CHAPTER 2

Strongly Regular Gamma - Near Rings

2.1 Introduction

In [35], Mason introduced the notion of strongly regular near-rings. After the appearance of [35], several papers appeared which improved the results and answered some open questions (cf.[36],[41],[43]). In this chapter we will use the notions of 0-prime and completely prime ideals to characterize strongly regular $\Gamma$-near rings.

In Section 2.2, we shall prove that if $N$ is arbitrary $\Gamma$-near ring then $P_0(N)$ (0-prime radical of $N$) coincides with the intersection of the 0-prime left ideals of $N$ and $P_c(N)$ (completely prime radical of $N$) coincides with the intersection of the completely prime left ideals.
of $N$. If $N$ is zero-symmetric $\Gamma$ -near ring, then $\mathcal{P}_\nu(N)$ ($\nu$ -prime radicals of $N$, $\nu = 2, 3$) coincides with the $\nu$- prime left ideals of $N$, for $\nu = 2, 3$. Section 2.3 deals with some basic properties of left weakly $\Gamma$ - near rings and in Section 2.4 we give some characterization of strongly regular $\Gamma$-near rings.

2.2 $\nu$- prime ideals

**Definition 2.2.1.** If $N$ is a $\Gamma$ -near ring and $I$ is an ideal of $N$, then $I$ is said to be $0 - (1-, 2-)$ prime if $A, B$ ideals (left ideals, left $\Gamma$-subgroups) of $N$, $A \Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. If $a, b \in N$, $a \Gamma N \Gamma b \subseteq I$ implies $a \in I$ or $b \in I$, then $I$ is called a $3$-prime ideal of $N$.

**Definition 2.2.2.** If $N$ is a $\Gamma$- ring and $I$ is an ideal of $N$, then $I$ is said to be $0 - (1-, 2-)$ semiprime if for any ideal $A$ (left ideal, left $\Gamma$- subgroup) of $N$, $A \Gamma A \subseteq I$ implies $A \subseteq I$. If $a \in N$, $a \Gamma N \Gamma a \subseteq I$ implies $a \in I$, then $I$ is called a $3$-semiprime ideal of $N$.

**Definition 2.2.3.** Let $N$ be a $\Gamma$ -near ring and $I$ be an left ideal of $N$. We define $I^* = \sum \{ A \triangleleft N/A \subseteq I \}$ that is $I^*$ is the largest two sided ideal contained in $I$.

**Lemma 2.2.4.** Let $N$ be a $\Gamma$ -near ring and $I$ be an left ideal of $N$.Then $(I : N) = \{m \in N/m \Gamma N \subseteq I \}$ is a two sided ideal of $N$. 
Proof. First we shall prove that \((I : N)\) is a right ideal of \(N\). Let \(x \in (I : N)\) and \(m \in N\). It is enough to prove that \(xam\Gamma N \subseteq I\). Since \(x \in (I : N)\), \(x\Gamma N \subseteq I\). Now \(xam\Gamma N \subseteq x\alpha N \subseteq I \forall \alpha \in \Gamma\). Hence \(xam \in (I : N)\). Thus \((I : N)\) is a right ideal. On the other hand, let \(a, b \in N\) and \(m \in (I : N)\). We shall prove that \(a\alpha (b + m) - a\alpha b \in (I : N)\). It is enough to prove that \([a\alpha (b + m) - a\alpha b] \beta N \subseteq I \forall \alpha, \beta \in \Gamma\).

Now

\[ [a\alpha (b + m) - a\alpha b] \beta N = a\alpha (b\beta N + m\beta N) - (a\alpha b) \beta N \]
\[ \subseteq a\alpha (b\beta N + I) - a\alpha (b\beta N) \]
\[ \subseteq I \quad \forall \alpha, \beta \in \Gamma. \]

Thus \((I : N)\) is a left ideal of \(N\) and consequently \((I : N)\) is a two sided ideal of \(N\).

Lemma 2.2.5. Let \(N\) be a \(\Gamma\)-near ring and \(I\) be a \(0\)-prime left ideal of \(N\). Then \(I^*\) is a \(0\)-prime ideal of \(N\) and \(I^* = (I : N)\).

Proof. Let \(A\) and \(B\) be any two ideals in \(N\) such that \(A, B \not\subseteq I^*\), By the definition, it follows that \(A, B \not\subseteq I\). Since \(I\) is \(0\)-prime, \(A\Gamma B \not\subseteq I\) and hence \(A\Gamma B \not\subseteq I^*\). Thus \(I^*\) is \(0\)-prime.

