CHAPTER – VI
CHAPTER VI
ON GENERALIZED LINE GRAPHS OF
PLANAR GRAPHS WITH CROSSING NUMBER ONE

6.1. Introduction

Generalized line graphs were first studied by Hoffman in 1968, and it was studied later by Cvetkovic, Rao and others (see, [6], [28]). In fact, this class of graphs are a natural generalization of the well-known concept of line graphs. For a nontrivial connected graph $G$, let $L(G, v)$ denote the complete graph on the set of all edges incident to a vertex $v$ of $G$. With this notation, the line graph $L(G)$ of a graph $G$ is also defined as $\bigcup_{v \in V(G)} L(G, v)$. A $(2n - 2)$-regular graph of order $2n$ is called a cocktail party graph $CP(n)$, and is obtained by removing a 1-factor from $K_{2n}$. Obviously, $CP(0)$ is an empty graph without vertices, $CP(1) = \overline{K_2}$ and $CP(2) = C_4$. For a function $f : V(G) \rightarrow \mathbb{N}^*$, the set of nonnegative integers, let $\{CP(f(v)) : v \in V(G)\}$ be the family of cocktail party graphs disjoint from each other and from $G$ as well as $L(G)$. The generalized line graph $L(G, f)$ of a graph $G$ is the graph: $\bigcup_{v \in V(G)} \{L(G, v) + CP(f(v))\}$. Clearly, $L(G) \subseteq L(G, f)$, and $L(G, f) = L(G)$ if and only if $f(v) = 0$ for each vertex $v$ of $G$. 
One of the spectral properties of line graphs is that the eigenvalues of the 0-1 adjacency matrix are bounded below by -2. The line graphs are not the only graphs with this property – the cocktail party graphs and generalized line graphs are also having the eigenvalue no smaller than -2 (see, [6]). In 1985, Syslo and Topp[31], have determined the conditions for the generalized line graphs to have crossing number 0. Recently, Jendrol and Klešč[12] have revised the conditions (stated by Kulli, Akka and Beineke[14]) for line graphs of planar graphs to have crossing number one. Now, our aim is to study the generalized line graphs of planar graphs as well as nonplanar graphs to have crossing number one.

In this chapter, we obtain a necessary and sufficient condition for planar graphs whose generalized line graphs have crossing number 1. Our result will give a natural generalization for line graphs of planar graphs given by Kulli et al.[14] and, the revised result by Jendrol et al.[12].

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6.2. Basic Results

The following results are useful to obtain our main result:

THEOREM 6.2.1.[31] For any cocktail party graph $CP(n)$, 
$cr(CP(n)) \geq 1$ for $n \geq 4$.

THEOREM 6.2.2.[12] For any planar graph $G$, $cr(L(G)) = 1$ if and only if (a) or (b) holds:

a) $\Delta(G) = 4$, and there is a unique noncutvertex of degree 4.

b) $\Delta(G) = 5$, every vertex of degree 4 is a cutvertex, there is a unique vertex of degree 5, and it has at most three incident edges in any block.

Now, we establish few simple lemmas for our immediate use:

LEMMA 6.2.3. Let $G$ be a graph, and let $f : V(G) \rightarrow N^*$, be a function. If $d(v) + f(v) \geq 6$ for a vertex $v$ of $G$, then $cr(L(G, f)) \geq 2$.

PROOF. Let $v \in V(G)$. We distinguish 4 cases depending on $d(v)$:

CASE 1. If $1 \leq d(v) \leq 2$, then $K_2 \subseteq G$. Immediately, $K_1 \subseteq L(G, v)$. Since $d(v) + f(v) \geq 6$, it follows that $f(v) \geq 4$. In view of Theorem 6.2.1, $CP(f(v))$ is nonplanar, and hence by Kuratowski's theorem it has a subgraph homeomorphic to ei-
ther $K_5$ or $K_{3,3}$. By definition, $L(G, f) \supseteq L(G, v) + CP(f(v)) \supseteq K_1 + CP(f(v))$. It is easy to see that $K_1 + CP(f(v))$ has a subgraph homeomorphic to $K_6$ or $K_{3,4}$.

