CHAPTER – VIII
8.1. Introduction

Some of the important variations of a line graph are the following graph-valued functions:

1. The *middle graph* $M(G)$ (or the *semitotal-edge graph*) of a graph $G$ (due to Sampathkumar et al.[29] and Hamada et al.[1]) is the graph whose vertex set is $V(G) \cup E(G)$, and two vertices of $M(G)$ are adjacent if they are adjacent edges of $G$ or one is a vertex and the other is an edge of $G$ incident with it.

The *endedge-graph* $G^+$ of a graph $G$ is the graph obtained from $G$ by adjoining an endedge $u_iu_i'$ at each vertex $u_i$ of $G$. Thus, $G^+ = G \circ K_1$ (corona of $G$ and $K_1$). Hamada and Yoshimura [8] showed that $M(G) = L(G^+)$. If we call $M(G, v) = L(G, v)^+ < v >$, then $M(G) = \bigcup_{v \in V(G)} M(G, v)$.

2. The *total graph* $T(G)$ of a graph $G$ (due to Behzad [2]) is the graph with vertex set $V(G) \cup E(G)$, and two vertices are adjacent if and only if they correspond to two adjacent vertices of $G$ or two adjacent edges of $G$ or to a vertex and an edge incident to it in $G$. 
In 1985, Syslo and Topp [31] defined the generalized middle graph $M(G, f)$, and the generalized total graph $T(G, f)$ of a non-trivial connected graph $G$ as follows:

$$M(G, f) = \bigcup_{v \in V(G)} \{L(G, v) + [CP(f(v)) \cup < v >]\}$$

and

$$T(G, f) = \bigcup_{v \in V(G)} \{L(G, v) + [CP(f(v)) \cup < v >]\} \cup G,$$

where $< v >$ is a trivial graph on $\{v\}$. Moreover, they obtained characterizations of graphs whose generalized line, middle and total graphs have crossing number 0. In the previous chapters we have already studied the graphs whose generalized line graphs have crossing number one. The purpose of this chapter is to introduce the notion of generalized semitotal-vertex graph and obtain its criterion for planarity. In addition, we establish a characterization of graphs whose generalized middle graphs (resp. the generalized total graphs) to have crossing number one. Our result will generalize those for middle graphs, total graphs or semitotal graphs (see, [18], [15], [29]).

3. The semitotal-vertex graph $N(G)$ of a graph $G$ (introduced by Sampathkumar et al. [29]), is the graph having its vertex set $V(G) \cup E(G)$, and in which two vertices are adjacent if and only if either they are adjacent vertices of $G$ or one is a vertex and the
other is an edge of \( G \) incident to it. We call this, a semitotal-graph and is obtained either by (I) or (II):

I. On each edge \( e = uv \) of \( G \), draw a triangle having vertices \( u, v, \) and \( e \) (where \( e \) is a new vertex corresponding to the edge \( e \))

II. \( N(G) = \bigcup_{v \in V(G)} \{ \overline{L(G, v)} + < v > \} \cup G. \)

We first introduce the generalized semitotal graph \( N(G, f) \) of a connected nontrivial graph \( G \) as follows:

\[
N(G, f) = \bigcup_{v \in V(G)} \{ \overline{L(G, v)} + [CP(f(v)) \cup < v >] \} \cup G.
\]

It is easy to see that \( X(G) \subseteq X(G, f) \), and \( X(G, f) = X(G) \iff f(v) = 0 \) for each vertex \( v \) of \( G \), where \( X \) is one of (the graph-valued functions) \( M, T \) and \( N. \)

The \textit{neighbourhood} of a vertex \( v \) in a graph \( G \), denoted \( N(v) \), is the set of all vertices adjacent to \( v \) in \( G \). The \textit{closed neighbourhood} \( N[v] = N(v) \cup \{ v \}. \) Let \( G_1 \) and \( G_2 \) be any two graphs. Then the \textit{product} of \( G_1 \) and \( G_2 \) (denoted by \( G_1 \times G_2 \)) is the graph with vertex set \( V(G_1) \times V(G_2) \), and two vertices \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) in \( V(G_1) \times V(G_2) \) are adjacent in \( G_1 \times G_2 \) whenever \((u_1 = v_1 \text{ and } u_2 \text{ is adjacent to } v_2) \) or \((u_2 = v_2 \text{ and } u_1 \text{ is adjacent to } v_1) \).
8.2. Crossings in Generalized Middle Graphs

In this section, we mainly characterize graphs whose generalized middle graphs have crossing number one. First, we establish six lemmas:

**Lemma 8.2.1.** Let $G$ be a graph and $f : V(G) \rightarrow N^*$ be a function. If $d(v) + f(v) \geq 5$ for a vertex $v$ of $G$, then $cr(M(G, f)) \geq 2$.

