CHAPTER - V
5.1. Introduction

In 1963, Harary and Hill (see, [10]) introduced one of the topological invariants, viz., the crossing number of a graph: The crossing number \( \text{cr}(G) \) of a graph \( G \) is the minimum number of pairwise intersections of its edges when \( G \) is drawn in the plane. Obviously, \( \text{cr}(G) = 0 \) if and only if \( G \) is planar. For a nonplanar graph \( G \), \( \text{cr}(G) \geq 1 \). Kuratowski's theorem on planar graphs may be stated in the following manner: A graph \( G \) has crossing number 0 if and only if \( G \) has no subdivision of either \( K_5 \) or \( K_{3,3} \). However, it is well-known that \( \text{cr}(K_{3,3}) = 1 \), \( \text{cr}(K_{3,4}) = 2 \), \( \text{cr}(K_5) = 1 \) and \( \text{cr}(K_6) = 3 \).

Gross and Harary have suggested in [7] for the calculation of the exact values of the crossing number for the interesting class of graphs. However, Sedláček [30] have already determined the conditions for the well-known class of graphs viz., line graphs, to have crossing number 0. In [17], some results on join graphs of specialized graphs to have crossing number 1 were presented. In this chapter, we mainly establish the characterizations of sequential join graphs to have crossing number 1, either in terms of sequence of graphs or in terms of forbidden sequence-subgraphs.
However, the main results of this chapter have been presented in the *Conference on Graph Theory and its Applications*, Anna University, Chennai during March (14-16), 2001.

5.2. Crossings in Join Graphs

In this section, we obtain characterizations of the join graphs having crossing number 1. Before doing so, we first establish the following lemmas which are useful to develop the main results.

**Lemma 5.2.1.** Let $G$ be a nonplanar graph. Then $cr(K_1 + G) \geq 2$.

**Proof.** Since $G$ is nonplanar, by Kuratowski's theorem $G$ has a subgraph $H$ homeomorphic to either $K_5$ or $K_{3,3}$. Consequently, $K_1 + H$ contains a subgraph homeomorphic to either $K_6$ or $K_{3,4}$. In either case, $cr(K_1 + H) \geq 2$. Since $K_1 + H \subseteq K_1 + G$, $cr(K_1 + G) \geq 2$. ■

**Lemma 5.2.2.** Let $G_1$ and $G_2$ be two disjoint graphs such that $|G_1| \geq 3$ and $|G_2| \geq 4$. Then $cr(G_1 + G_2) \geq 2$.

**Proof.** Since $K_3 \subseteq G_1$, and $K_4 \subseteq G_2$, it follows that $K_3 + K_4 = K_{3,4} \subseteq G_1 + G_2$. However $cr(K_{3,4}) = 2$. Consequently, $cr(G_1 + G_2) \geq 2$. ■

The *inner vertex number* $i(G)$ of a planar graph $G$ (introduced by Kulli (see, [13])) is the minimum number of vertices not be-
longing to the boundary of the exterior region in any embedding of \( G \) in the plane. Clearly, \( G \) is outerplanar if and only if \( i(G) = 0 \).

A planar graph \( G \) is minimally nonouterplanar if \( i(G) = 1 \), and is an \( r \)-nonouterplanar graph if \( i(G) = r \) for \( r \geq 2 \). We denote an \( r \)-nonouterplanar graph by \( G^*_r \), and an outerplanar graph by \( G_0 \).

Now, we introduce a new class of graphs called \( k \) edge minimal graphs as follows:

**DEFINITION 5.2.3.** A minimally nonouterplanar graph \( G \) is called \( k \) edge minimal graph (for \( k \geq 1 \)) if \( G \) has exactly \( k \) edges \( e_1, e_2, \ldots, e_k \) such that the removal of these \( k \) edges results an outerplanar subgraph but the removal of \( m \) edges for \( m < k \) results a minimally nonouterplanar subgraph.

We denote a \( k \) edge minimal graph by \( G(e_1, e_2, \ldots, e_k) \), and call the edges \( e_1, e_2, \ldots, e_k \), as minimal edges of \( G(e_1, e_2, \ldots, e_k) \). For \( k = 1 \), we simply call \( G(e_1) \), an edge minimal graph, and denote it by \( G(e) \).

**LEMMA 5.2.4.** Let \( G \) be a 2-nonouterplanar graph. Then \( cr(K_1 + G) \geq 2 \).

**PROOF.** Since \( G \) is a 2-nonouterplanar graph, in any plane embedding of \( G \) there is a plane subgraph \( H \) having exactly two inner vertices \( u \) and \( v \) which cannot be placed on the boundary of the exterior region of \( H \). It is not difficult to check that in \( K_1 + H \),
the two edges joining $K_1$, and both $u$ and $v$ contribute at least 2 crossings with the edges of $H$. Since $K_1 + H \subseteq K_1 + G$, it follows that $cr(K_1 + G) \geq 2$. ■

**Lemma 5.2.5.** Let $e_1, e_2, \ldots, e_k$ for $k \geq 1$, be the minimal edges of a graph $G(e_1, e_2, \ldots, e_k)$. Then $cr(K_1 + G(e_1, e_2, \ldots, e_k)) = k$.

**Proof.** Let $K = G(e_1, e_2, \ldots, e_k)$ be a plane graph having exactly one inner vertex $u$ which does not lie on the boundary of the exterior region in any embedding of $K$ in the plane. Since $e_1, e_2, \ldots, e_k$ are minimal edges of $K$, it follows that $M = K - \{e_1, e_2, \ldots, e_k\}$ is outerplanar. But $K_1 + M$ is clearly planar, and $K_1 + K$ is nonplanar because the edge joining a vertex $v$ of $K_1$ to the inner vertex $u$ of $K$ intersects each $e_i$ once, $1 \leq i \leq k$. Consequently, $cr(K_1 + K) = k$. ■

**Lemma 5.2.6.** Let $G(e)$ be a subgraph of a graph $G$. Then $cr(K_2 + G) \geq 2$.

**Proof.** Obviously, $K_2 + G(e) = K_1 + G(e) + K_1$. In view of Lemma 5.2.5, $cr(K_1 + G(e)) = 1$. Since $K_1 + G(e) + K_1 = (K_1 + G(e)) \cup (G(e) + K_1)$, it follows that $cr(K_2 + G(e)) \geq 2$. But, $K_2 + G(e) \subseteq K_2 + G$, and hence $cr(K_2 + G) \geq 2$. ■
DEFINITION 5.2.7. Let $G$ and $H$ be two disjoint graphs, and let $T$ be a triangle in $H$. Then we say that $G$ is skew-homeomorphic to $H$ with respect to a triangle $T$ of $H$ (in short, $G$ is $T$-skew to $H$), (denoted $G_T(H)$) if $G$ is obtained from $H$ by a finite sequence of subdivisions of any edges of $H$ except edges of $T$.

LEMMA 5.2.8. Let $G$ be a graph $T$-skew to $K_2 + \overline{K}_2$. Then $\sigma (K_2 + G) \geq 2$.