**CASE 2.** If $3 \leq d(v) \leq 4$, then $G \supseteq K_{1,3}$. Immediately, $L(G, v) \supseteq K_3$. Since $d(v) + f(v) \geq 6$, it follows that $f(v) \geq 2$. By definition, $L(G, f) \supseteq L(G, v) + CP(2) \supseteq K_3 + C_4$. It is easy to see that $K_3 + C_4$ contains a subgraph homeomorphic to $K_6$.

**CASE 3.** If $d(v) = 5$, then $K_{1,5} \subseteq G$. Hence $L(G, v) \supseteq K_5$. Clearly $f(v) \geq 1$ since $d(v) + f(v) \geq 6$. Now, $L(G, f) \supseteq L(G, v) + CP(1) \supseteq K_5 + K_2 \supseteq K_6$.

**CASE 4.** If $d(v) = 6$, then $K_{1,6} \subseteq G$, and hence $L(G, v) \supseteq K_6$. However, $L(G, f) \supseteq L(G, v) \supseteq K_6$.

In each case, $L(G, f)$ contains a subgraph homeomorphic to either $K_{3,4}$ or $K_6$. Since $cr(K_6) = 3$, and $cr(K_{3,4}) = 2$, it follows that $cr(L(G, f)) \geq 2$.

**LEMMA 6.2.4.** Let $G$ be a graph containing a noncutvertex $v$ of degree 2, and let $f : V(G) \rightarrow N^*$ be a function such that $d(v) + f(v) \geq 4$. Then $cr(L(G, f)) \geq 2$.

**PROOF.** Since $d(v) = 2$ in $G$, there exist 2 edges: $e_i = vv_i$ (for $i = 1, 2$) incident to $v$ in $G$. Since $v$ is a noncutvertex, $G$ has 2 paths: $Q_1 = v_1vv_2$ and $Q_2 = v_1v_3 \ldots v_2$ (not containing $v$). More-
over \( f(v) \geq 2 \), we have \( L(G, f) \supseteq \{[L(G, v) + CP(2)] \cup L(Q_1 \cup Q_2)\} \supseteq (K_2 + C_4) \cup L(Q_1 \cup Q_2) \). However, it is not difficult to see that the cycle \( L(Q_1 \cup Q_2) \) meets exactly 2 edges of \( C_4 \) in an optimal drawing of \( (K_2 + C_4) \cup L(Q_1 \cup Q_2) \) in the plane. Thus, \( cr(L(G, f)) \geq 2 \).

**Lemma 6.2.5.** Let \( G \) be a graph having a vertex \( v \) of degree at least one, and let \( f : V(G) \rightarrow N^* \) be a function such that \( f(v) \geq 3 \). Then \( cr(L(G, f)) \geq 3 \).

**Proof.** Since \( d(v) \geq 1 \) in \( G \), \( K_2 \subseteq G \). Consequently, \( L(G, v) \supseteq K_1 \). But \( f(v) \geq 3 \). Thus, we have \( L(G, f) \supseteq L(G, v) + CP(f(v)) \supseteq K_1 + CP(3) \). In view of Theorem 6.2.1, \( CP(3) \) is planar, and further it is easy to see that the inner vertex number \( i(CP(3)) = 3 \). Now, it is not difficult to check that the crossing number of \( K_1 + CP(3) \) is 3. Hence, we have \( cr(L(G, f)) \geq 3 \).

**Lemma 6.2.6.** Let \( G \) be a graph, and let \( f : V(G) \rightarrow N^* \) be a function. If \( G \) has at least 2 noncutvertices \( v_i \) (for \( i = 1, 2 \)), of degree \( \geq 3 \) such that \( d(v_i) + f(v_i) = 4 \), then \( cr(L(G, f)) \geq 2 \).

**Proof.** First note that by Theorem 6.2.2, \( G \) contains at most one noncutvertex of degree 4. We consider two cases depending on the existence of a noncutvertex of degree 4.
Fig. 6.2.1. The vertices marked by $\odot$ induce $CP(f(v_2))$. 

$H = K_1 + P_4$. 