We now prove that \(I^* = (I : N)\), let \(a \in I^*, m \in N\). Then \(a\alpha m \in I^* \subseteq I \forall \alpha \in \Gamma\), so that \(a \in (I : N)\). Hence \(I^* \subseteq (I : N)\). Now
(I : N) is two sided ideal in N, so that (I : N) \Gamma N \subseteq I. Since I is 0-prime, (I : N) \subseteq I. Hence (I : N) \subseteq I^* and consequently I^* = (I : N).

**Theorem 2.2.6.** If N be a \(\Gamma\) -near ring, then \(\mathcal{P}_0(N) = \cap\{I/I is a 0-prime left ideal of N\}.

**Proof.** Let \(Q = \cap\{I/I is a 0-prime left ideal of N\}\). Since every 0-prime two sided ideal of N is a 0-prime left ideal, \(Q \subseteq \mathcal{P}_0(N)\).

Let I be 0-prime left ideal of N. By Lemma 2.2.5, \(I^*\) is a 0-prime ideal of N. Then \(\mathcal{P}_0(N) \subseteq I^* \subseteq I\). Taking intersection as I runs through the 0-prime left ideals of N, \(\mathcal{P}_0(N) \subseteq Q\). Thus \(\mathcal{P}_0(N) = Q\).

**Definition 2.2.7.** A left ideal \(I\) of a \(\Gamma\) -near ring N is said to be completely prime if for \(a, b \in N\) with \(a \alpha b \in I \forall \alpha \in \Gamma\) and \(b \notin I\) implies that \(a \in I^*\).

**Lemma 2.2.8.** Let N be a \(\Gamma\) -near ring and I is a completely prime left ideal of N, then I is a 0-prime left ideal of N.

**Proof.** Suppose I is completely prime ideal of N. We have to prove that I is 0-prime. Let A and B be any two ideals in N such that \(A \Gamma B \subseteq I\). If \(B \nsubseteq I\), then there exists \(b \in B\) such that \(b \notin I\). Since \(b \in B\), \(A \Gamma b \subseteq I\). Again since I is completely prime and \(b \notin I\), \(A \subseteq I^* \subseteq I\), so \(A \subseteq I\). Therefore I is 0-prime.
Lemma 2.2.9. Let $N$ be a $\Gamma$-near ring and $I$ a completely prime left ideal of $N$. Then $I^*$ is a completely prime ideal of $N$.

**Proof.** Suppose that $I$ is a completely prime left ideal of $N$. We shall prove that $I^*$ is a completely prime ideal. Let $a, b \in N$ such that $aob \in I^* \forall \alpha \in \Gamma$ and $b \notin I^*$. Then there exists $m \in N$ such that $b\beta m \notin I$. Since $aob \in I^* \forall \alpha \in \Gamma$, $(a\alpha b)\beta m \in I \forall \alpha \in \Gamma$. Now $I$ is completely prime and $b\beta m \notin I$, it follows that $a \in I^*$. Thus $I^*$ is a completely prime ideal of $N$.

Theorem 2.2.10. If $N$ is a $\Gamma$-near ring, then $P_c(N) = \cap \{I/I$ is a completely prime left ideal of $N\}$.

**Proof.** It follows from Lemma 2.2.9.

Lemma 2.2.11. Let $N$ be a zero symmetric $\Gamma$-near ring. If $I$ is a 2-prime (3-prime) left ideal of $N$, then $I^*$ is a 2-prime (3-prime) ideal of $N$.