PROOF. Let $H = K_2 + \overline{K}_2$. But $K_2 + H = K_6 - e$, where $e \in E(K_6)$. It is easy to check that $\sigma (K_6 - e) = 2$. Since $K_2 + G$ is $T$-skew to $K_2 + H$, it follows that $\sigma (K_2 + G) \geq 2$. ■

LEMMA 5.2.9. Suppose $G$ contains at least two disjoint cycles, then $\sigma (\overline{K}_2 + G) \geq 2$.

PROOF. Let $H$ be a subgraph of $G$ isomorphic to $C_m \cup C_n$ for $m, n \geq 3$. Then $H$ is homeomorphic to $2K_3$. It is easy to check that $\overline{K}_2 + 2K_3$ is the union of two edge-disjoint copies of $K_5 - e$ whose union produces 2 crossings in an optimal drawing in the plane. Consequently, $\sigma (\overline{K}_2 + G) \geq 2$. ■

The corona $G_1 \circ G_2$ of two graphs $G_1$ and $G_2$ is the graph $G$ obtained by taking one copy of $G_1$ and $|G_1|$ copies of $G_2$, and then joining the $i$th vertex of $G_1$ to every vertex in the $i$th copy of $G_2$. 
Fig. 5.2.1. The graph $K_1 + S(K_{1,3}) + K_1$.

Fig. 5.2.2. The graph $K_1 + (K_3 \circ K_1) + K_1$. 
Fig. 5.2.3. The graph $K_1 + (C_4 \cup \overline{K_2}) + K_1$. 
LEMMA 5.2.10. If a graph $G$ has a subgraph isomorphic to one of the graphs: $S(K_{1,3}), K_3 \circ K_1$, and $\overline{K}_2 \cup C_n; n \geq 4$, then $cr(\overline{K}_2 + G) \geq 2$.

PROOF. Suppose $G$ contains a subgraph $H$ isomorphic to one of the graphs: $S(K_{1,3}), K_3 \circ K_1$ and $\overline{K}_2 \cup C_n$ for $n \geq 4$. Then $\overline{K}_2 + H$ is the 3-sequential join graph $K_1 + H + K_1$. We consider 3 cases depending on $H$:

1. If $H = S(K_{1,3})$, then it is easy to see that $K_1 + S(K_{1,3}) + K_1$ is nonplanar, and its crossing number is 2, (see, Figure 5.2.1).

2. If $H = K_3 \circ K_1$, then it is easy to see that $K_1 + (K_3 \circ K_1) + K_1$ is nonplanar, and its crossing number is 3, (see, Figure 5.2.2).

3. If $H = \overline{K}_2 \cup C_n; n \geq 4$, then it is easy to see that $K_1 + (\overline{K}_2 \cup C_n) + K_1$ is nonplanar and its crossing number is 2, (see, Figure 5.2.3).

In either case, $cr(\overline{K}_2 + H) \geq 2$. Since $\overline{K}_2 + H \subseteq \overline{K}_2 + G$, it follows that $cr(\overline{K}_2 + G) \geq 2$. ♦

LEMMA 5.2.11. If $G$ has a subgraph isomorphic to $K_1 \cup C_n$ for $n \geq 4$, then $cr(K_2 + G) \geq 2$.

PROOF. Since $K_1 \cup C_n$ for $n \geq 4$, is a subgraph of $G$, it follows that $K_1 + (K_1 \cup C_n)$ is a 2-nonouterplanar graph. In view of Lemma 5.2.4, $cr(K_1 + [K_1 + (K_1 \cup C_n)]) \geq 2$. In fact,
\(c_t(K_1 + [K_1 + (K_1 \cup C_n)]) = 2. \) But \(K_1 + [K_1 + (K_1 \cup C_n)] = K_2 + (K_1 \cup C_n)\) which is in \(K_2 + G\). Consequently, \(c_t(K_2 + G) \geq 2.\)

Next, we restate Theorem 2.2.2 which characterizes the join graphs to have crossing number zero, one in terms of pairs of graphs and the other in terms of forbidden subgraph-sequences.

**THEOREM 5.2.12.** For any two disjoint graphs \(G_1\) and \(G_2\), the following assertions are equivalent:

a) \(c_t(G_1 + G_2) = 0.\)

b) \((G_1, G_2)\) is either of the type \((A, B)\) or \((B, A)\) such that

\((A, B) \in \Theta_1,\) where

\[
\Theta_1 = \{(K_1, G_0), (K_2, C_p); p \geq 3, (\overline{K}_2, mP_n),
\]

\[(K_2, mP_n); m, n \geq 1\},
\]

c) \((G_1, G_2)\) is free from either \((A, B)\) or \((B, A)\) so that one of \((A, B)\) and \((B, A)\) is isomorphic (resp. homeomorphic) to a pair of graphs in \(\Theta_2\) (resp. \(\Theta_3\), where

\[
\Theta_2 = \{(\overline{K}_2, K_{1,3}), (K_2, C_p); p \geq 3, (\overline{K}_3, \overline{K}_3)\},
\]

\[
\Theta_3 = \{(K_1, K_4), (K_1, K_{2,3}), (\overline{K}_2, K_1 \cup K_3)\}.
\]
Before proceeding further, we first define three sets of graphs as follows:

\[ \Gamma_1 = \{ \overline{K}_2, K_2 \} . \]

\[ \Gamma_2 = \{ \alpha_i \cup mP_n : 1 \leq i \leq 3 \text{ and } m \geq 0; n \geq 1 \} \]

where,

\( \alpha_1 \) - denotes a graph which consists of a path \( P_n \) for \( n \geq 3 \) together with a new vertex joined to some vertex of degree 2.

\( \alpha_2 \) - denotes a graph that consists of a path \( P_n \) for \( n \geq 3 \) together with a new vertex joined to both vertices of some end-edge of \( P_n \).

\( \alpha_3 \) - denotes a graph that consists of a path \( P_n \) for \( n \geq 4 \) together with a new vertex joined to both vertices of some nonend-edge of \( P_n \).

\[ \Gamma_3 = \{ \overline{K}_3, K_2 \cup K_1, P_3, K_3 \} \]

\( K_{3,3} - C_4 \) to denote the graph obtained from \( K_{3,3} \) by deleting all edges (but not vertices) of the cycle \( C_4 \).

The graph operation \( G \bullet H \) is a graph obtained by identifying any vertex of \( G \) with an arbitrary vertex of \( H \) in a unique manner (up to isomorphism).
The following lemmas settle the special case of our main result.