**Proof.** Let $v \in V(G)$. We discuss 2 cases depending on $d(v) + f(v)$:

**Case 1.** Consider $d(v) + f(v) = 5$. If $d(v) = 5$, then $f(v) = 0$ and subsequently, $G$ contains a subgraph isomorphic to $K_{1,5}$. Immediately, $K_5 \subseteq L(G, v)$. But by definition, $M(G, v) = L(G, v) + <v> \supseteq K_5 + K_1 = K_6$. So, $K_6 \subseteq M(G, v) \subseteq M(G, f)$. Since $cr(K_6) = 3$, it follows that $cr(M(G, f)) \geq 3$. Next, if $d(v) \leq 4$, then $f(v) \geq 1$. In view of Theorem 6.3.1, $cr(L(G, f)) \geq 2$. Since $L(G, f) \subseteq M(G, f)$, it follows that $cr(M(G, f)) \geq 2$.

In both the possibilities $cr(M(G, f)) \geq 2$.

**Case 2.** If $d(v) + f(v) \geq 6$, then in view of Theorem 6.3.1., $cr(L(G, f)) \geq 2$, and consequently $cr(M(G, f)) \geq 2$.

In either case, $cr(M(G, f)) \geq 2$. ■
LEMMA 8.2.2. Let $G$ be a graph and $f : V(G) \rightarrow N^*$ be a function. If $G$ contains a noncutvertex $v$ of degree 3 such that $d(v) + f(v) = 4$, then $cr(M(G, f)) \geq 2$.

PROOF. Now $d(v) = 3$ in $G$. Let $e_i = vv_i$ ($1 \leq i \leq 3$) be the edge incident to $v$. Since $v$ is a noncutvertex of $G$, there exists a path $Q$ joining $v_1$ and $v_3$ through $v_2$ that does not contain $v$. Thus, $G$ contains a subgraph homeomorphic to $K_1 + P_3$. Consequently, the vertices $v, e_1, e_2$ and $e_3$ in $L(G, v) + <v>$ induce a subgraph isomorphic to $K_4$. Moreover, the two vertices of $CP(f(v)) = CP(1) = \overline{K_2}$ are adjacent to the vertices $e_1, e_2$ and $e_3$ in $L(G, v)$. Consequently, $\cup_{u \in V(Q)} L(G, u) \cup \{L(G, v) + [CP(f(v)) \cup <v>]\} \subseteq M(G, f)$, and it contains a nonplanar subgraph homeomorphic to $K_{3,4}$, which has 2 crossings in an optimal drawing. Thus, $cr(M(G, f)) \geq 2$. ■

The following results of [18] are needed for our immediate use:

THEOREM 8.2.3. A graph $G$ has a middle graph with crossing number $k$ for $k=1$ or 2 if and only if the following hold: $G$ is planar; $\Delta(G) = 4$, and $G$ has exactly $k$ vertices of degree 4, and each of which is a cutvertex.

THEOREM 8.2.4. If a graph $G$ has at least one noncutvertex of degree 4, then $cr(M(G)) \geq 3$. 
LEMMA 8.2.5. Let $G$ be a graph and $f : V(G) \rightarrow N^*$ be a function. If $G$ contains at least 2 vertices, one is a cutvertex $u_1$ of degree 4, and the other is a vertex $u_2$ of degree $\geq 3$ such that $d(u_i) + f(u_i) = 4$ for $i=1,2$, then $cr(M(G,f)) \geq 2$.

PROOF. We distinguish 2 cases depending on the degree of $u_2$:

CASE 1. If $d(u_2) = 4$, then $f(u_2) = 0$. Now 2 subcases arise depending on the nature of $u_2$:

SUBCASE 1.1. Suppose $u_2$ is a cutvertex in $G$. In this situation, $G$ has at least 2 cutvertices $u_i$ ($i=1,2$) of degree 4. In view of Theorem 8.2.3, $cr(M(G)) \geq 2$. Since $M(G) \subseteq M(G,f)$, it follows that $cr(M(G,f)) \geq 2$.

SUBCASE 1.2. Suppose $u_2$ is a noncutvertex in $G$. Immediately, $cr(M(G)) \geq 3$ due to Theorem 8.2.4. Thus, $cr(M(G,f)) \geq 3$.

In either of the subcases, we have $cr(M(G,f)) \geq 2$.

CASE 2. If $d(u_2) = 3$, then $f(u_2) = 1$. Since $d(u_1) = 4$ in $G$, $f(u_1) = 0$. Consequently, $M(G,u_1) = L(G,u_1) + < u_1 > = K_5$. (i) Suppose $u_2$ is a noncutvertex of $G$. Then in view of Lemma 8.2.2, we have $cr(M(G,f)) \geq 2$. (ii) Suppose $u_2$ is a cutvertex of $G$. Let $e_j = u_2v_j (1 \leq j \leq 3)$ be the edge incident to $u_2$. Immediately, $u_2$ and 3 of its neighbouring vertices
$v_1, v_2$ and $v_3$ lie in a subgraph of $G$ isomorphic to $K_{1,3}$, whose middle graph contains a subgraph isomorphic to $K_4$. In addition to this, the 2 vertices of $CP(f(u_2)) = CP(1) = \overline{K_2}$ are adjacent to the vertices $e_1, e_2$ and $e_3$ in $L(G, u_2)$. Consequently, the graph $H = L(G, u_2) + [CP(f(u_2)) \cup <u_2>]$ contains a subgraph isomorphic to $K_{3,3}$ in $M(G, f)$. Since $M(G, u_1)$ and $H$ are edge-disjoint subgraphs of $M(G, f)$, it follows that there are at least 2 edge-disjoint nonplanar subgraphs in $M(G, f)$. This implies that $\text{cr}(M(G, f)) \geq 2$.

