, $g(K_{r,s} \boxtimes K_n) = 4$.*

Proof. If $r \geq 4$, then it is easily seen that $g(K_{r,s}) = og(K_{r,s}) = 4$ and so by Theorem 4.44, $g(K_{r,s} \boxtimes K_n) = 4$. If $r = 3$, then $g(K_{r,s}) = 3$ and $og(K_{r,s}) = 4$. Hence it follows from Corollary 4.43 and Theorem 4.44 that $g(K_{r,s} \boxtimes K_n) = 4$. Now, let $r = 2$. Let (X, Y) be the partite sets of $K_{2,s}$ with $|X| = 2$. Now, X and Y are geodetic sets of $K_{2,s}$. Let S be any geodetic set of $K_{2,s}$. If $S = X$ or Y, then $S^o = \phi$ and so $n|S| - (n-1)|S^o| = n|S| \geq 4$. Assume that $S \neq X, Y$. Then $|S| \geq 3$. If $|S| = 3$, then $|S^o| = 1$ and so $n|S| - (n-1)|S^o| = 2n + 1 \geq 5$. If $|S| \geq 4$, then $S^o = S$ or $|S^o| = 1$. If $|S^o| = 1$, then $n|S| - (n-1)|S^o| \geq 3n + 1 \geq 7$. If $S^o = S$, then $n|S| - (n-1)|S^o| = |S|$. Now, let $S = \{x_1, x_2, y_1, y_2\}$, where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then S is a geodetic set of $K_{2,s}$ with $S^o = S$. Hence it follows from Corollary 4.42 that $g(K_{2,s} \boxtimes K_n) = 4$. ∎

Geodetic number and double domination

In this section, we obtain an upper bound for the geodetic number of some strong product graphs in terms of the open geodetic number and double domination number of the factor graphs. This upper bound is also improved for certain classes of graphs.

Theorem 4.46 *Let G and H be connected graphs such that G has no extreme vertices. Then $g(G \boxtimes H) \leq og(G)\gamma_{\times 2}(H) - min\{og(G), \gamma_{\times 2}(H)\}$.*

Proof. Let $S = \{g_1, g_2, \ldots, g_p\}$ be an og-set of G and $T = \{h_1, h_2, \ldots, h_q\}$ a $\gamma_{\times 2}$-set of H. Let $r = min\{p, q\}$ and $U = S \times T - \bigcup_{i=1}^{r}\{(g_i, h_i)\}$. Then $|U| = pq - r$. We show that U is a geodetic set of $G \boxtimes H$. Let $(g, h) \in V(G \boxtimes H)$. Since

S is an *og*-set of G and G has no extreme vertices, g lies on a $g_i - g_j$ geodesic $P : g_i = u_0, u_1, \ldots, u_s = g, u_{s+1} \ldots, u_t = g_j$ for some $1 \leq s \leq t-1$ with $g_i, g_j \in S$. Also, since T is a $\gamma_{\times 2}$-set of H, it follows that h lies on a $h_k - h_l$ path $Q : h_k, h, h_l$ of length at most 2 with $1 \leq k \neq l \leq m$. Note that if $l(Q) = 1$, then either $h = h_k$ or $h = h_l$.

Case 1. $i = k$. Then $i \neq l$ and $j \neq k$. Hence $(g_i, h_l), (g_j, h_k) \in U$. It follows from Theorem 1.34 that $P' : (g_i, h_l) = (u_0, h_l), (u_1, h_l), \ldots, (u_{s-1}, h_l), (u_s, h) = (g, h), (u_{s+1}, h_k), \ldots, (u_t, h_k) = (g_j, h_k)$ is a geodesic in $G \boxtimes H$ that contains the vertex (g, h). Hence U is a geodetic set of $G \boxtimes H$.

Case 2. $i \neq k$. We consider the following two subcases.

Subcase 2.1. $j = l$. Then $i \neq l$ and $j \neq k$. Then as in Case 1, U is a geodetic set of $G \boxtimes H$.

Subcase 2.2. $j \neq l$. Then $(g_i, h_k), (g_j, h_l) \in U$ and it follows from Theorem 1.34 that $P'' : (g_i, h_k) = (u_0, h_k), (u_1, h_k), \ldots, (u_{s-1}, h_k), (u_s, h) = (g, h), (u_{s+1}, h_l), \ldots, (u_t, h_l) = (g_j, h_l)$ is a geodesic in $G \boxtimes H$ that contains the vertex (g, h). Hence U is a geodetic set of $G \boxtimes H$. ∎

Definition 4.47 Let G be a connected graph. A double dominating set $S = \{g_1, g_2, \ldots, g_p\}$ of G is *linear* if for each $g \in V(G)$, there exists an index i with $1 \leq i < n$ such that $g_i, g_{i+1} \in N[g]$.

For the graph G in Figure 4.6, the set $S = \{v_1, v_2, v_3\}$ is a linear minimum double dominating set. Any double dominating set consisting of exactly two elements is always linear. For the graph $G = K_{r,s}$ ($r = 1$ and $s \geq 3$), the set of all vertices of

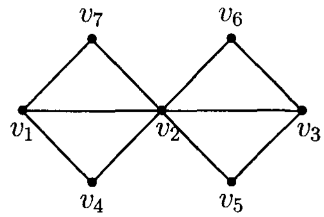

Figure 4.6: G

G is the unique double dominating set, which is not linear. For the graph $G = K_{r,s}$ $(r, s \geq 3)$, let S be a set of four vertices obtained by selecting the first two vertices from one partite set and the last two vertices from the other. Then S is a linear minimum double dominating set of G. The graph $K_{r,s}$ $(r = 2, s \geq 2)$ does not admit a linear minimum double dominating set.