**LEMMA 5.2.13.** Let $G$ be an outerplanar graph with $\Delta(G) \geq 3$. Then the following statements are equivalent:

a) $cr(\overline{K_2} + G) = 1$.

b) $G \in \Psi_1$ where,

$$\Psi_1 = \{a_i \cup mP_n, \ 1 \leq i \leq 3; m \geq 0, n \geq 1, \ G_T(K_1 + P_3), C_p \bullet K_2; \ p \geq 4\}$$

c) $G$ has no subgraph either isomorphic to a member of $\Psi_2$ or homeomorphic to $K_{3,3} - C_4$ where,

$$\Psi_2 = \{K_{1,4}, 2K_{1,3}, 2K_3, C_p \cup \overline{K_2}; \ p \geq 4, K_3 \cup K_{1,3}, G_T(K_1 + P_3) \cup K_1, S(K_{1,3})\}$$

**PROOF.** (a) implies both (b) and (c): Suppose $cr(\overline{K_2} + G) = 1$. If $\Delta(G) \geq 4$, then $K_{1,4} \subseteq G$. Clearly, $\overline{K_2} + G \supseteq \overline{K_2} + K_{1,4} \supseteq K_{3,4}$. But $cr(K_{3,4}) = 2$ and hence, $cr(\overline{K_2} + G) \geq 2$. This is a contradiction to the fact that $cr(\overline{K_2} + G) = 1$. Thus, $\Delta(G) \leq 3$. Since $\Delta(G) \geq 3$, it follows that $\Delta(G) = 3$. Consequently, $G$ is $K_{1,4}$-free.

Let us first assume that $G$ is a forest. Suppose $G$ has at least two vertices $u$ and $v$ of degree 3. We distinguish two cases:
CASE 1. If \( u \) and \( v \) lie in different components of \( G \), then \( 2K_{1,3} \subseteq G \), and hence \( \overline{K_2, 2K_{1,3}} \subseteq (\overline{K_2}, G) \). It is easy to check that \( \overline{K_2} + 2K_{1,3} \) contains two edge-disjoint copies of \( K_{3,3} \), and hence \( \overline{\sigma(\overline{K_2} + 2K_{1,3})} = 2 \). This shows that \( \overline{\sigma(\overline{K_2} + G)} \geq 2 \). Equivalently, this implies that \( 2K_{1,3} \) is a forbidden subgraph of \( G \) and hence \( G \) must be \( 2K_{1,3} \)-free.

CASE 2. If \( u \) and \( v \) lie in a single component of \( G \), then \( G \) contains a subgraph homeomorphic to \( K_{3,3} - C_4 \). Therefore, \( (\overline{K_2}, G) \) contains a subgraph homeomorphic to \( (\overline{K_2}, K_{3,3} - C_4) \). However, it is easy to check that \( \overline{\sigma(\overline{K_2} + (K_{3,3} - C_4))} = 2 \). Consequently, \( \overline{\sigma(\overline{K_2} + G)} \geq 2 \). Equivalently, this proves that \( G \) can not contain a subgraph homeomorphic to \( K_{3,3} - C_4 \).

In either case, we arrived at a contradiction. So, \( G \) must contain only one vertex \( w \) of degree 3. Consequently, \( G \) contains a subgraph homeomorphic to \( K_{1,3} \) rooted at \( w \). Moreover, in view of Lemma 5.2.10, there can not be a subgraph of \( G \) isomorphic to \( S(K_{1,3}) \) rooted at \( w \). Thus, it follows that \( S(K_{1,3}) \) is a forbidden subgraph of \( G \) and further, one of the components of \( G \) must be \( \alpha_1 \), and each of its remaining components are all paths if it exists. This shows that \( (\overline{K_2}, \alpha_1 \cup mP_k) \) for \( m \geq 0 \) and \( k \geq 1 \), is the possible solution for \( \overline{\sigma(\overline{K_2} + G)} = 1 \).
Finally, assume that $G$ is not a forest. Then $G$ has at least one cycle. Assume that $G$ contains two or more cycles. We discuss two cases depending on whether cycles are disjoint or not:

1) If the cycles are disjoint in $G$, then by Lemma 5.2.9 (or 5.2.10), $cr(K_2 + G) \geq 2$. Subsequently, this implies that $C_n \cup \overline{K}_2$ for $n \geq 4$ and $2K_3$ are the forbidden subgraphs of $G$.

2) If some cycles $C_m$ and $C_n$ for $m, n \geq 3$, in $G$ are non-disjoint, then these cycles can not have a vertex in common. Otherwise $\Delta(G) \geq 4$, which is not possible. Moreover, these cycles must contain an edge in common because of the outerplanarity of $G$.

Next, observe that both $m$ and $n$ can not be $\geq 4$, since otherwise a subgraph homeomorphic to either $K_{3,3} - C_4$ or $C_k \cup \overline{K}_2$ for $k = m$ or $n \geq 4$; appears as a subgraph in $G$. But $cr(K_2 + (C_k \cup \overline{K}_2)) = 2$ by Lemma 5.2.10, and therefore, $cr(K_2 + G) \geq 2$. This shows that both $K_{3,3} - C_4$ and $C_p \cup \overline{K}_2; p \geq 4$ are forbidden subgraphs of $G$.

In either case, we arrived at a contradiction. This shows that $G$ must contain either a cycle $C_n$ for $n \geq 3$ or a subgraph which is isomorphic to $G_T(K_1 + P_3)$.

Suppose $G$ has a unique cycle $C_n$ for $n \geq 3$. Immediately, $G$ is free from $C_n \cup \overline{K}_2$ for $n \geq 4$ and $K_3 \cup K_{1,3}$; otherwise, $cr(K_2 + (C_n \cup \overline{K}_2)) = 2$ or $cr(K_2 + (K_3 \cup K_{1,3})) = 2$, and hence, $cr(K_2 + G) \geq 2$. Moreover, all vertices of degree 3 in $G$ lie on $C_n$. 
We discuss 2 possibilities depending on n:

i) If \( n = 3 \), then all vertices of \( C_n \) in \( G \) cannot be cutvertices; since otherwise, \( K_3 \circ K_1 \) appears in \( G \) and by Lemma 5.2.10, \( cr(\overline{K_2 + G}) \geq 2 \), and hence \( G \) is free from \( K_3 \circ K_1 \). This shows that \((\overline{K_2}, \alpha_i \cup mP_k)\) for \( i = 2, 3; m \geq 0, k \geq 1 \), is the possible solution for \( cr(\overline{K_2 + G}) = 1 \).

ii) If \( n \geq 4 \), then \( G = C_n \bullet K_2 \); since otherwise, as before \( 2K_{1,3} \), or a subgraph homeomorphic to \( K_{3,3} - C_4 \) or \( C_n \cup \overline{K_2} \) appears as a subgraph in \( G \). Therefore, \((\overline{K_2}, C_n \bullet K_2)\) is the possible solution for \( cr(\overline{K_2 + G}) = 1 \).

Suppose \( G \) contains a subgraph isomorphic to \( G_T(K_1 + P_3) \). Then obviously \( G = G_T(K_1 + P_3) \); since otherwise, \( G_T(K_1 + P_3) \cup K_1 \) will appear as a subgraph in \( G \) and also it is not difficult to check that \( cr(\overline{K_2 + [G_T(K_1 + P_3) \cup K_1]}) = 2 \) and therefore, \( cr(\overline{K_2 + G}) \geq 2 \). Hence, \( G \) is free from \( G_T(K_1 + P_3) \cup K_1 \), and \((\overline{K_2}, G_T(K_1 + P_3))\) is the possible solution for \( cr(\overline{K_2 + G}) = 1 \). This proves both (b) and (c).