**Lemma 8.2.6.** Let $G$ be a graph and $f : V(G) \rightarrow N^*$ be a function. If $G$ contains at least 2 vertices $u_i$ (for $i=1,2$) of degree 3 such that $d(u_i) + f(u_i) = 4$, then $\text{cr}(M(G, f)) \geq 2$.

**Proof.** Suppose one of the vertices $u_i$ for $i=1,2$, is a noncutvertex of degree 3 satisfying $d(u_i) + f(u_i) = 4$. Then by Lemma 8.2.2, $\text{cr}(M(G, f)) \geq 2$. Next, assume that both $u_1$ and $u_2$ are cutvertices of $G$. Then $f(u_i) = 1$. Proceeding as in the proof of Case 2 (ii) of Lemma 8.2.5, $M(G, f)$ contains the subgraph $M_i = L(G, u_i) + [CP(f(u_i)) \cup <u_i>]$ in which $K_{3,3}$ appears. Since $M_1$ and $M_2$ are edge-disjoint subgraphs in $M(G, f)$, it follows that there are at least 2 edge-disjoint nonplanar subgraphs (each isomorphic to $K_{3,3}$) in $M(G, f)$. So, $\text{cr}(M(G, f)) \geq 2$. ■
LEMMA 8.2.7. Every nonplanar graph has a generalized middle graph with crossing number at least 2.

PROOF. Let G be a nonplanar graph. Then it has a subgraph H homeomorphic to either $K_5$ or $K_{3,3}$. If $H$ is homeomorphic to $K_5$, then in view of Theorem 8.2.4, $\text{cr}(M(H)) \geq 3$. Suppose $H$ is homeomorphic to $K_{3,3}$. Since $M(H) = L(H^+)$, each vertex of $H$ is a cutvertex in $H^+$. Moreover, $H^+$ has no vertex of degree 2. Since $H^+$ fails to satisfy the conditions of Theorem 7.2.1, $\text{cr}(L(H^+)) \neq 1$. But, in view of Theorem 7.2.2, and by the fact that $\text{cr}(H^+) = 1$, $\text{cr}(L(H^+)) = \text{cr}(M(H)) \geq 2$.

In both the possibilities, since $M(H) \subseteq M(G) \subseteq M(G, f)$, $\text{cr}(M(G, f)) \geq 2$. ■

LEMMA 8.2.8. Let $G$ be a graph, and $f : V(G) \rightarrow N^*$ be a function. If $G$ has a vertex $v$ of degree at most 2 satisfying the condition: $d(v) + f(v) = 4$, then $\text{cr}(M(G, f)) \geq 2$.

PROOF. We distinguish two cases depending on the degree of the vertex $v$:

1. Assume $d(v) = 1$. Then $f(v) = 3$. From Theorem 6.3.1., we have $\text{cr}(L(G, f)) > 1$. Therefore, $\text{cr}(M(G, f)) \geq 2$.

2. Assume $d(v) = 2$. Then $f(v) = 2$. In view of Theorem 6.3.1, $v$ must be a cutvertex since otherwise, $\text{cr}(L(G, f)) \geq 2$. 
and hence \( \text{cr}(M(G, f)) \geq 2 \). Let \( e_j = vu_j \) for \( 1 \leq j \leq 2 \), be the edge incident to \( v \). Immediately, \( v \) and 2 of its neighbouring vertices \( v_1 \) and \( v_2 \) lie in a subgraph of \( G \) isomorphic to \( K_{1,2} \), whose middle graph contains a subgraph isomorphic to \( K_3 \). Moreover, the 4 vertices of \( CP(f(v)) = CP(2) = C_4 \), are adjacent to the vertices \( e_1 \) and \( e_2 \) of \( L(G, v) \). Consequently, \( H = L(G, v) + [CP(f(v)) \cup <v>] \) contains a subgraph isomorphic to the sequential join graph \( K_1 + K_2 + C_4 \). It is not difficult to check that \( H \) has at least 2 crossings in an optimal drawing in a plane. Therefore, \( \text{cr}(M(G, f)) \geq 2 \).

In either case, we arrive at \( \text{cr}(M(G, f)) \geq 2 \). ■

We now present the main result. For this, we need a result of [31] and is as follows:

**THEOREM 8.2.9.** For a graph \( G \), and a function \( f : V(G) \rightarrow N^* \), \( \text{cr}(M(G, f)) = 0 \) if and only if \( G \) is planar, and \( d(v) + f(v) \leq 3 \) for every vertex \( v \) of \( G \).