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CASE 1. If $G$ has a noncutvertex $v_1$ of degree 4, then $v_2$ is a noncutvertex of degree 3 in $G$ and $f(v_2) = 1$. In this case, $G$ has a subgraph $H$ homeomorphic to $K_1 + P_4$. It is sufficient to consider the subgraph $H = K_1 + P_4$ with $V(H) = \{v_1, v_2, u, a, b\}$, where $d(v_1) = 4, d(v_2) = d(u) = 3$ and $d(a) = d(b) = 2$. Further, $f(u), f(a), f(b)$ all are $\geq 0$. Then $L(G, f)$ contains a subgraph homeomorphic to the graph $H_1 = \cup_{x \in \{v_1, v_2, u, a, b\}}[L(H, x) + CP(f(x))]$, (see, Fig. 6.2.1).

CASE 2. If $G$ has only cutvertices of degree 4, then since $G$ has at least 2 noncutvertices $v_i$ (for $i = 1, 2$) of degree $\geq 3$, it has a subgraph $H$ homeomorphic to $K_4 - e$. It is sufficient to consider $H = K_4 - e$ with $V(H) = \{v_1, v_2, a, b\}$ such that $d(v_i) = 3$, and $d(a) = d(b) = 2$. Moreover, since $d(v_i) + f(v_i) = 4$, for $i = 1, 2$, it follows that $f(v_i) = 1$. So, $CP(f(v_i)) = CP(1) = K_2$. Further, $f(a)$ and $f(b) \geq 0$. Clearly, $L(G, f)$ has a subgraph homeomorphic to the graph $H_2 = \cup_{x \in \{v_1, v_2, a, b\}}[L(H, x) + CP(f(x))]$, (see, Fig. 6.2.2).

In both the cases, it is not difficult to see that $H_1$ or $H_2$ has crossing number 2. Since $L(G, f)$ contains a subgraph homeomorphic to one of the graphs $H_1$ and $H_2$, $cr(L(G, f)) \geq 2$. ■
Fig. 6.2.2.

$H = K_4 - e.$

$\bigcirc$ induce $CP(f(v_1))$ and $\bigcirc$ induce $CP(f(v_2))$. 
LEMMA 6.2.7. Let G be a graph and let \( f : V(G) \rightarrow N^* \) be a function. If \( d(v) \leq 4 \) and \( d(v) + f(v) = 5 \) for a vertex \( v \) of \( G \), then \( cr(L(G, f)) \geq 2 \).

PROOF. Let \( v \in V(G) \). There are 3 cases to discuss depending on \( d(v) \):

1. If \( d(v) = 4 \), then \( f(v) = 1 \), and immediately \( K_{1,4} \subseteq G \), and hence \( K_4 \subseteq L(G, v) \). But \( L(G, f) \supseteq L(G, v) + CP(f(1)) \supseteq K_4 + K_2 = K_6 - e \). Since \( cr(K_6 - e) = 2 \), it follows that \( cr(L(G, f)) \geq 2 \).

2. If \( d(v) = 3 \), then \( f(v) = 2 \), and \( K_{1,3} \subseteq G \). So, \( K_3 \subseteq L(G, v) \). Now, \( L(G, f) \supseteq K_3 + CP(2) \supseteq K_3 + C_4 \), which contains a subgraph homeomorphic to \( K_6 \). As before, \( cr(L(G, f)) \geq 3 \).

3. If \( d(v) \leq 2 \), then \( f(v) \geq 3 \). From Lemma 6.2.5, \( cr(L(G, f)) \geq 3 \).

The following theorem of [31] is used to prove our main result.

THEOREM 6.2.8. For a graph \( G \), and a function \( f : V(G) \rightarrow N^* \), \( cr(L(G, f)) = 0 \) if and only if \( G \) is planar, and conditions (a) and (b) hold: (a) \( d(v) + f(v) \leq 4 \) for every vertex \( v \) of \( G \).

(b) If \( d(v) + f(v) = 4 \) for any vertex \( v \) of \( G \), then \( d(v) \geq 3 \), and \( v \) is a cutvertex of \( G \).