(b) \( \implies \) (a). Suppose (b) holds. It is easy to check that in an optimal drawing of \( \overline{K_2 + G} \) in the plane, for \( G \in \Psi_1 \), there is exactly one crossing. This proves (a).
(c) \implies (b). Suppose (c) holds. Then no subgraph of $G$ is either isomorphic to a member of $\Psi_2$ or homeomorphic to $K_{3,3} - C_4$. Since $K_{1,4}$ is a forbidden subgraph of $G$, it follows that $\Delta(G) = 3$.

Two cases now arise, depending on the structure of $G$:

**CASE 1.** Suppose $G$ is a forest. Since $G$ fails to contain a subgraph homeomorphic to $K_{3,3} - C_4$, and also $G$ is $2K_{1,3}$-free, there exist a unique vertex of degree 3 in $G$. Subsequently, $G = \alpha_i \cup mP_n$ for $m \geq 0$ and $n \geq 1 \in \Psi_1$. Otherwise, a forbidden subgraph isomorphic to $S(K_{1,3})$ will appear in $G$.

**CASE 2.** Suppose $G$ is not a forest. Then $G$ has at least one cycle. Since $G$ fails to have a subgraph either isomorphic to one of the graphs: $K_{1,4}, 2K_{1,3}, K_3 \cup K_{1,3}, C_n \cup \overline{K_2}$ for $n \geq 4$ and $G_T(K_1 + P_3) \cup K_1$ or homeomorphic to $K_{3,3} - C_4$, it follows that $G$ has either only one cycle $C_p (p \geq 3)$ having all vertices of degree 3 in $G$ or $G = G_T(K_1 + P_3)$.

2.1) If $p = 3$, then $G = \alpha_i \cup mP_n$ for $i = 2$ or $3$; $(m \geq 0, n \geq 1)$ in $\Psi_1$. Otherwise, a forbidden subgraph isomorphic to $K_3 \circ K_1$ appears in $G$.

2.2) If $p \geq 4$, then since $G$ is $(C_p \cup \overline{K_2})$-free, it follows that $G = C_p \cdot K_2 \in \Psi_1$. This proves (b).
LEMMA 5.2.14. Let $G_1$ and $G_2$ be two graphs such that $|G_1| = 2$ and $\Delta(G_2) \leq 2$. Then the following statements are equivalent:

a) $cr(G_1 + G_2) = 1$.

b) $(G_1, G_2)$ is isomorphic to a pair of $\xi_1$ where,

$$\xi_1 = \{(\overline{K_2}, C_n \cup K_1), (K_2, C_n); n \geq 4, (\overline{K_2}, K_3 \cup tP_s), (K_2, K_3 \cup lP_s)\} \text{ for } l \geq 0 \text{ and } t, s \geq 1.$$

c) $(G_1, G_2)$ is free from $(H_1, H_2)$ so that $(H_1, H_2)$ is a pair in $\Theta_1 \cup \xi_2$ where,

$$\Theta_1 = \{(K_1, G_0), (\overline{K_2}, C_p); p \geq 3, (\overline{K_2}, mP_n), (K_2, mP_n); m, n \geq 1\}.$$

$$\xi_2 = \{(\overline{K_2}, C_n \cup \overline{K_2}), (K_2, C_n \cup K_1); n \geq 4, (\overline{K_2}, 2K_3)\}.$$

PROOF. (a) implies both (b) and (c): Suppose $cr(G_1 + G_2) = 1$.

We consider two cases depending on the structure of $G_2$.

CASE 1. Suppose $G_2$ is a forest. Then since $\Delta(G_2) \leq 2, G_2 = tP_s$ for $t, s \geq 1$. By Theorem 5.2.12, $cr(G_1 + G_2) = 0$. This is a contradiction to the fact that $cr(G_1 + G_2) = 1$. Thus $(G_1, G_2) \not\in \Theta_1$. 
CASE 2. Suppose $G_2$ is not a forest. Then $G_2$ contains a cycle. Since $\Delta(G_2) \leq 2$, all cycles in $G_2$ are disjoint. Suppose $G_2$ has two or more disjoint cycles. Then $G_2 \supseteq C_m \cup C_n$ for $m, n \geq 3$. If either $m$ or $n$ is $\geq 4$, then $(K_2, C_p \cup K_2); p \geq 4$ will appear as a subgraph-sequence in $(G_1, G_2)$. But by Lemma 5.2.10, $cr(\overline{K_2} + (C_n \cup \overline{K_2})) = 2$. Hence $cr(G_1 + G_2) \geq 2$, a contradiction. This shows that, $m=n=3$. Again, by Lemma 5.2.9, $cr(\overline{K_2} + 2K_3) = 2$. Hence, $cr(G_1 + G_2) \geq 2$, a contradiction. So, $(G_1, G_2)$ has no subgraph-sequence isomorphic to either $(\overline{K_2}, 2K_3)$ or $(\overline{K_2}, C_n \cup \overline{K_2})$ for $n \geq 4$. The above contradictions leads to the fact that $G_2$ contains a unique cycle $C_n$ for $n \geq 3$. Now, we consider two cases depending on $n$:

SUBCASE 2.1. Assume $n \geq 4$.

2.1.1) Suppose $G_2$ contains three or more components. Then $C_n \cup \overline{K_2} \subseteq G_2$. By Lemma 5.2.10, $cr(\overline{K_2} + (C_n \cup \overline{K_2})) = 2$. So, $cr(G_1 + G_2) \geq 2$, a contradiction. Hence no solution exist for which $cr(G_1 + G_2) = 1$.

2.1.2) Suppose $G_2$ contains exactly two components. Then $G_2 = C_n \cup P_m$ for $m \geq 1$. In this situation $m = 1$, otherwise $C_n \cup \overline{K_2} \subseteq G_2$, and as above, the solution does not exist. If $G_1 = \overline{K_2}$, then it is easy to check that $cr(G_1 + G_2) = 1$. Consequently, $(\overline{K_2}, C_n \cup K_1)$ is the solution of $cr(G_1 + G_2) = 1$. If $G_1 = K_2$ then
since \(\sigma(K_2 + (C_n \cup K_1)) = 2\) by Lemma 5.2.11, no solution exist in this case. Hence \((G_1, G_2)\) has no subgraph-sequence isomorphic to \((K_2, C_n \cup K_1)\).

2.1.3) Suppose \(G_2\) contains only one component. Then \(G_2 = C_n; n \geq 4\). If \(G_1 = \overline{K_2}\) then by Theorem 5.2.12, \(\sigma(G_1 + G_2) = 0\), and consequently no solution exist for \(\sigma(G_1 + G_2) = 1\). Hence \((G_1, G_2) \notin \Theta_1\). If \(G_1 = K_2\) then \(G_1 + G_2 = K_2 + C_n\) is homeomorphic to \(K_5\) whose crossing number is 1, and hence \((K_2, C_n)\) is the possible solution for which \(\sigma(G_1 + G_2) = 1\).