**THEOREM 8.2.10.** For a nontrivial connected graph \( G \), and a function \( f : V(G) \rightarrow N^* \), \( \text{cr}(M(G, f)) = 1 \) if and only if the following conditions hold:
a) $G$ is planar.

b) $G$ contains a unique cutvertex $v$ of degree 3 or 4 such that $d(v) + f(v) = 4$.

c) For a vertex $u \neq v$ of $G$, $d(u) + f(u) \leq 3$.

**Proof.** Suppose $cr(M(G, f)) = 1$. In view of Lemma 8.2.7, $G$ is planar. However, by hypothesis, $M(G, f)$ is nonplanar, in view of Theorem 8.2.9, $G$ contains a vertex $u$ such that $d(u) + f(u) \geq 4$. On the other hand, in view of Lemma 8.2.1, $d(u) + f(u) \leq 4$ for every vertex $u$ of $G$. Consequently, $G$ must contain a vertex $v$ satisfying $d(v) + f(v) = 4$. Further, in view of Lemma 8.2.8, $d(v)$ cannot be 2 or less. So, $d(v) \geq 3$. We distinguish two cases:

**Case 1.** Assume $d(v) = 4$. Then $f(v) = 0$. Since $M(G) \subseteq M(G, f)$, it follows from Theorem 8.2.3 that $G$ contains a unique vertex of degree 4, which is a cutvertex. However, for any vertex $u \neq v$ in $G$ we have $d(u) + f(u) \leq 4$. Immediately, $d(u) \leq 3$; since otherwise Theorem 8.2.3 leads to $cr(M(G)) > 1$, and hence $cr(M(G, f)) > 1$. We discuss 2 subcases depending on the degree $d(u)$ of $u$:

**Subcase 1.1.** Suppose $d(u) = 3$ for any vertex $u$ of $G$. Then $f(u) \leq 1$. If $f(u) = 1$, then $u$ is a cutvertex; since otherwise Lemma 8.2.2 leads to $cr(M(G, f)) > 1$. Furthermore, $u$ is a unique cutvertex of degree 3; since otherwise Lemma 8.2.6
leads to \( cr(M(G, f)) \geq 2 \). Finally, the application of Lemma 8.2.5 implies that \( cr(M(G, f)) > 1 \). This is a contradiction. Therefore, \( f(u) = 0 \). Consequently, we have \( d(u) + f(u) = 3 \) in this case.

**SUBCASE 1.2.** Suppose \( d(u) \leq 2 \) for any vertex \( u \) of \( G \). If \( d(u) + f(u) = 4 \), then by Lemma 8.2.8, \( cr(M(G, f)) \geq 2 \), a contradiction. This implies that \( d(u) + f(u) \leq 3 \).

**CASE 2.** Assume that \( d(v) = 3 \). Then \( f(v) = 1 \). In view of Lemma 8.2.2, \( v \) must be a cutvertex. Further, from Lemma 8.2.6, \( v \) must be unique. Next, for any vertex \( u \neq v \) in \( G \), we have \( d(u) + f(u) \leq 4 \). Suppose \( d(u) + f(u) = 4 \). Then \( d(u) \leq 2 \) since otherwise, when \( d(u) = 3 \) Lemma 8.2.6 directly implies that \( cr(M(G, f)) \geq 2 \); if \( d(u) = 4 \), then either Theorem 8.2.4 or Lemma 8.2.5 leads to \( cr(M(G, f)) \geq 2 \). Since \( d(u) \leq 2 \), in view of Lemma 8.2.8, \( cr(M(G, f)) \geq 2 \). This proves that \( d(u) + f(u) \leq 3 \).

Conversely, assume that a planar graph \( G \) satisfies the conditions (b) and (c). By Theorem 8.2.9, \( cr(M(G, f)) \geq 1 \). We now show that \( cr(M(G, f)) \leq 1 \). Since \( G \) holds (b), the unique cutvertex \( v \) of degree 3 or 4 satisfies the condition \( d(v) + f(v) = 4 \). We discuss two cases depending on \( d(v) \):

**CASE 1.** If \( d(v) = 3 \), then \( v \) and 3 of its neighbouring vertices lie in a subgraph of \( G \) isomorphic to \( K_{1,3} \). Immediately, \( \{L(G, v) + <v>\} = K_4 \), appears in \( M(G, f) \). Since \( d(v) = 3 \), it follows that
\( f(v) = 1 \). But the 2 vertices of \( CP(f(v)) = \overline{K}_2 \) are joined to each vertex of \( L(G, v) \). So, \( L(G, v) + [<v> \cup CP(f(v))] = K_3 + [K_1 \cup \overline{K}_2] \) appears uniquely in \( M(G, f) \). Further, it is easy to check that \( cr(K_1 + K_3 + \overline{K}_2) = 1 \).

**CASE 2.** If \( d(v) = 4 \), then \( v \) and 4 of its neighbouring vertices lie in a subgraph of \( G \) isomorphic to \( K_{1,4} \), and \( f(v) = 0 \). Hence, \( L(G, v) + [<v> \cup CP(f(v))] = K_5 \) appears uniquely in \( M(G, f) \), and it has crossing number 1.