Now, we present the main result which will give a natural generalization of Theorem 6.2.2.
6.3. Main Result

THEOREM 6.3.1. For a planar graph $G$, and a function $f : V(G) \to \mathbb{N}^*$, $cr(L(G, f)) = 1$ if and only if conditions (a) and (b) hold:

(a) $G$ has a unique vertex $v$ satisfying one of the facts:

(i) $d(v) = 5$ and $f(v) = 0$, $v$ has at most 3 incident edges in any block;

(ii) $v$ is either a noncutvertex of degree $\geq 3$ or a cutvertex of degree 2 satisfying $d(v) + f(v) = 4$.

(b) For any vertex $u$ of $G$ other than $v$ if $d(u) + f(u) = 4$, then $u$ is a cutvertex of degree $\geq 3$ or $d(u) + f(u) \leq 3$.

PROOF. Necessity: Suppose $cr(L(G, f)) = 1$. In view of Theorem 6.2.8, $G$ contains a vertex $v$ such that $d(v) + f(v) \geq 4$. On the other hand, in view of Lemma 6.2.3, we have $d(u) + f(u) \leq 5$ for each vertex $u$ of $G$. This implies $\Delta(G) \leq 5$.

We consider five cases depending on $\Delta(G)$:

CASE 1. $\Delta(G) = 5$. Then $G$ contains a vertex $v$ of degree 5 and hence a subgraph isomorphic to $K_{1,5}$ appears in $G$. So, $K_5 \subseteq L(G)$. Consequently, $cr(L(G)) = 1$. By Theorem 6.2.2, $v$ is unique. Moreover, $v$ has at most 3 edges in any block, and every vertex of degree 4 is a cutvertex. However, for any vertex $u \neq v$
in *G*, we have \(d(u) \leq 4\). Further, if \(d(u) + f(u) = 5\) for any vertex \(u\) in *G*, then by Lemma 6.2.7, \(\text{cr}(L(G, f)) \geq 2\), a contradiction. This implies that \(d(u) + f(u) \leq 4\). We discuss four subcases depending on \(d(u)\):

1.1) Suppose \(d(u) = 4\) for any \(u \in V(G)\). Then as above, \(u\) is a cutvertex, and consequently \(f(u) = 0\).

1.2) Suppose \(d(u) = 3\) for any \(u \in V(G)\). Clearly \(f(u) \leq 1\). In view of Lemma 6.2.6, *G* has at most one noncutvertex of degree 3 such that \(d(v_i) + f(v_i) = 4\) for \(v_i \in V(G)\). Let \(u\) be a noncutvertex of degree 3. Then \(f(u) = 1\). Thus, *G* contains a subgraph \(H\) which is homeomorphic to either \((\overline{K}_2 + K_1 + P_3)\) or \((K_1 + K_2 + K_1 + \overline{K}_3)\) having both vertices \(u\) and \(v\). It is sufficient to consider \(H\) which is either \((\overline{K}_2 + K_1 + P_3)\) or \((\overline{K}_3 + K_1 + K_2 + K_1)\).

i) If \(H = \overline{K}_2 + K_1 + P_3\) with \(d(v) = 5\), \(d(u) = 3\), and \(d(a) = d(b) = 2\) in *H*, then \(L(G, f)\) contains a subgraph which is homeomorphic to the graph \(K = \cup_{x \in \{u, v, a, b\}}[L(H, x) + CP(f(x))],\) where \(f(a), f(b) \geq 0\) (see, Fig. 6.2.3).

ii) If \(H = \overline{K}_3 + K_1 + K_2 + K_1\) with \(d(v) = 5\), \(d(u) = d(b) = 3\) and \(d(a) = 2\) in *H*, then \(L(G, f)\) contains a subgraph which is homeomorphic to the graph \(K = \cup_{x \in \{u, v, a, b\}}[L(H, x) + CP(f(x))],\) where \(f(a), f(b) \geq 0\) (see, Fig. 6.2.4).
Fig. 6.2.3. Vertices marked by $\odot$ induce $CP(f(u))$. 

$$H = \overline{K_2} + K_1 + P_3$$
Fig. 6.2.4. The vertices marked by $\odot$ induce $CP(f(u))$. 

$$H = \overline{K}_3 + K_1 + K_2 + K_1.$$
In either case, it is not difficult to see that \( cr(K) = 2 \). Hence, \( cr(L(G, f)) \geq 2 \), a contradiction. This shows that either every vertex \( u \) of degree 3 such that \( d(u) + f(u) = 4 \), is a cutvertex or \( u \) holds the condition: \( d(u) + f(u) = 3 \).