**Proof.** Suppose that $I$ is a 2-prime left ideal of $N$. We shall prove that $I^*$ is a 2-prime ideal of $N$. Since $N$ is a zero-symmetric, $I$ is 0-prime and hence $I^* = (I : N)$ by Lemma 2.2.5. Let $A$ and $B$ be left $\Gamma$-subgroups of $N$ such that $A, B \not\subset I^*$. Then there exist $m, n \in N$ such that $A\alpha m, B\beta n \notin I$ for $\alpha, \beta \in \Gamma$. Since $A\alpha m, B\beta n$ are left $\Gamma$-subgroups of $N$ and $I$ is 2-prime, $A\alpha m\Gamma B\beta n \not\subset I$. Hence
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\[ A\alpha m \Gamma B \notin I^{*}, \text{ so that } A\alpha B \notin I^{*}. \text{ Consequently } I^{*} \text{ is 2-prime. Now suppose that } I \text{ is a 3-prime left ideal of } N. \text{ Again } I \text{ is a 0-prime and so } I^{*} = (I : N). \text{ To prove that } I^{*} \text{ is 3-prime, let } a, b \notin I^{*}. \text{ Then there exist } m, n \in N \text{ such that } a\alpha m, b\beta n \notin I. \text{ Since } I \text{ is 3-prime, there exist } r \in N \text{ such that } (a\alpha m) \gamma r \delta (b\beta n) \notin I, \text{ i.e., } (a\alpha m) \gamma r \delta b \notin I^{*}. \text{ Hence } I^{*} \text{ is 3-prime.} \]

Using an argument similar to that used in the proof of Theorem 2.2.6, we have the following theorem.

**Theorem 2.2.12.** Let \( N \) be a zero symmetric \( \Gamma \)-near ring. Then \( \mathcal{P}_{\nu}(N) = \cap \{I/I \text{ is a } \nu \text{-prime left ideal of } N\} \) for \( \nu = 2, 3 \).

### 2.3 Weakly regular \( \Gamma \)-near rings

In this section we give some properties of weakly regular \( \Gamma \)-near rings.

**Definition 2.3.1.** A \( \Gamma \)-near ring \( N \) is said to be **left** (respectively **right**) **weakly regular** if \( a \in \langle a \rangle \Gamma a \) (respectively \( a \in a\Gamma \langle a \rangle \)) for all \( a \in N \). \( N \) is said to be **weakly regular** if it is both left and right weakly regular.
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Definition 2.3.2. A $\Gamma$-near ring $N$ is said to be left (respectively right) pseudo $\pi$-regular if for every $x \in N, \gamma \in \Gamma$, there exists a natural number $n = n(x)$ such that $x^n = \gamma x \gamma \ldots \gamma x \in \langle x \rangle \Gamma x^n$ (respectively, $x^n = \gamma x \gamma \ldots \gamma x \in x^n \Gamma \langle x \rangle$).

Proposition 2.3.3. Let $N$ be a $\Gamma$-near ring, then

(i) $a^k \in \langle a \rangle \Gamma a^{k+1}$ for some integer $k$ if and only if the descending chain $\langle a \rangle \Gamma a \supseteq \langle a \rangle \Gamma a^2 \supseteq \ldots$ stabilizes after a finite number of steps;

(ii) If $N$ has the descending chain condition on left $\Gamma$ subgroups, then $N$ is left pseudo $\pi$-regular;

(iii) If $N$ is finite, then $N$ is left and right pseudo $\pi$-regular.

Proof. (i) Suppose that $a^k \in \langle a \rangle \Gamma a^{k+1}$.

Now

\begin{align*}
\langle a \rangle \Gamma a^k &\subseteq \langle a \rangle \Gamma (\langle a \rangle \Gamma a^{k+1}) \\
&= (\langle a \rangle \Gamma \langle a \rangle) \Gamma a^{k+1} \\
&\subseteq \langle a \rangle \Gamma a^{k+1} \\
&= \langle a \rangle \Gamma (a \Gamma a^k) \\
&= (\langle a \rangle \Gamma a) \Gamma a^k \\
&\subseteq \langle a \rangle \Gamma a^k.
\end{align*}
Hence $< a > \Gamma a^k = < a > \Gamma a^{k+1}$. Therefore the descending chain $< a > \Gamma a \supseteq < a > \Gamma a^2 \supseteq \cdots$ stabilizes after a finite number of steps.

Conversely, assume that $< a > \Gamma a^m = < a > \Gamma a^{m+1}$, then for each $\alpha \in \Gamma$, $a^{m+1} = a\alpha a^m \in < a > \Gamma a^m$ implies that $a^{m+1} \in < a > \Gamma a^{m+1}$ by assumption. Now

$$a^{m+1} \in < a > \Gamma a^{m+1}$$

$$= ( < a > \Gamma a^m ) \Gamma a$$

$$= ( < a > \Gamma a^{m+1} ) \Gamma a$$

$$= < a > \Gamma a^{m+2}.$$  

Thus, $a^{m+1} \in < a > \Gamma a^{m+2}$. Take $k = m + 1$. Hence $a^k \in < a > \Gamma a^{k+1}$.