Since no vertex \( u \) other than \( v \) of \( G \) satisfies \( d(u) \geq 3 \), and \( d(u) + f(u) = 4 \), each vertex of \( G - v \) holds \( d(u) + f(u) \leq 3 \). In view of Theorem 8.2.8, it is easy to see that \( cr(M(G - v), f) = 0 \).

### 8.3. Crossings in Generalized Total Graphs

In this section, we obtain a characterization of graphs whose generalized total graphs have crossing number one. Our result generalizes the following theorem of [15].

**THEOREM 8.3.1.** For any graph \( G \), \( cr(T(G)) = 1 \) if and only if \( \Delta(G) = 4 \), every vertex of degree either 3 or 4 is a cutvertex, and there is a unique vertex of degree 4 in \( G \).

The following result of [31] will be applied in sequel.
**Theorem 8.3.2.** For a graph $G$, and a function $f : V(G) \to N^*$, $cr(T(G, f)) = 0$ if and only if the following conditions hold:

- $d(v) + f(v) \leq 3$ for each vertex $v$ of $G$; and if $d(v) = 3$, then $v$ is a cutvertex of $G$.

**Theorem 8.3.3.** For a nontrivial connected graph $G$ and a function $f : V(G) \to N^*$, $cr(T(G, f)) = 1$ if and only if the following conditions hold:

- a) $G$ contains a unique cutvertex $v$ of degree 3 or 4 such that $d(v) + f(v) = 4$.
- b) For a vertex $u \neq v$ in $G$, $d(u) + f(u) \leq 3$.
- c) If $d(u) + f(u) = 3$ for a vertex $u$ of $G$, then $d(u) = 3$ is a cutvertex.

**Proof.** Suppose that $cr(T(G, f)) = 1$. Since $M(G, f) \subseteq T(G, f)$, by Lemma 8.2.1, $d(v) + f(v) \leq 4$ for every vertex $v$ of $G$. Otherwise, $cr(M(G, f)) > 1$, and hence $cr(T(G, f)) \geq 2$, a contradiction. On the other hand, $T(G, f)$ is nonplanar. In view of Theorem 8.3.2, either (i) $d(v) + f(v) \geq 4$ for some vertex $v$ of $G$ or (ii) $G$ has a noncutvertex $v$ of degree 3. Assume (i) holds. Immediately, $G$ has a vertex $v$ such that $d(v) + f(v) = 4$. From Theorem 8.2.9, we have $cr(M(G, f)) \neq 0$. Since $cr(T(G, f)) = 1$ and $M(G, f) \subseteq T(G, f)$, it follows that $cr(M(G, f)) = 1$. By Theorem 8.2.10, $G$ is planar, and it has a unique cutvertex $v$ of degree...
either 3 or 4 satisfying the condition: \( d(v) + f(v) = 4 \). Finally, assume that (ii) holds. Then \( G \) contains a non-cut vertex \( v \) of degree 3. By Theorem 8.3.1, \( cr(T(G)) \geq 2 \). Since \( T(G) \subseteq T(G, f) \), it follows that \( cr(T(G, f)) \geq 2 \), a contradiction. This shows that every vertex of degree 3 in \( G \) must be a cut vertex. This proves the necessity part.

Conversely, assume that \( G \) satisfies the hypothesis of the theorem. If \( d(v) = 3 \) then since \( d(v) + f(v) = 4 \), \( f(v) = 1 \). Now, in view of Theorem 8.3.2, \( T(G, f) - CP(f(v)) \) is planar. Moreover, the vertices of \( L(G, v) \) appear on the exterior face of \( T(G, f) - CP(f(v)) \). Now, \( L(G, v) + CP(f(v)) \) produces exactly one crossing in \( T(G, f) \). If \( d(v) = 4 \) then \( K_{1,4} \subseteq G \). Clearly, \( L(G, v) + <v> = K_4 + K_1 = K_5 \) has exactly one crossing. Moreover, \( G \) has no vertex of degree 3, and also \( f(v) = 0 \). Thus, the introduction of \( G \) in \( \bigcup_{u \in N[v]} \{L(G, u) + <u>\} \) does not increase the crossing. Thus, \( cr(T(G, f)) = 1 \). 

8.4. Crossings in Generalized Semitotal Graphs

In this section, first we characterize planar generalized semitotal graphs. This will generalize the following result of [29].

**Theorem 8.4.1.** The semitotal graph of a graph \( G \) is planar if and only if \( G \) is planar.

First, we establish three lemmas for our immediate use.
LEMMA 8.4.2. The semitotal graph of any nonplanar graph has crossing number at least 4.

PROOF. Since $G$ is nonplanar, it has a subgraph $H$ homeomorphic to either $K_5$ or $K_{3,3}$. But we know that $H$ has crossing number 1. By the construction of $N(G)$ as stated in (I), we see that the introduction of new vertices corresponding to the crossed edges in $H$ constitute 2 overlapping triangles with at least 4 crossings in $N(G)$. Thus, $cr(N(G)) \geq 4$. 