1.3) Assume \( d(u) = 2 \) for any vertex \( u \) of \( G \). Then \( f(u) \leq 2 \). Suppose \( f(u) = 2 \) for each vertex \( u \) of \( G \). In view of Lemma 6.2.4, \( u \) must be a cutvertex. Consequently, \( G \) contains a subgraph \( H \) homeomorphic to the 4-sequential join graph \( \overline{K_4} + K_1 + K_1 + K_1 \). It is enough to consider \( H = \overline{K_4} + K_1 + K_1 + K_1 \) with \( d(v) = 5 \) and \( d(u) = 2 \) in \( H \). Now, it is not difficult to check that crossing number of \( K = \cup_{x \in \{u, v\}} [L(H, x) + CP(f(x))] \) is 2 (see, Fig. 6.2.5). Immediately, \( cr(L(G, f)) \geq 2 \), a contradiction. This proves \( f(u) \leq 1 \). Thus, we showed that \( d(u) + f(u) \leq 3 \) for each vertex \( u \) of degree 2 in \( G \).

1.4) Assume \( d(u) = 1 \) for every vertex \( u \) of \( G \). Then \( f(u) \leq 3 \). Suppose \( f(u) = 3 \). From Lemma 6.2.5, \( cr(L(G, f)) \geq 3 \), a contradiction. This implies that \( f(u) \leq 2 \). In this case, we have \( d(u) + f(u) \leq 3 \).

**CASE 2.** \( \Delta(G) = 4 \). Then \( d(u) \leq 4 \) for every vertex \( u \) of \( G \). In view of Lemma 6.2.7, \( d(u) + f(u) \leq 4 \). Since \( L(G) \subseteq L(G, f) \), by Theorem 6.2.2, \( G \) contains at most one noncutvertex \( v \) of degree 4 satisfying the condition: \( d(v) + f(v) = 4 \).
$H = \overline{K_4} + K_1 + K_1 + K_1.$

Fig. 6.2.5. The vertices marked by $\oplus$ induce $CP(f(u)).$
We consider two subcases:

**SUBCASE 2.1.** Assume $G$ has a noncutvertex $v$ of degree 4 such that $d(v) + f(v) = 4$. For each vertex $u \neq v$ of degree 4 in $G$, suppose $d(u) + f(u) = 4$ holds. Then $u$ must be a cutvertex because of Theorem 6.2.2. We discuss 3 possibilities depending on $d(u)$:

2.1.1) Assume $d(u) = 3$ for any vertex $u$ of $G$. Clearly $f(u) \leq 1$. If $f(u) = 1$, then $u$ must be a cutvertex; since otherwise, it leads to Lemma 6.2.6, and thereby $cr(L(G, f)) \geq 2$. Thus, vertices of degree 3 in $G$ satisfying the conditions: $d(u) + f(u) = 4$, are all cutvertices. Otherwise, vertices of degree 3 in $G$ holds $d(u) + f(u) = 3$.

2.1.2) Assume $d(u) = 2$ for any vertex $u$ of $G$. Obviously, $f(u) \leq 2$. If $f(u) = 2$, then from Lemma 6.2.4, $u$ must be a cutvertex. Then $G$ contains a subgraph $H$ homeomorphic to either $B$ or $C$, where $B$ (resp. $C$) is the graph obtained from the union of 2 disjoint graphs $K = (K_1 + P_4)$ and $P_3$ such that any one pendant vertex of $P_3$ is identified with a vertex of degree 2 (resp. 3) of $K$.

We consider two possibilities:

i) $H = B$. Clearly, $\cup_{x \in \{u, v, l, m, a, b\}}[L(B, x) + CP(f(x))]$, where $d(a) = d(b) = 2, d(l) = d(m) = 3$, in $H$, and $f(y) \geq 0$ for $y \in \{a, b, l, m\}$ has crossing number 2, (see, Fig. 6.2.6).
The graph $B$.