SUBCASE 2.2. Assume \(n = 3\). Then \(G_2 = K_3 \cup tP_s\) for \(t \geq 0, s \geq 1\). If \(G_1 = \overline{K_2}\), then \(G_2 = K_3 \cup tP_s, t, s \geq 1\) since otherwise by Theorem 5.2.12, \(\sigma(G_1 + G_2) = 0\). Consequently, \((\overline{K_2}, K_3 \cup tP_s)\) for \(t, s \geq 1\) is the possible solution for \(\sigma(G_1 + G_2) = 1\). Thus \((G_1, G_2) \notin \Theta_1\). If \(G_1 = K_2\) then it is easy to check that \((K_2, K_3 \cup tP_k)\) for \(t \geq 0, k \geq 1\) is the possible solution of \(\sigma(G_1 + G_2) = 1\). This proves both (b) and (c).

(b) \(\implies\) (a). Suppose (b) holds. It is easy to check that \(\sigma(G_1 + G_2) = 1\) for which \((G_1, G_2) \in \xi_1\). This proves (a).

c) \(\implies\) (b). Suppose (c) holds. Then \((G_1, G_2)\) has no subgraph-sequence isomorphic to a member of \(\Theta_1\) or \(\xi_2\). We consider two cases:
CASE 1. $G_2$ is a forest. Then $G_2 = lP_s$ for $l, s \geq 1$. Then the forbidden subgraph-sequence either $(\overline{K_2}, lP_s)$ or $(K_2, lP_s)$ of $\Theta_1$ appears in $(G_1, G_2)$, a contradiction.

CASE 2. $G_2$ has a cycle. Since $(\overline{K_2}, 2K_3)$ and $(\overline{K_2}, C_n \cup \overline{K_2})$ for $n \geq 4$ are the forbidden subgraph-sequences of $(G_1, G_2)$, it follows that $G_2$ must have a unique cycle. Further $(\overline{K_2}, C_n \cup \overline{K_2}), (K_2, C_n \cup K_1)$ for $n \geq 4$ and $(\overline{K_2}, C_m)$ for $m \geq 3$, are the forbidden subgraph-sequences of $(G_1, G_2)$. Hence $(G_1, G_2)$ is isomorphic to one of the graphs: $(\overline{K_2}, C_n \cup K_1); n \geq 4, (K_2, C_n); n \geq 4, (\overline{K_2}, K_3 \cup tP_s); t, s \geq 1, (K_2, K_3 \cup lP_s); l \geq 0, s \geq 1$. Consequently these are members of $\xi_1$. This proves (b). ■

In the following theorem, we obtain a necessary and sufficient condition for a join graph to have crossing number one in terms of pairs of graphs.

THEOREM 5.2.15. Let $G_1$ and $G_2$ be two disjoint graphs. Then $\text{cr}(G_1 + G_2) = 1$ if and only if one of the following holds:

a) $(G_1, G_2)$ is isomorphic to a pair of

$$[(\Gamma_3 \times \Gamma_3) - \{(P_3, K_3), (K_3, P_3), (K_3, K_3)\}]$$

b) $(G_1, G_2)$ is either $(A, B)$ or $(B, A)$ so that one of $(A, B)$ and $(B, A)$ is a pair of $(\Gamma_1 \times \Gamma_2) \cup \zeta \cup \xi_1$. 


where,
\[
\zeta = \{(K_1, G(e)), (K_2, C_p \bullet K_2); p \geq 4, (K_2, G_T(K_1 + P_3))\}
\]
\[
\xi_1 = \{(\overline{K_2}, C_n \cup K_1), (K_2, C_n); n \geq 4, (\overline{K_2}, K_3 \cup tP_s), (K_2, K_3 \cup lP_s)\} \text{ for } l \geq 0 \text{ and } t, s \geq 1.
\]

**PROOF.** Suppose \( cr(G_1 + G_2) = 1 \). Then \( G_1 + G_2 \) is nonplanar. In view of Lemma 5.2.1, \( G_i \) for \( i = 1, 2 \), is planar. Moreover, from Lemma 5.2.2, one of the following facts holds:

A. \( |G_1| \leq 2, \text{ and } |G_2| \geq 1. \)

B. \( |G_1| \geq 3, \text{ and } |G_2| \leq 3. \)

Suppose the inequalities in (A) holds: We discuss two cases depending on \( |G_1| \):

**CASE 1.** If \( |G_1| = 1 \), then \( G_1 = K_1 \). Clearly, \( G_2 \) is nonouterplanar because if \( G_2 \) were outerplanar, then in view of Theorem 5.2.12, \( cr(G_1 + G_2) = 0 \), a contradiction. However \( G_2 = G(e) \), since otherwise by Lemmas 5.2.4 (or 5.2.5), \( cr(G_1 + G_2) \geq 2. \) Thus, \( (K_1, G(e)) \) is the possible solution for \( cr(G_1 + G_2) = 1. \)

**CASE 2.** If \( |G_1| = 2 \), then \( G_1 \) is either \( \overline{K_2} \) or \( K_2 \). Now \( G_2 \) must be outerplanar; since otherwise in view of Lemma 5.2.6, \( cr(\overline{K_2} + G_2) \geq 2. \)
We need to discuss three subcases depending on $\Delta(G_2)$:

**SUBCASE 2.1.** $\Delta(G_2) \geq 3$. Immediately, two possibilities arise depending on the structure of $G_1$:

2.1.1) Assume $G_1 = \overline{K}_2$. But, $G_2$ is outerplanar with $\Delta(G_2) \geq 3$. Since $cr(\overline{K}_2 + G_2) = 1$, in view of Lemma 5.2.13, $G_2 \in \Psi_1$. Consequently, $(\overline{K}_2, \alpha_i \cup mP_n)$ for $1 \leq i \leq 3; m \geq 0, n \geq 1; (\overline{K}_2, C_p \cdot K_2)$ for $p \geq 4$ and $(\overline{K}_2, G_T(K_1 + P_3))$ are the possible solutions of $cr(G_1 + G_2) = 1$.

2.1.2) Assume $G_1 = K_2$. Since $\overline{K}_2 \subseteq K_2$, we first need to solve the equation $cr(\overline{K}_2 + G_2) = 1$. As in (2.1.1) above, the solution $G_2 \in \Psi_1$. But, we see that $G_2 \neq C_p \cdot K_2$. Otherwise $C_p \cup K_1 \subseteq G_2$, by Lemma 5.2.11, $cr(K_2 + G_2) \geq 2$. Further, $G_2 \neq G_T(K_1 + P_3)$; since otherwise $cr(K_2 + G_2) > 1$ by Lemma 5.2.8. Consequently, $(K_2, \alpha_i \cup mP_n)$ for $1 \leq i \leq 3; m \geq 0; n \geq 1$, is the possible solution for $cr(K_2 + G_2) = 1$.

**SUBCASE 2.2.** $\Delta(G_2) \leq 2$. Since $cr(G_1 + G_2) = 1$, and $|G_1| = 2$, in view of Lemma 5.2.14, $(G_1, G_2) \in \xi_1$ is the possible solution for $cr(G_1 + G_2) = 1$.