(ii) Clearly $< a > \Gamma a^i$, $\forall i = 1, 2, \cdots$ are left $\Gamma$-subgroups and by hypothesis $< a > \Gamma a \supseteq < a > \Gamma a^2 \supseteq \cdots$ stabilizes after a finite number of steps. Hence from (i) for every $a \in N$,

$$a^k \in < a > \Gamma a^{k+1}$$

$$= < a > \Gamma ( a \Gamma a^k )$$

$$= ( < a > \Gamma a ) a^k$$

$$\subseteq < a > \Gamma a^k$$

i.e., $a^k \in < a > \Gamma a^k$. 
Hence $N$ is left pseudo $\pi-$ regular.

(iii) If $N$ is finite, then $< a > \Gamma a \supseteq < a > \Gamma a^2 \supseteq \cdots$ stabilizes after a finite number of steps. Therefore by (i) there exists a positive integer $k$ such that $a^k \in < a > \Gamma a^{k+1}$. Since $< a > \Gamma a^{k+1} \subseteq < a > \Gamma a^k$, $a^k \in < a > \Gamma a^k$. Thus $N$ is left pseudo $\pi-$ regular. Similarly $N$ is right pseudo $\pi-$ regular.

**Definition 2.3.4.** A $\Gamma-$ near ring $N$ is said to be **left quasi duo** if every maximal left ideal is a two sided ideal; and **strict left quasi duo** if every maximal left ideal is closed under right multiplication.

**Proposition 2.3.5.** If $N$ is a left quasi duo $\Gamma-$near ring with left unity $e$, and $k, n$ are natural numbers, then $a^n \in < a^k > \Gamma a^n$ if and only if $N = < a^k > + (0 : a^n) \forall a \in N$.

**Proof.** Let $a^n \in < a^k > \Gamma a^n$ for $a \in N$. Then

$$N\Gamma a^n \subseteq N \Gamma < a^k > \Gamma a^n$$

$$\subseteq < a^k > \Gamma a^n$$

$$\subseteq N\Gamma a^n .$$

Consequently,

$$N\Gamma a^n = < a^k > \Gamma a^n. \quad (*)$$

We claim $N = < a^k > + (0 : a^n) \forall a \in N$. If not, there exists a maximal left ideal $M$ such that $< a^k > + (0; a^n) \subseteq M$. Since $N$ is
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left quasi duo, $M$ is also a two sided ideal. Since $< a^k > \subseteq M$, we have

$$< a^k > \Gamma a^n \subseteq M \Gamma a^n \subseteq N \Gamma a^n = < a^k > \Gamma a^n.$$

Therefore $M \Gamma a^n = < a^k > \Gamma a^n$. Hence, there exists $x \in M$ such that $a^n = e \Gamma a^n = x \Gamma a^n$. From this, we have $(e - x) \Gamma a^n = 0$, and therefore, $(e - x) \in (0 : a^n) \subseteq M$. Hence $e = (e - x) + x \in M$. This is not possible.

Hence $N = < a^k > + (0 : a^n)$.

Conversely, suppose that $N = < a^k > + (0 : a^n) \forall a \in N$. We shall prove that there exist natural numbers $k$ and $n$ such that $a^n \in < a^k > \Gamma a^n$. Since $e \in N$, there exist $t < a^k >$ and $\ell \in (0 : a^n)$ such that $e = t + \ell$. Hence for each $\alpha \in \Gamma$, $a^n = e \alpha a^n = (t + \ell) \alpha a^n = t \alpha a^n + \ell \alpha a^n = t \alpha a^n \in < a^k > \Gamma a^n$.

\[\Box\]

Definition 2.3.6. A $\Gamma$– near ring $N$ is said to be left (resp. right) weakly $\pi$– regular if for every $x \in N$, $\alpha \in \Gamma$, there exists a positive integer $n$ such that

$$x^n = x \alpha x \alpha \cdots \alpha x \in < x^n > \Gamma x^n.$$

Corollary 2.3.7. If $N$ is a left quasi duo $\Gamma$– near ring with left unity then

(i) $N$ is left weakly $\pi$– regular, if and only if $N = < a^k > + (0 : a^k) \forall a \in N$ and some natural number $k$. 

(ii) \( N \) is left weakly regular if and only if \( N = \langle a \rangle + (0 : a) \forall a \in N \).