Fig. 6.2.6. The vertices marked by $\odot$ induce $CP(f(u))$. 
The vertices marked by $\odot$ induce $CP(f(u))$. 

Fig. 6.2.7. The vertices marked by $\odot$ induce $CP(f(u))$. 

The graph $C$
ii) $H = C$. In this case, $\cup_{x \in \{u,v,l,m,a,b\}} [L(C,x) + CP(f(x))]$, where $d(a) = d(b) = 2, d(l) = 4, d(m) = 3$ in $H$, and $f(y) \geq 0$ for $y \in \{a,b,l,m\}$, has crossing number 2 (see, Fig. 6.2.7).

2.1.3) Assume $d(u) = 1$ for any vertex $u$ of $G$. Clearly, $f(u) \leq 3$. If $f(u) = 3$, then from Lemma 6.2.5, $cr(L(G,f)) \geq 3$, a contradiction. This shows that $f(u) \leq 2$. Thus, every vertex $u$ of degree 1 in $G$ holds $d(u) + f(u) \leq 3$.

**SUBCASE 2.2.** Assume each vertex $u$ of degree 4 in $G$ is a cutvertex. In this case, certainly $G$ contains at most one non-cutvertex $v$ of degree 3 satisfying $d(v) + f(v) = 4$; since otherwise in view of Lemma 6.2.6, $cr(L(G,f)) \geq 2$.

2.2.1) Suppose $G$ contains a noncutvertex $v$ of degree 3 satisfying $d(v) + f(v) = 4$. Clearly, all other vertices $w$ of degree 3 either satisfying $d(w) + f(w) = 4$ are cutvertices or they hold the equality $d(w) + f(w) = 3$.

2.2.2) Suppose each vertex $w$ of degree 3 in $G$ satisfying $d(w) + f(w) = 4$, is a cutvertex. If $v \in V(G)$ is different from both $u$ and $w$, then we have $d(v) \leq 2$. Suppose $d(v) = 2$. Then $f(v) \leq 2$. If $f(v) = 2$, then by Lemma 6.2.4, $v$ must be a cutvertex. Otherwise the vertex $v$ holds the inequality: $d(v) + f(v) \leq 3$, and in this case by Theorem 6.2.8, $cr(L(G,f)) = 0$, a contradiction.
Finally assume $d(v) = 1$. Then by Lemma 6.2.5, $f(v) \leq 2$. Consequently, $d(v) + f(v) \leq 3$. As above, $cr(L(G, f)) = 0$, a contradiction.

**CASE 3.** $\Delta(G) = 3$. Then $d(u) \leq 3$ for every vertex $u$ of $G$, and also $d(u) + f(u) \leq 4$. In view of Theorem 6.2.8, any vertex $v$ of $G$ satisfying $d(v) + f(v) = 4$, is either a noncutvertex of degree 3 or a cutvertex of degree 2. Further from Lemma 6.2.6, a noncutvertex $v$ of degree 3 must be unique or from Lemma 6.2.4, a vertex $v$ of degree 2 must be a cutvertex. Next, we claim that the cutvertex of degree 2 is unique. Suppose $v_i, i = 1, 2$ be a cutvertex of degree 2 such that $d(v_i) + f(v_i) = 4$. Then $G$ contains a subgraph homeomorphic to $P_4$. Then $\cup_{x \in \{v_1, v_2\}}[L(G, x) + CP(f(x))]$ has at least 2 crossings in an optimal drawing of $L(G, f)$, in the plane (see, Fig. 6.2.8), a contradiction.

**CASE 4.** $\Delta(G) = 2$. Then $d(u) \leq 2$ for every vertex $u$ of $G$, and moreover $d(u) + f(u) \leq 4$. If $d(u) + f(u) \leq 3$ for every vertex $u$ of $G$, then by Theorem 6.2.8, $cr(L(G, f)) = 0$. This is a contradiction. Thus, $G$ must contain a vertex $v$ of degree 2 such that $d(v) + f(v) = 4$, and by Lemma 6.2.4, $v$ is a cutvertex. The argument as in Case 3 above shows that $v$ is unique.