LEMMA 8.4.3. The graph $\overline{K}_n + CP(k)$ for $n \geq 1$, is planar if and only if

$$(n+k) \begin{cases} 
\geq 1 & \text{if } k = 0. \\
\geq 2 & \text{if } k = 1. \\
\leq 4 & \text{if } k = 2.
\end{cases}$$

PROOF. Assume that $H(n, k) = \overline{K}_n + CP(k)$ for $n \geq 1$, is planar. Since $CP(k)$ for $k \geq 4$ is nonplanar, $H(n, k)$ is nonplanar. This is a contradiction. So, $k \leq 3$. If $k = 3$, then $CP(3) = \overline{K}_2 + C_4$. It is easy to check that the graph $H(1, 3)$ is nonplanar, and it has at least 3 crossings in its optimal drawing in a plane. Since $H(1, 3) \subseteq H(n, 3)$, it follows that $H(n, 3)$ is nonplanar. This is a contradiction, and so, $k \leq 2$.

1. If $k = 2$, then $CP(2) = C_4$. In this case, $n \leq 2$ since otherwise $H(3, 2) \subseteq H(n, 2)$, and immediately $H(3, 2)$ contains a subgraph
isomorphic to $K_{3,4}$. But $cr(K_{3,4}) = 2$. This implies that $H(n,2)$ is nonplanar. Thus, we have $n + k \leq 4$.

2. If $k \leq 1$, then we have

$$CP(k) = \begin{cases} K_2 & \text{if } k = 1; \\ \Phi & \text{if } k = 0. \end{cases}$$

In this case, $n \geq 1$ because $H(n,k)$ is either $H(n,0)$ or $H(n,1)$. But $H(n,0) = K_n$; $H(n,1) = K_n + K_2 = K_{n+2}$ and both are obviously planar graphs. Thus, we have

$$(n + k) \geq \begin{cases} 1 & \text{if } k = 0. \\ 2 & \text{if } k = 1. \end{cases}$$

The converse is obvious. $\blacksquare$

**LEMMA 8.4.4.** The graph $\overline{K_n} + [CP(k) \cup K_1]$ for $n \geq 1$, is planar if and only if

$$(n + k) \begin{cases} \geq 1 & \text{if } k = 0; \\ \leq 3 & \text{if } 1 \leq k \leq 2. \end{cases}$$

**PROOF.** Assume that $G(n,k) = \overline{K_n} + [CP(k) \cup K_1]$ is planar. Let $H(n,k) = \overline{K_n} + CP(k)$. Since $H(n,k) \subseteq G(n,k)$, $H(n,k)$ is also planar. By Lemma 8.4.3,
1. If $k = 0$, then by (*), we have $n \geq 1$, and $G(n, 0) = \overline{K}_n + K_1 = K_{1,n}$, which is planar. In this case, $n + k \geq 1$.

2. If $k = 1$, then by (*), we have $n \geq 1$. In this case $n \leq 2$ since otherwise $G(3, 1) \subseteq G(n, 1)$. But $G(3, 1) = \overline{K}_3 + [CP(1) \cup K_1] = \overline{K}_3 + (\overline{K}_2 \cup K_1) = K_{3,3}$, which is nonplanar, and hence, $G(n, 1)$ is nonplanar. Thus, $n + k \leq 3$.

3. If $k = 2$, then by (*), we have $n \leq 2$. However, as above $n$ cannot be $\geq 3$, and also $n$ cannot be 2. Otherwise, $G(2, 2) \subseteq G(n, 2)$, but it is easy to check that $G(2, 2) = \overline{K}_2 + [CP(2) \cup K_1] = \overline{K}_2 + [C_4 \cup K_1]$ contains a subgraph homeomorphic to $K_5$, and hence, $G(n, 2)$ is nonplanar. This implies that $n = 1$. Thus, $n + k = 3$ in this case.

The converse is obvious. ☐

We now characterize planar semitotal graphs in the following theorem:
THEOREM 8.4.5. For a nontrivial connected graph $G$, and a function $f : V(G) \rightarrow \mathbb{N}^*$, $N(G, f)$ is planar if and only if the following conditions hold:

a) $G$ is planar.

b) For a vertex $v$ of $G$, if $f(v) \neq 0$, then $d(v) + f(v) \leq 3$. Otherwise, $d(v) + f(v) \geq 1$.

PROOF. Suppose $N(G, f)$ is planar. Then its subgraphs $N(G)$ and $\{L(G, v) + [CP(f(v)) \cup < v>]\} \cup G$ for every vertex $v$ of $G$, are also planar. In view of Lemma 8.4.2, $G$ must be planar. Further, for any vertex $u$ of degree $d(u)$ in $G$, we have $K_{d(u)} + [CP(f(u)) \cup < u>] = L(G, u) + [CP(f(u)) \cup < u>] \subset N(G, f)$. But Lemma 8.4.3 gives, for each vertex $u$ of $G$, $d(u) + f(u) \geq 1$ with $f(u) = 0$, and also $d(u) + f(u) \leq 3$ with $1 \leq f(u) \leq 2$.