Now, we suppose the inequalities in (B) holds. Then $|G_1| \geq 3$. Let us split the second inequality: $|G_2| \leq 3$ as follows: $|G_2| = 3$ or $|G_2| \leq 2$. 
Suppose $|G_1| \geq 3$ and $|G_2| = 3$. (i) If $|G_1| = 3$, then $G_i \in \Gamma_3$ for $i=1,2$. However, one can directly check that the pair of graphs from $\Pi = \{(P_3, K_3), (K_3, P_3), (K_3, K_3)\}$ are not the solutions of $cr(G_1 + G_2) = 1$. Therefore, the possible solutions in this case are the pairs of $(\Gamma_3 \times \Gamma_3) - \Pi$. (ii) Suppose $|G_1| \geq 4$, and since $|G_2| = 3$, in view of Lemma 5.2.2, we get $cr(G_2 + G_1) \geq 2$. Hence, the solution does not exist in this case.

Suppose $|G_1| \geq 3$ and $|G_2| \leq 2$. Then using the graph-transformation: replace $G_1$ by $G_2$, and $G_2$ by $G_1$ one can get directly the inequalities in (A). However for (A), we have already determined solutions: $(G_1, G_2)$ of $cr(G_1 + G_2) = 1$. Since $cr(G_1 + G_2) = cr(G_2 + G_1)$, it follows that the solutions of $cr(G_1 + G_2) = 1$ are also all pair-graphs $(G_2, G_1)$.

Sufficiency: It is not difficult to check that each pair $(G_1, G_2)$ of $(\Gamma_1 \times \Gamma_2) \cup \xi_1 \cup \zeta$ satisfies the following conditions:

i) $G_1 + G_2$ contains a unique nonplanar subgraph $H$ and is homeomorphic to either $K_5$ or $K_{3,3}$

ii) There exists an edge $e \in H$ such that $(G_1 + G_2) - e$ is planar.

Consequently, $cr(G_1 + G_2) = 1$. ■

Next, we obtain a characterization of a join graph to have crossing number one in terms of forbidden subgraph-sequences:
**THEOREM 5.2.16.** Let $G_1$ and $G_2$ be two disjoint graphs. Then $\sigma'(G_1 + G_2) = 1$ if and only if $(G_1, G_2)$ is free from either $(A, B)$ or $(B, A)$ so that one of $(A, B)$ and $(B, A)$ is isomorphic to a pair of graphs in $\Theta_1 \cup \Phi$ or homeomorphic to $(\overline{K_2}, K_{3,3} - C_4)$, where

\[ \Theta_1 = \{(K_1, G_0), (\overline{K_2}, C_p); p \geq 3, (\overline{K_2}, mP_n), (K_2, mP_n); m, n \geq 1\} . \]

\[ \Phi = \{(\overline{K_3}, \overline{K_4}), (K_1, G^*_2), (K_1, G(e_0, e_2)), (K_1, 2G(e)), (\overline{K_2}, K_{1,4}), (\overline{K_2}, G(e)), (\overline{K_2}, 2K_{1,3}), (\overline{K_2}, G_T(K_1 + P_3) \cup K_1), (\overline{K_2}, 2K_3), (\overline{K_2}, K_3 \circ K_1), (\overline{K_2}, S(K_1, 3)), (\overline{K_2}, \overline{K_2} \cup C_p), (\overline{K_2}, K_{1,3} \cup K_3), (K_2, G_T(K_1 + P_3)), (K_2, K_1 \cup C_p), (P_3, K_3)\} \text{ for } p \geq 4 . \]

**PROOF.** Necessity: Suppose $\sigma'(G_1 + G_2) = 1$. Then any subgraph-sequence $(H_1, H_2)$ of $(G_1, G_2)$ holds the inequality: $\sigma'(H_1 + H_2) \leq 1$. Therefore, all we need to show is that a pair either $(A, B)$ of $\Theta_1 \cup \Phi$ or $(A, B)$ homeomorphic to $(\overline{K_2}, K_{3,3} - C_4)$ satisfies the condition: $\sigma'(A + B) \neq 1$. This follows because a pair-graph $(A, B)$ of $\Theta_1$ is such that $\sigma'(A + B) = 0$ by Theorem 5.2.12, and moreover, $\Phi \neq (\Gamma_1 \times \Gamma_2) \cup \xi_1 \cup \zeta$ and so, by Theorem 5.2.15, each pair $(A, B)$ of $\Phi$ holds the inequality $\sigma'(A + B) \geq 2$, and also a pair $(A, B)$ homeomorphic to $(\overline{K_2}, K_{3,3} - C_4)$ is not in $(\Gamma_1 \times \Gamma_2) \cup \xi_1 \cup \zeta$ and hence $\sigma'(G_1 + G_2) \geq 2$.
Sufficiency: Assume that $(G_1, G_2)$ satisfies the hypothesis of the theorem. Since a pair $(K_3, K_4)$ is a forbidden subgraph-sequence of $(G_1, G_2)$, it follows that the inequalities either of (A) or (B) holds:

A. $|G_1| \leq 2$ and $|G_2| \geq 1$.

B. $|G_1| \geq 3$ and $|G_2| \leq 3$.

Suppose the inequalities in (A) hold. We consider two cases depending on $|G_1|$:

**CASE 1.** Assume $|G_1| = 1$. Then $G_1 = K_1$. Since $(K_1, G_2^*)$, $(K_1, 2G(e))$ and $(K_1, G(e_1, e_2))$ are forbidden subgraph-sequences of $(G_1, G_2)$, it follows that $(G_1, G_2)$ is isomorphic to $(K_1, G(e))$, and consequently $(K_1, G(e))$ is a solution of $cr(G_1 + G_2) = 1$.

**CASE 2.** Assume $|G_1| = 2$. Then $G_1 = \overline{K_2}$ or $K_2$. Since $(K_1, G_2^*)$, $(K_1, 2G(e))$ and $(K_1, G(e_1, e_2))$ are forbidden pairs in $(G_1, G_2)$, $G_2$ must be $G(e)$. However, $(\overline{K_2}, G(e))$ is a forbidden subgraph-sequence of $(G_1, G_2)$. It follows that $G_2$ must be outer-planar. Next, assume that $G_1 = \overline{K_2}$, and also $\Delta(G_2) \geq 3$. If $G_2$ contains a subgraph $H$ either isomorphic to a member of $\Psi_2$ as in Lemma 5.2.13 or homeomorphic to $K_{3,3} - C_4$ then $(\overline{K_2}, H)$ is a forbidden subgraph sequence of $(G_1, G_2)$. This is a contradiction. Therefore, $G_2$ has no subgraph $H$ either isomorphic to a member
of $\Psi_2$ or homeomorphic to $K_{3,3} - C_4$. In view of Lemma 5.2.13, 
$$cr(K_2 + G_2) = 1.$$ 

Since $K_2 \subseteq K_2, (K_2, G_2)$ with $G_2$ having subgraph isomorphic 
to $\lambda_2$ or homeomorphic to $K_{3,3} - C_4$ is also a forbidden subgraph. 
Further, if $G_2$ has a subgraph either isomorphic to $C_n \cdot K_2; n \geq 4$ 
or $G_T(K_1 + P_3)$, then the forbidden subgraphs $(K_2, G_T(K_1 + P_3))$ 
and $(K_2, C_n \cdot K_2); n \geq 4$ appear in $(G_1, G_2)$, which is a contradiction. 
Thus $G_2 = \alpha_i \cup mP_n$ for $m \geq 0$ and $n \geq 1$. Consequently, 
$$cr(K_2 + G_2) = 1.$$ 

**SUBCASE 2.2.** Assume $\Delta(G_2) \leq 2$. Since $(G_1, G_2)$ has none 
of the following as its isomorphic subgraph-sequence: $(\overline{K_2}, C_n \cup \overline{K_2}), (K_2, C_n \cup K_1)$ for $n \geq 4$ and $(\overline{K_2}, C_m); m \geq 3, (K_2, tP_s), (\overline{K_2}, tP_s)$, by Lemma 5.2.14, $(G_1, G_2) \in \xi_1$. Consequently, $cr(G_1 + G_2) = 1$.