Definition 4.48 Let G be a connected graph. An open geodetic set $S = \{g_1, g_2, \ldots, g_p\}$ of G is linear if for each $g \notin Ext(G)$, there exists an index i with $1 \leq i < n$ such that g lies as an internal vertex of a g_i-g_{i+1} geodesic in G.

For the graph G in Figure 4.7, the set $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a linear minimum open geodetic set of G. For the graph $G = K_{r,s}$ $(r, s \geq 2)$, let S be a set of four

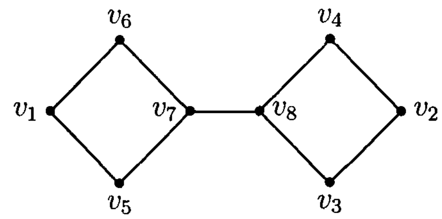

Figure 4.7: G

vertices obtained by selecting the first two vertices from one partite set and the last two vertices from the other. Then S is a linear minimum open geodetic set of G.

The following theorem gives an improved upper bound of Theorem 4.46.

Theorem 4.49 *Let G and H be connected graphs such that G has no extreme vertices. If G has a linear og-set and H has a linear $\gamma_{\times 2}$-set, then*

$$g(G \boxtimes H) \leq \lfloor \frac{og(G).\gamma_{\times 2}(H)}{2} \rfloor.$$

Proof. Let $S = \{g_1, g_2, \ldots, g_p\}$ be a linear og-set of G and $T = \{h_1, h_2, \ldots, h_q\}$ a linear $\gamma_{\times 2}$ of H. Let $U = S \times T - \bigcup_{i+j \text{ even}} \{(g_i, h_j)\}$. Then $|U| = \lfloor \frac{pq}{2} \rfloor$. We prove that U is a geodetic set of $G \boxtimes H$. Let $(g, h) \in V(G \boxtimes H)$. Since G has no extreme vertices and S is a linear og-set of G, it follows that g lies on a g_i-g_{i+1} geodesic $P : u_0, u_1, \ldots, u_s = g, u_{s+1}, \ldots, u_t = g_{i+1}$ with $1 \leq s \leq t-1$ for some $1 \leq i < p$. Also, since T is a linear $\gamma_{\times 2}$-set of H, h lies on a h_j-h_{j+1} path $Q : h_j, h, h_{j+1}$ of length at most 2 with $1 \leq j < q$.

Suppose that $i+j$ is odd. Then $(i+1)+(j+1)$ is odd and so $(g_i, h_j), (g_{i+1}, h_{j+1}) \in U$. Now, it follows from Theorem 1.34 that $P' : (g_i, h_j) = (u_0, h_j), (u_1, h_j), \ldots, (u_{s-1}, h_j),$ $(u_s, h) = (g, h), (u_{s+1}, h_{j+1}), \ldots, (u_t, h_{j+1}) = (g_{i+1}, h_{j+1})$ is a geodesic in $G \boxtimes H$ that contains (g, h). Hence U is a geodetic set of $G \boxtimes H$.

Next, suppose that $i + j$ is even. Then $i + (j + 1)$ and $(i + 1) + j$ are odd and so $(g_i, h_{j+1}), (g_{i+1}, h_j) \in U$. Now, it follows from Theorem 1.34 that $P'' : (g_i, h_{j+1}) =$ $(u_0, h_{j+1}), (u_1, h_{j+1}), \ldots, (u_{s-1}, h_{j+1}), (u_s, h) = (g, h), (u_{s+1}, h_j), \ldots, (u_t, h_j) = (g_{i+1}, h_j)$ is a geodesic in $G \boxtimes H$ that contains (g, h). Hence U is a geodetic set of $G \boxtimes H$. ∎

Corollary 4.50 *Let G be a connected graph such that G has no extreme vertices and G has a linear og-set. Then, for integers $r, s \geq 3$, $g(G \boxtimes K_{r,s}) \leq 2\ og(G)$. Moreover, $g(K_{r_1,s_1} \boxtimes K_{r_2,s_2}) \leq 8$ for $r_i, s_i \geq 3, i = 1, 2$.*

Proof. For the graph $K_{r,s}$ $(r, s \geq 3)$, let S be a set of four vertices obtained by selecting the first two vertices from one partite set and the last two vertices from the other. Then S is both a linear og-set as well as a linear $\gamma_{\times 2}$-set of $K_{r,s}$. Hence the corollary follows from Theorem 4.49. ∎

Remark 4.51 Let $r_i, s_i \geq 3$ for $i = 1, 2$. It follows from Corollary 4.37 that $g(K_{r_1,s_1} \boxtimes K_{r_2,s_2}) \leq 9$ if one of r_i or s_i is equal to 3 for $i = 1, 2$ and $g(K_{r_1,s_1} \boxtimes K_{r_2,s_2}) \leq 16$ for all $r_i, s_i \geq 4$ for $i = 1, 2$. However, Corollary 4.50 gives a better bound for $g(K_{r_1,s_1} \boxtimes K_{r_2,s_2})$.