Conversely, assume that a planar graph $G$ and $f$ satisfy the given condition (b). We distinguish 2 cases depending on the value of $f(u)$.

**CASE 1.** If $f(u) = 0$ for every vertex $u$ of $G$, then $d(u) \geq 1$. Consequently, $N(G, f) = N(G)$. Since $G$ is planar, $N(G)$ and hence $N(G, f)$, is planar by Theorem 8.4.1.

**CASE 2.** If $f(u) \neq 0$ for some vertex $u$ of $G$, then since $1 \leq f(u) \leq 2$, we have $d(u) \leq 2$. 
There are two possibilities to discuss:

2.1) Suppose \( d(u) = 1 \). Then \( f(u) \leq 2 \). Clearly a planar subgraph \( \overline{L(G, u)} + [CP(f(u)) \cup < u >] \) isomorphic to either \( K_{1,3} \) or \( K_1 + (C_4 \cup K_1) \) appears in \( N(G, f) \). Further, \( G \) is planar, it is easy to see that \( N(G, f) \) is planar in this case.

2.2) Suppose \( d(u) = 2 \). Then \( f(u) = 1 \), and also there exist 2 edges \( e_1 \) and \( e_2 \) in \( G \) such that \( e_i = u_i u \) for \( 1 \leq i \leq 2 \). Obviously, an outerplanar induced subgraph \( < \{ u, e_1, e_2, u_1, u_2 \} > \) isomorphic to \( K_2 + K_1 + K_2 \) appears in \( N(G, f) \), and in addition to this each vertex of \( CP(f(u)) = \overline{K_2} \) is adjacent to both vertices \( e_1 \) and \( e_2 \), thereby producing a planar subgraph in \( N(G, f) \). Moreover, \( G \) is planar, it is easy to see that \( N(G, f) \) is planar.

Finally, we find a necessary and sufficient condition for graphs whose generalized semitotal graphs have crossing number one. For this, we establish 3 lemmas.

**Lemma 8.4.6.** Let \( G \) be a graph and let \( f : V(G) \to N^* \) be a function. If \( d(v) + f(v) \geq 5 \) for a vertex \( v \) of \( G \) such that \( f(v) \neq 0 \), then \( cr(N(G, f)) \geq 2 \).

**Proof.** If \( f(v) \geq 3 \), then \( CP(f(v)) \supseteq CP(3) \). Since \( G \) is a nontrivial graph, \( N(G, f) \supseteq K_1 + CP(3) \). But, it is easy to check that \( cr(K_1 + CP(f(v))) = 3 \). Thus, \( cr(N(G, f)) \geq 3 \). Now, define \( H = \overline{L(G, v)} + [CP(f(v)) \cup < v >] \).
We consider 2 cases:

**CASE 1.** If \( f(v) = 1 \), then \( d(v) \geq 4 \). In this situation, we have

\[
H \supseteq \overline{K_{d(v)}} + [CP(1) \cup K_1].
\]
\[
\supseteq \overline{K_4} + [\overline{K_2} \cup K_1].
\]
\[
= K_{3,4}.
\]

**CASE 2.** If \( f(u) = 2 \), then \( d(u) \geq 3 \). In this case,

\[
H \supseteq \overline{K_3} + [C_4 \cup K_1].
\]
\[
\supseteq K_{3,4}.
\]

In either case, \( H \subseteq N(G, f) \). Since \( cr(K_{3,4}) = 2 \), and \( K_{3,4} \subseteq H \), it follows that \( cr(N(G, f)) \geq 2. \]

**LEMMA 8.4.7.** Let \( G \) be a graph with a noncutvertex \( u \) of degree 3, and let \( f : V(G) \rightarrow N^* \) be a function such that \( d(u) + f(u) = 4 \), then \( cr(N(G, f)) \geq 4 \).

**PROOF.** Since \( u \) is a noncutvertex of degree 3 in \( G \), it follows that \( G \) has a subgraph \( H \) homeomorphic to \( K_4 - e \). Since \( d(u) + f(u) = 4 \), \( f(u) = 1 \). So, \( CP(f(u)) = \overline{K_2} \). Consequently, \( N(G, f) \) contains a subgraph \( K \), where

\[
K = N(H) \cup [\overline{L(H, u)} + CP(f(u))]
\]

But, it is easy to check that \( cr(K) = 4 \). Thus, \( cr(N(G, f)) \geq 4 \).
LEMMA 8.4.8. Let $G$ be a graph with at least 2 cutvertices $u_i$ (for $i = 1, 2$) of degree 3, and let $f : V(G) \rightarrow N^*$ be a function such that $d(u_i) + f(u_i) = 4$, then $\sigma(N(G, f)) \geq 2$.

PROOF. Since $G$ has at least two cutvertices $u_1, u_2$ of degree 3, it has a subgraph $H$ homeomorphic to either the graph $A$ or $K_2 \circ K_2$, where $A = K_1 + K_2 \circ K_1 = (K_1 + K_2) \cup (K_2 \circ K_1)$.