**Proof.** This is an easy consequence of Proposition 2.3.5.

**Definition 2.3.8.** A \( \Gamma \)-near ring \( N \) is said to be **strict left weakly regular** if \( a \in (N \Gamma a) \Gamma (N \Gamma a) \forall a \in N \).

**Definition 2.3.9.** A \( \Gamma \)-near ring \( N \) is said to be **strict left weakly \( \pi \)-regular** if \( a^n \in (N \Gamma a^n) \Gamma (N \Gamma a^n) \forall a \in N \).

**Proposition 2.3.10.** If \( N \) is a zero - symmetric and strict left quasi duo \( \Gamma \)-near ring with left unity \( e \), then

(i) \( N \) is strict left weakly regular if and only if \( N = N \Gamma a + (0 : a) \forall a \in N \).

(ii) \( N \) is strict left weakly \( \pi \)-regular if and only if \( N = N \Gamma a + (0 : a^n) \forall a \in N \) and some natural number \( n \).

**Proof.** (i) Suppose that \( N \) is strict left weakly regular and let \( a \in N \).

We have to prove that \( N = N \Gamma a + (0 : a) \). If not, there is a maximal left \( \Gamma \)-subgroup \( M \) of \( N \) such that \( N \Gamma a + (0 : a) \subseteq M \). Since \( N \) is strict left weakly regular, \( a \in (N \Gamma a) \Gamma (N \Gamma a) \). Hence \( a = x \Gamma a \) for some \( x \in N \Gamma a \Gamma N \). Since \( M \) is closed under multiplication from the right, \( N \Gamma a \Gamma N \subseteq M \) and consequently \( x \in M \). Since \( a = e \Gamma a \), it
follows that \(e - x \in (0 : a)\). Hence \(e = (e - x) + x \in M\). This is not possible and hence \(N = N\Gamma a + (0 : a)\).

Conversely, suppose that \(N = N\Gamma a + (0 : a)\) for every \(a \in N\). We shall prove that \(a \in (N\Gamma a) \Gamma (N\Gamma a)\). Now

\[
N = N\Gamma N = (N\Gamma a) \Gamma N + (0 : a) \Gamma N
\]

\[= N\Gamma a \Gamma N.
\]

Then \((N\Gamma a) \Gamma (N\Gamma a) = N\Gamma a\). Since \(N\) has left unity \(e\), \(a = e\alpha a \in N\Gamma a, \forall \alpha \in \Gamma\). Hence \(a \in (N\Gamma a) \Gamma (N\Gamma a)\).

(ii) Suppose that \(N\) is strict left weakly \(\pi\)-regular and \(a \in N\). We shall prove that \(N = N\Gamma a + (0 : a^n)\) where \(n\) is a natural number. If not, there is a maximal \(\Gamma\)-subgroup \(M\) of \(N\) such that \(N\Gamma a + (0 : a^n) \subseteq M\). By a similar argument as in (i), we can show that \(e \in M\) and consequently \(N = N\Gamma a + (0 : a^n)\).

Conversely, suppose that \(N = N\Gamma a + (0 : a^n)\) for every \(a \in N\) and some natural number \(n\). We have \(N\Gamma a^n = N\Gamma a^{n+1} \forall a \in N\). Let \(b \in N\) and \(b^n = x\Gamma b^{n+1}\) for some \(x \in N\). Now \(b^n = xab^nab = x\alpha (x\alpha b^{n+1}) ab = x^2\alpha b^nab^2 = \cdots = x^{n+1}\alpha b^nab^{n+1} \in N\Gamma b^n\Gamma b^n, \alpha \in \Gamma\), i.e., \(b^n \in N\Gamma b^n\Gamma N\Gamma b^n \forall b \in N\) and consequently \(N\) is strict left weakly \(\pi\)-regular.

Proposition 2.3.11. Let \(N\) be a \(\Gamma\)-near ring. If \(N\) is left weakly regular with left unity \(e\) and has the IFP, then \(N\) is simple if and only
if \( N \) is integral.

**Proof.** Suppose that \( N \) is an integral. We shall prove that \( N \) is simple. Let \( I \) be a non-zero ideal of \( N \). Then there exists an element \( x \neq 0 \in I \). Since \( N \) is left weakly regular, \( x \in \langle x \rangle \Gamma x \). Hence there exists \( t \in \langle x \rangle \) such that \( x = t\alpha x \ \forall \alpha \in \Gamma \). Since \( e \) is left identity, \( x = e\gamma x \ \forall \gamma \in \Gamma \). Therefore \( x = e\alpha x = t\alpha x \) and hence \( (e-t)\alpha x = 0 \) for all \( \alpha \in \Gamma \). Since \( x \in I \), \( \langle x \rangle \subseteq I \). Therefore \( e \in I \). Consequently \( I = N \). Hence \( N \) is simple.