Assume the inequalities in (B) hold. Let us split the first inequality: $|G_1| \geq 3$ as follows: $(|G_1| \geq 4$ or $|G_1| = 3)$ and $|G_2| \leq 3$.

Suppose $|G_1| \geq 4$ and $|G_2| \leq 3$. (i) If $|G_2| = 3$, then $(G_1, G_2)$ contains a forbidden subgraph-sequence $(\overline{K_4}, \overline{K_3})$, and hence no solution exist in this case. (ii) If $|G_2| \leq 2$, and since $|G_1| \geq 4$, then as usual replace $G_1$ by $G_2$ and $G_2$ by $G_1$, This can be trans-
formed to produce the inequalities in (A), and in this case all solutions \((G_1, G_2)\) of \(cr(G_1 + G_2) = 1\) are determined earlier. Since \(cr(G_1 + G_2) = cr(G_2 + G_1)\), the above solutions of \((G_1, G_2)\) of \(cr(G_1 + G_2) = 1\) are also the solutions of \((G_2, G_1)\) in this case.

Suppose \(|G_1| = 3\) and \(|G_2| \leq 3\). (i) If \(|G_2| = 3\), then \(G_i \in \Gamma_3\) for \(i=1\) or 2. Since \((P_3, K_3)\) and \((K_3, P_3)\) are the forbidden subgraph-sequences in \((G_1, G_2)\), the possible solutions are the members of \([\Gamma_3 \times \Gamma_3 - \{(P_3, K_3), (K_3, P_3), (K_3, K_3)\}]\).

(ii) If \(|G_2| \leq 2\), then the graph-transformation replace \(G_1\) by \(G_2\) and \(G_2\) by \(G_1\), produces the inequalities in (A). This completes the proof. ■

5.3. Crossings in Sequential Join Graphs

In this section, we establish a necessary and sufficient condition for a 3-sequential join graph to have crossing number one in terms of triples of graphs.

**THEOREM 5.3.1.** Let \(G_1, G_2\) and \(G_3\) be three disjoint graphs. Then \(cr(G_1 + G_2 + G_3) = 1\) if and only if \((G_1, G_2, G_3)\) is either \((A, B, C)\) or \((C, B, A)\) so that one of \((A, B, C)\) and \((C, B, A)\) is a triple in \(\Omega\).
where,
\[
\Omega = \{(G_0, K_1, G(e)), (K_1, Y, K_1), (Y, X, mP_n); m, n \geq 1.
\]
\[
(K_1, P, K_1), (K_3 \cup tP_n, X, mP_n); t \geq 0, m, n, k \geq 1,
\]
\[
(K_1, \overline{K_2}, C_p); p \geq 4, (K_1, Z, X),
\]
for \( X \in \Gamma_1, Y \in \Gamma_2, Z \in \Gamma_3, P \in \mathcal{Y} \)

and
\[
\mathcal{Y} = \{C_p \cdot K_2, C_p \cup K_1; p \geq 4, G_T(K_1 + P_3), K_3 \cup mP_n; m, n \geq 1\}.
\]

**Proof.** Suppose \( cr(G_1 + G_2 + G_3) = 1 \). Then \( G_1 + G_2 + G_3 \) is nonplanar. In view of Lemma 5.2.1, \( G_i \) for \( i = 1, 2, 3 \) is planar. Moreover since \( G_1 + G_2 + G_3 = (G_1 \cup G_3) + G_2 \) from Lemma 5.2.2, one of the following inequalities holds:

1) \( |G_2| \leq 2 \) and \( |G_1 \cup G_3| \geq 2 \).

2) \( |G_2| \geq 3 \), and \( |G_1 \cup G_3| \leq 3 \).

Suppose the inequalities in (1) hold. We discuss two cases depending on \( |G_2| \).

**CASE 1.** Suppose \( |G_2| = 1 \). Then \( G_2 = K_1 \). Clearly \( G_1 \cup G_3 \) is nonouterplanar, since otherwise if \( G_1 \cup G_3 = G_0 \), an outerplanar graph, then \( G_1 + G_2 + G_3 = (G_1 \cup G_3) + G_2 = G_0 + K_1 \). But in view of Theorem 5.2.12, \( cr(G + K_1) = 0 \). Further, \( G_1 \cup G_3 \) is minimally nonouterplanar, and is isomorphic to \( G(e) \cup G_0 \). Otherwise by Lemma 5.2.4 (or 5.2.5), \( cr(G_1 + G_2 + G_3) = cr((G_1 \cup G_3) + G_2) \geq 2 \).
Consequently, either \((G(e), K_1, G_0)\) or \((G_0, K_1, G(e))\) is the possible solution for \(\text{cr}(G_1 + G_2 + G_3) = 1\).

**CASE 2.** Suppose \(|G_2| = 2\). Then \(G_2\) is either \(\overline{K_2}\) or \(K_2\). In this case \(G_1 \cup G_3\) is outerplanar. Otherwise, by Lemma 5.2.4, 5.2.5 or 5.2.6, \(\text{cr}((G_1 \cup G_3) + G_2) \geq 2\).

Let us first assume that \(\Delta(G_1 \cup G_3) \geq 3\). We discuss two subcases depending on the structure of \(G_2\):

**SUBCASE 2.1.** Suppose \(G_2 = \overline{K_2}\). Since \(\text{cr}(G_1 + \overline{K_2} + G_3) = \text{cr}((G_1 \cup G_3) + \overline{K_2}) = 1\), from Lemma 5.2.13(b), \((G_1 \cup G_3) \in \Psi_1\). However, the graphs \(C_p \cdot K_2\) (\(p \geq 4\)), and \(G_T(K_1 + P_3)\) are all in \(\lambda_1\), and by Lemma 5.2.13(c) the crossing number of each of the graphs: \(((C_p \cdot K_2) \cup K_1) + \overline{K_2}\) and \((G_T(K_1 + P_3) \cup K_1) + \overline{K_2}\) can not be 1. Therefore, the possible solutions are: \((\alpha_i \cup mP_n, \overline{K_2}, tP_s), (tP_s, \overline{K_2}, \alpha_i \cup mP_n); 1 \leq i \leq 3, m \geq 0; t, n, s \geq 1,\) for which \(\text{cr}(G_1 + G_2 + G_3) = 1\).