**CASE 5.** $\Delta(G) = 1$. Then $d(u) \leq 1$ for each vertex $u$ of $G$, and as before $d(u) + f(u) \leq 4$. In view of Lemma 6.2.5, $f(u) \leq 2$. 
Fig. 6.2.8.

○ induce $CP(f(v_1))$ and ◯ induce $CP(f(v_2))$. 

$H = P_4$. 
Consequently $d(u) + f(u) \leq 3$ holds for all vertices $u$ of $G$. From Theorem 6.2.8, $cr(L(G, f)) = 0$, a contradiction.

For the sufficiency, assume that a planar graph $G$ and $f$ holds condition (a) and (b). Then by Theorem 6.2.8, $cr(L(G, f)) \geq 1$. First assume (a)(i) and (b) hold. Then $v$ is a unique cutvertex of degree 5 such that the edges at $v$ can be partitioned into 2 sets $E_1$ and $E_2$ with $|E_i|$ is either 2 or 3 (for $i=1,2$), and no edges in different sets lies in the same block, and $f(v) = 0$. Let $e_j = va_j (1 \leq j \leq 5)$ be the edge incident at $v$. Then $G$ has a unique subgraph $H$ isomorphic to $K_{1,5}$ rooted at $v$, and further by Theorem 6.2.8, $cr(L(G - va_j, f)) = 0$ for some $j(1 \leq j \leq 5)$. But however, the vertices $va_1, va_2, va_3, va_4$ and $va_5$ are mutually adjacent in $L(G, f)$, and produce a unique optimal drawing of $L(H)$ with only one crossing.

Assume (a)(ii) and (b) holds. Then $2 \leq \Delta(G) \leq 4$. We distinguish three cases depending on the vertex $v$:

1. If $v$ is unique noncutvertex of degree 4 in $G$, then $f(v) = 0$. Obviously, $G$ has a unique subgraph $H$ isomorphic to $K_1 + P_4$, where $P_4 = a_1a_2a_3a_4$, and by Theorem 6.2.8, $cr(L(G - e, f)) = 0$, where $e = a_i a_{i+1}$ for $i=1$ or 2. However, the vertices $va_1, va_2, va_3$ and $va_4$ are mutually adjacent in $L(G, f)$, and produce a unique optimal drawing of $L(H)$ with only one crossing.
2. If \( v \) is a unique noncutvertex of degree 3 in \( G \), and satisfies the condition \( d(v) + f(v) = 4 \), then \( f(v) = 1 \). For any vertex \( u \neq v \) of \( G \), if \( d(u) + f(u) = 4 \), then \( u \) is a cutvertex of degree \( \geq 3 \). Clearly, \( G \) contains a subgraph \( H \) isomorphic to \( K_1 + P_3 \), where \( P_3 = a_1a_2a_3 \). From Theorem 6.2.8, \( cr(L(G - a_1a_2), f)) = 0 \). However, the vertices \( va_1, va_2 \) and \( va_3 \) are mutually adjacent in \( L(G, f) \), and produce a unique optimal drawing of \( L(H, \hat{f}) \), (where \( \hat{f} : V(H) \rightarrow N^* \) is a function such that \( \hat{f}(u) = f(u) \) for all vertices \( u \) of \( H \)) with only one crossing.

3. If \( v \) is a unique cutvertex of degree 2 satisfying \( d(v) + f(v) = 4 \), then \( f(v) = 2 \). So, \( CP(f(v)) = C_4 \). For any vertex \( u \neq v \) of \( G \), if \( d(u) + f(u) = 4 \), then \( u \) is a cutvertex of degree \( \geq 3 \). It is easy to see that \( L(G, f) = [L(G, v) + CP(2)] \cup L(G_1, f_1) \) (where \( G_1 \) is the subgraph of \( G \) induced by \( V(G) - \{v\} \), and \( f_1 : V(G_1) \rightarrow N^* \) be a function such that \( f_1(u) = f(u) \) for a vertex \( u \) in \( G_1 \), has only one crossing in its optimal drawing. This proves the sufficiency part.■