Conversely, suppose that \( N \) is simple. We shall prove that \( N \) is an integral. Let \( a, b \in N \) such that \( a\Gamma b = 0 \). If \( a = 0 \) then we are done. Suppose that \( a \neq 0 \). Since \( a \in (0 : b) \) and \( N \) has IFP, \( (0 : b) \) is non-zero two sided ideal. Again since \( N \) is simple, \( (0 : b) = N \) and consequently \( b \in (0 : b) \), this implies that \( \langle b \rangle \subseteq (0 : b) \). Since \( N \) is left weakly regular, \( b \in \langle b \rangle \Gamma b \). But \( \langle b \rangle \Gamma b \subseteq (0 : b) \Gamma b = 0 \). Hence \( b = 0 \).

### 2.4 Strongly regular \( \Gamma \)-near rings

In this section we shall prove that the characterization of strongly regular \( \Gamma \)-near ring. Throughout this section \( N \) stands for zero symmetric \( \Gamma \)-near ring.

**Proposition 2.4.1.** \( N \) is left strongly regular if and only if it is regular
and has the IFP.

**Proof.** From the definition of left strongly regular it follows that \( N \) is regular. First we have to prove that \( N \) is reduced. Let \( a\gamma a = 0 \), for all \( \gamma \in \Gamma \). Since \( N \) is left strongly regular, there exists \( x \in N \) such that

\[
a = x\gamma a^2 = x\gamma 0 = 0 \quad \forall \gamma \in \Gamma.
\]

Now to prove that IFP holds, let \( a, b \in N \) such that \( a\gamma b = 0 \). Our claim is that \( a\gamma m\gamma b = 0 \) \( \forall m \in N \). Now

\[
(a\gamma m\gamma b)^2 = (a\gamma m\gamma b)\gamma (a\gamma m\gamma b)
\]

\[
= a\gamma m\gamma (b\gamma a)\gamma m\gamma b
\]

\[
= a\gamma m\gamma 0\gamma m\gamma b = 0
\]

Since \( N \) is reduced, \( a\gamma m\gamma b = 0 \). Hence IFP holds.

Conversely, suppose that \( N \) is regular and has the IFP. For any idempotent \( f \) of \( N \) and any \( a \in N, \gamma \in \Gamma \), we have

\[
(a - a\gamma f)\gamma f = a\gamma f - (a\gamma f)\gamma f
\]

\[
= a\gamma f - a\gamma (f\gamma f)
\]

\[
= a\gamma f - a\gamma f = 0
\]

Since \( N \) has the IFP, for any \( m \in N \), we have \((a - a\gamma f)\gamma m\gamma f = 0\). Then

\[
a\gamma m\gamma f = (a\gamma f)\gamma (m\gamma f).
\]
Since $x\gamma a, a\gamma x$ are idempotent, we have,

\[ x\gamma a = (x\gamma a) \gamma x\gamma a \]
\[ = [x\gamma (x\gamma a)] \gamma [a\gamma (x\gamma a)] \]
\[ = [(x\gamma x) \gamma a] \gamma a \]
\[ = (x\gamma x) \gamma (a\gamma a) = x^2\gamma a^2 \]

and

\[ a\gamma x = (a\gamma a) \gamma (x\gamma x) = a^2\gamma x^2. \quad (***) \]

Since $N$ is regular, there exists, $x \in N$ such that $a = a\gamma_1 x\gamma_2 a$ for every pair of non zero elements $\gamma_1$ and $\gamma_2$ in $\Gamma$. It follows from (***)

\[ a = a\gamma_1 x\gamma_2 a = a\gamma_1 x^2\gamma_2 a^2 = y\gamma_2 a^2, \quad \text{where} \quad y = a\gamma_1 x^2 \]

and

\[ a\gamma_1 y\gamma_2 a = a\gamma_1 a\gamma_1 x^2\gamma_2 a = a\gamma_1 x\gamma_2 a = a. \]

Thus $N$ is left strongly regular.