**SUBCASE 2.2.** Suppose \(G_2 = K_2\). Now, we obtain the solutions of \(\text{cr}(G_1 + K_2 + G_3) = 1\). By Subcase 2.1, the solutions of \(\text{cr}(G_1 + (K_2 - e) + G_3) = 1\) are just the triples \((\alpha_i \cup mP_n, (K_2 - e), tP_s)\) and \((tP_s, (K_2 - e), \alpha_i \cup mP_n); 1 \leq i \leq 3, t, m \geq 0; n, s \geq 1\). Further, it is easy to check that the possible solutions of \(\text{cr}(G_1 + K_2 + G_3) = 1\) are the triples \((\alpha_i \cup mP_n, K_2, tP_s)\) and \((tP_s, K_2, \alpha_i \cup mP_n); 1 \leq i \leq 3, t, m \geq 0; n, s \geq 1\).
Finally, let us assume that $\Delta(G_1 \cup G_3) \leq 2$. Since $cr(G_1 + G_2 + G_3) = cr(G_2 + (G_1 \cup G_3)) = 1$, by Lemma 5.2.14, $(G_2, G_1 \cup G_3) \in \xi_1$. Thus, $(G_1, G_2, G_3)$ is isomorphic to one of the graphs: $(C_p, \overline{K}_2, K_1), (K_1, \overline{K}_2, C_p); p \geq 4, (mP_n, K_2, K_3 \cup tP_s)$ and $(K_3 \cup tP_s, K_2, mP_n); t \geq 0, m, n, s \geq 1$. But it is easy to check that these graphs are solutions of $cr(G_1 + G_2 + G_3) = 1$.

Finally, suppose the inequalities in (2) hold: We distinguish two cases depending on $|G_1 \cup G_3|:

**CASE 1.** Assume $|G_1 \cup G_3| = 2$. Then $G_1 = G_3 = K_1$. Now, $G_2$ is outerplanar; since otherwise by Lemma 5.2.4, 5.2.5 or 5.2.6, $cr((G_1 \cup G_3) + G_2) \geq 2$.

Let us first consider $\Delta(G_2) \geq 3$. Since $G_2$ is outerplanar, by Lemma 5.2.13 the solutions for $cr(\overline{K}_2 + G_2) = 1$ are all members of $\Psi_1$. The solutions of $cr(G_1 + G_2 + G_3) = 1$ are $(K_1, G_2, K_1)$, where $G_2 \in \Psi_1$. If $\Delta(G_2) \leq 2$, then since $|G_1 \cup G_3| = 2$, and $cr(G_1 + G_2 + G_3) = cr((G_1 \cup G_3) + G_2)) = 1$, in view of Lemma 5.2.14, $(G_1 \cup G_3, G_2) \in \xi_1$. Hence $(G_1, G_2, G_3)$ is isomorphic to one of the graphs: $(K_1, C_p \cup K_1, K_1); p \geq 4$ and $(K_1, K_3 \cup mP_n, K_1); m, n \geq 1$. It is easy to check that these triples are the solutions of $cr(G_1 + G_2 + G_3) = 1$. 
CASE 2. Assume $|G_1 \cup G_3| = 3$. Suppose $|G_2| \geq 4$. Then $\overline{K_4} \subseteq G_2$. Since $G_1 + G_2 + G_3 = (G_1 \cup G_3) + G_2 \supseteq \overline{K_3} + \overline{K_4} = K_{3,4}$, and $cr(K_{3,4}) = 2$, $cr(G_1 + G_2 + G_3) \geq 2$. Hence, $|G_2| \leq 3$. Again, for $|G_2| \leq 2$, $G_1 + G_2 + G_3 = (G_1 \cup G_3) + G_2 \subseteq (K_1 \cup K_2) + K_2$, and $cr((K_1 \cup K_2) + K_2) = 0$, a contradiction. Hence $|G_1 \cup G_3| = |G_2| = 3$. Since $(\overline{K_3}, \overline{K_3}) \subseteq (G_1 \cup G_3, G_2) \subseteq (K_1 \cup K_2, K_3)$, and $cr(\overline{K_3} + \overline{K_3}) = cr(K_{3,3}) = cr((K_1 \cup K_2) + K_3) = 1$, we get $cr(G_1 + G_2 + G_3) = 1$. This proves the necessity part.

Conversely, suppose $(G_1, G_2, G_3)$ is either $(A, B, C)$ or $(C, B, A)$ for each triple $(A, B, C) \in \Omega$, then it is easy to check that $cr(G_1 + G_2 + G_3) = 1$. This proves the sufficiency part. ■

Next, we deduce a characterization of a 3-sequential join graph to have crossing number 1, in terms of forbidden subgraph-sequences.

THEOREM 5.3.2. Let $G_1, G_2$ and $G_3$ be three disjoint graphs. Then $cr(G_1 + G_2 + G_3) = 1$ if and only if $(G_1, G_2, G_3)$ is free from either $(A, B, C)$ or $(C, B, A)$ so that one of $(A, B, C)$ and $(C, B, A)$ is a member of $\Phi_1 \cup \Phi_2$ or (i) $A \cup C$ is either isomorphic to a member of $\Phi_3$ or homeomorphic to $(K_{3,3} - C_4) \cup K_1$, and $B = \overline{K_2}$ or (ii) $A \cup C = \overline{K_2}$, and $B$ is either isomorphic to a member of $\Psi_2$ or homeomorphic to $K_{3,3} - C_4$. 

where,

\[ \Phi_1 = \{(mP_n, X, tP_s), (K_1, mP_n, K_1); m, n, t, s \geq 1, \]
\[ (G, K_1, H), (K_1, C_p, K_1); p \geq 3 \} \]

for \( X \in \Gamma_2 \) and \( G, H \) outerplanar graphs.

\[ \Phi_2 = \{(K_1, K_1, G^*_2), (K_1, K_1, G(e_1, e_2)), (K_1, K_1, 2G(e)), \]
\[ (K_1, \overline{K}_2, G(e)), (K_1, G(e), K_1), (G(e), K_1, G(e)), \]
\[ (K_1, \overline{K}_3, \overline{K}_3), (\overline{K}_2, \overline{K}_3, \overline{K}_2), (K_1, \overline{K}_4, \overline{K}_2), \]
\[ (K_1, K_2, C_p); p \geq 4 \}. \]

\[ \Phi_3 = \{K_{1,4} \cup K_1, 2K_{1,3}, 2K_3, C_p \cup \overline{K}_2; p \geq 4, K_3 \cup K_{1,3}, \]
\[ G_T(K_1 + P_3) \cup K_1, S(K_{1,3} \cup K_1) \} \]

\[ \Psi_2 = \{K_{1,4}, 2K_{1,3}, 2K_3, C_p \cup \overline{K}_2; p \geq 4, K_3 \cup K_{1,3}, \]
\[ G_T(K_1 + P_3) \cup K_1, S(K_{1,3}) \} \]

**Proof.** The proof follows from Theorem 5.2.12 and 5.2.16. Hence it is omitted.