We consider two cases:

CASE 1. Assume $H$ is homeomorphic to $A$. Now, since $d(u_i) = 3$, $f(u_i) = 1$ for $i = 1, 2$. Moreover, $CP(f(u_i)) = K_2$. Further, $H_i = \overline{L(G, u_i)} + [CP(f(u_i)) \cup < u_i >]$ is isomorphic to $K_{3,3}$. Also, $H_1$ is edge disjoint from $H_2$. Since $H_1 \cup H_2 \subseteq N(G, f)$, $\sigma(N(G, f)) \geq 2$.

CASE 2. Assume that $H$ is isomorphic to $K_2 \circ \overline{K_2}$. Then, by using the same argument as in Case 1 above, we see that $\sigma(N(G, f)) \geq 2$. $lacksquare$

THEOREM 8.4.9. For a nontrivial connected graph $G$, and a function $f : V(G) \rightarrow N^*$, $\sigma(N(G, f)) = 1$ if and only if the following conditions hold:

a) $G$ is planar.

b) If $d(v) + f(v) = 4$, then $v$ is a unique cutvertex of degree 3 in $G$. 
c) For a vertex \( u \neq v \) in \( G \), if \( f(u) \neq 0 \), then \( d(u) + f(u) \leq 3 \).

Otherwise \( d(u) + f(u) \geq 1 \).

**Proof.** Assume that \( cr(N(G, f)) = 1 \). Since \( N(G) \subseteq N(G, f) \), \( cr(N(G)) \leq 1 \), and in view of Lemma 8.4.2, \( G \) must be planar. Further, \( f(v) \neq 0 \) for at least one vertex \( v \) of \( G \); since otherwise \( N(G, f) = N(G) \) would become a planar graph. Now, in view of Lemma 8.4.6, if \( f(v) \neq 0 \) then \( d(v) + f(v) \leq 4 \) and, for \( u \neq v \in V(G) \), either \( d(u) + f(u) \geq 1 \) with \( f(u) = 0 \) or \( d(u) + f(u) \leq 4 \) with \( f(u) \neq 0 \). If \( d(v) + f(v) \leq 3 \) for every \( f(v) \neq 0 \), then we have a planar generalized semitotal graph by Theorem 8.4.5. Thus, there exist at least one vertex \( v \) of \( G \) such that \( f(v) \neq 0 \) satisfying \( d(v) + f(v) = 4 \). Clearly, \( f(v) \) cannot be 3; since otherwise \( L(G, v) + [CP(f(v)) \cup < v >] = K_1 + [CP(3) \cup K_1] \subseteq N(G, f) \). But, we know that \( cr(K_1 + CP(3)) = 3 \). Thus, \( cr(N(G, f)) \geq 3 \), a contradiction. Hence, \( f(v) \) is either 1 or 2. Consequently, \( d(v) = 3 \) or 2.

We consider two cases depending on \( d(v) \):

**Case 1.** \( d(v) = 3 \). Now \( f(v) = 1 \). In view of Lemma 8.4.7, \( v \) must be a cutvertex. Also, by Lemma 8.4.8, it must be unique. Moreover, for \( u \neq v \), if \( f(u) \neq 0 \) then \( d(u) + f(u) \leq 4 \) else \( d(u) + f(u) \geq 1 \). If \( d(u) + f(u) = 4 \) with \( f(u) \neq 0 \) then \( d(u) \leq 2 \). Also, note that existence of a vertex \( u \) in \( G \) such that \( d(u) = 1 \)
satisfying \( d(u) + f(u) = 4 \) is not possible. Thus, \( d(u) = 2 \). We consider two subcases depending on the nature of \( u \):

**SUBCASE 1.1.** \( u \) is a noncutvertex of \( G \). Then \( G \) will have a subgraph isomorphic to \( C_n \bullet K_2 \) for \( n \geq 3 \). Clearly, \( \overline{L(G, v)} + [CP(f(v)) \cup < v >] \) is isomorphic to \( K_{3,3} \), and \( \overline{L(G, u)} + [CP(f(u)) \cup < u >] = C_4 + \overline{K_2} + K_1 \), which is homeomorphic to \( K_5 \) appears in \( N(G, f) \). Notice that \( K_5 \) and \( K_{3,3} \) are edge disjoint subgraphs of \( N(G, f) \). Thus, \( cr(N(G, f)) \geq 2 \), a contradiction.

**SUBCASE 1.2.** \( u \) is a cutvertex of \( G \). Then \( G \) will have a subgraph isomorphic to the sequential join graph \( \overline{K_2} + K_1 + K_1 + K_1 \). Again by using a similar argument as in Subcase 1.1 above, we have \( cr(N(G, f)) \geq 2 \), a contradiction.

Thus, \( d(u) + f(u) \leq 3 \) for \( f(u) \neq 0 \) or \( d(u) + f(u) \geq 1 \) for \( f(u) = 0 \).

**CASE 2.** \( d(v) = 2 \). Now \( f(v) = 2 \). But it is easy to see that \( cr(N(G, f)) \geq 2 \), a contradiction. This proves the necessity part.

Converse is easy and hence it is omitted. ■