**Corollary 2.4.2.** $N$ is left strongly regular if and only if it is regular and reduced.

**Proof.** This is clear, since any reduced $\Gamma$-near ring has the IFR.

**Notation 2.4.3.** $O_p(N)$ ($C_p(N)$) denote the set of all 0-prime (respectively completely prime) left ideals of $N$. 

Lemma 2.4.4. If $N$ is left strongly regular $\Gamma$-near ring, then $O_p(N) \subseteq C_p(N)$.

Proof. Let $I \in O_p(N)$. We shall prove that $I$ is completely prime left ideal. Let $a \in I$ such that $a\alpha a \in I$, $\forall \alpha \in \Gamma$. Since $N$ is left strongly regular, there exists $x \in N$ such that $a = x\beta a\alpha a$, $\forall \alpha \neq 0, \beta \neq 0 \in \Gamma$. Since $I$ is left ideal, $a = x\beta a\alpha a, \in I$, $\forall \alpha \neq 0, \beta \neq 0 \in \Gamma$. Hence $I$ is completely semi-prime. Hence $O_p(N) \subseteq C_p(N)$. 

Definition 2.4.5. A left ideal $I$ of $N$ is said to be completely semiprime if $a\alpha a \in I, \forall \alpha \in \Gamma, a \in N$ implies that $a \in I^*$.

Theorem 2.4.6. The following statements are equivalent:

(i) $N$ is left strongly regular;

(ii) $N$ is 0-semiprime, $O_p(N) \subseteq C_p(N)$ and $\frac{N}{P}$ is regular for every completely prime ideal $P$ of $N$;

(iii) $a \in (a^2)_\Gamma, a = a\gamma_1 x\gamma_2 a$ for some $x \in N$, for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$ and for all $a \in N$;

(iv) Every $\Gamma$-subgroup of $N$ is completely semiprime and $a = a\gamma_1 x\gamma_2 a$ for some $x \in N$, for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$ and for all $a \in N$;

Proof. $(i) \Rightarrow (ii)$: Suppose that $N$ is left strongly regular. First we shall prove that $N$ is 0-semi prime. Let $A$ be an ideal in $N$ such that
$A \Gamma A = 0$. Our claim is that $A = 0$. Let $a \in A$. Since $N$ is left strongly regular, there exists $y \in N$ such that $a = ya^2a$, $\forall \alpha \neq 0 \in \Gamma$. Since $a^2 \in A \Gamma A = 0$, $a = 0$ and hence $A = 0$. Therefore $N$ is 0-semi prime.

Since $N$ is left strongly regular, from Lemma 2.4.4, $O_p(N) \subseteq C_p(N)$.

Now we shall prove that $N$ is regular. Let $x + p \in \frac{N}{P}$, where $x \in N$. Since $N$ is regular, there exists $a \in N$ such that $x = x\alpha_1a\alpha_2x$ for every pair of non-zero elements $\alpha_1$ and $\alpha_2$ in $\Gamma$. Now

$$x + P = x\alpha_1a\alpha_2x + P$$

$$= (x\alpha_1a + P)\alpha_2 (x + P)$$

$$= (x + P)\alpha_1 (a + P)\alpha_2 (x + P)$$

Hence $\frac{N}{P}$ is regular.

$(ii) \Rightarrow (i)$ : First we shall prove that $N$ is reduced. Let $a \in N$ such that $a^2 = 0$. Let $Q$ be any 0-prime ideal of $N$. Since $O_p(N) \subseteq C_p(N), Q$ is completely prime ideal and hence $Q$ is completely semiprime. Since $a^2 = 0 \in Q$, it follows that $a \in Q$. Hence $a \in P_0(N)$. Since $N$ is 0-semi prime, $P_0(N) = (0)$. Thus $N$ is reduced. Now we shall prove that $N$ is regular. Let $a \in N$ and $P$ be any completely prime ideal of $N$. Then by our assumption $\frac{N}{P}$ is regular. Since $\frac{N}{P}$ is regular, there exists $x + P \in \frac{N}{P}$ such that
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$a + P = (a + P)\gamma_1(x + P)\gamma_2(a + P)$ for every pair of non-zero elements $\gamma_1$ and $\gamma_2$ in $\Gamma$. Therefore $a - a\gamma_1 x\gamma_2 a + P = P$, i.e., $a - a\gamma_1 x\gamma_2 a \in P$. Hence $a - a\gamma_1 x\gamma_2 a \in \mathcal{P}_c(N)$. Since $N$ is reduced, $(0)$ is completely semiprime ideal in $N$. Hence $\mathcal{P}_c(N) = (0)$ and consequently $a = a\gamma_1 x\gamma_2 a$. Thus $N$ is left strongly regular.

$(i) \Rightarrow (iii)$: Obvious.

$(iii) \Rightarrow (i)$: Suppose that $a \in < a^2 >_\Gamma$ for every $a \in N$. It follows that $< a >_\Gamma = < a^2 >_\Gamma = N\Gamma a = N\Gamma a^2$. Hence $a \in N\Gamma a^2$. Then there exists $x \in N$ such that $a = a\gamma_1 x\gamma_2 a$ for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$ and for all $a \in N$. Thus $N$ is left strongly regular.

$(i) \Rightarrow (iv)$: Suppose that $N$ is left strongly regular. We shall prove that every $\Gamma-$ subgroup of $N$ is completely semiprime. Let $P$ be any $\Gamma-$ subgroup of $N$ and suppose that $a \in N$ such that $a\alpha a \in P$, $\forall \alpha \in \Gamma$. Since $N$ is left strongly regular, $P$ is a two sided ideal and there exist $x \in N$ such that $a = x\gamma_1 a\gamma_2 a$, for every pair of non-zero elements $\gamma_1$ and $\gamma_2$ in $\Gamma$. Since $P$ is a two sided ideal,

$$a = x\gamma_1 a\gamma_2 a \in P = P^*.$$ 

Hence $P$ is completely semiprime.

$(iv) \Rightarrow (i)$: Suppose that every $\Gamma-$ subgroup of $N$ is completely semiprime. We shall prove that $N$ is left strongly regular. Let $a \in N$. 

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Since \( a^2 \in < a^2 >_\Gamma \) and \( < a^2 >_\Gamma \) is completely semiprime, we have 
\( a \in < a^2 >_{\Gamma} \subseteq < a^2 >_{\Gamma} \). It is now easy to show that 
\[ < a >_{\Gamma} = < a^2 >_{\Gamma} = N\Gamma a = N\Gamma a^2. \]

Hence \( a \in N\Gamma a^2 \). Then there exists \( x \in N \) such that 
\( a = x\alpha a\beta a, \forall \alpha \neq 0, \beta \neq 0 \in \Gamma \). Thus \( N \) is left strongly regular.

**Definition 2.4.7.** For a \( \Gamma \)-near ring \( N \), \( N_c \) denotes the constant part of \( N \), that is 
\[ N_c = \{ a \in N/a\gamma 0 = a \text{ for all } \gamma \in \Gamma \} \]. \( N \) is called **strongly reduced** if \( a \in N \) and \( \gamma \in \Gamma, a\gamma a \in N_c \) implies \( a \in N_c \).

Obviously \( N \) is strongly reduced if and only if for \( a \in N \) and any positive integer \( n, a^n \in N_c \) implies \( a \in N_c \).

\( N \) is said to be **left** (resp. right) **strongly \( \pi \)-regular** if for each \( a \in N \) and \( \gamma \in \Gamma \), there exists a positive integer \( n = n(a) \) and an element \( x \in N \) such that 
\[ a^n = x\gamma a^{n+1} \text{ (resp., } a^n = a^{n+1}\gamma x) \], equivalently 
\[ a^n = y\gamma a^{2n} \text{ (resp., } a^n = a^{2n}\gamma y) \] for some \( y \in N \). \( N \) is said to be **strongly \( \pi \)-regular** if it is both left and right strongly \( \pi \)-regular.

An element \( a \) of a \( \Gamma \)-near ring \( N \) is called **\( C_N \)-regular** if \( a \in \langle a\gamma a > \) for all \( \gamma \in \Gamma \). Hence, \( N \) will be a \( C \)-regular \( \Gamma \)-near ring if 
\( a \in \langle a\gamma a > \) for all \( a \in N \) and \( \gamma \in \Gamma \). \( N \) is said to be **\( s \)-weakly regular** if for each \( a \in N, a \in \langle a\gamma a > \Gamma a \).

**Lemma 2.4.8.** Let \( N \) be a \( \Gamma \)-near ring. Then \( N \) is strongly reduced if and only if \( N \) is reduced.
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Proof. If \( a\gamma a \in N_c \) for all \( \gamma \in \Gamma \), then \( a\gamma a = a\gamma a\gamma 0 \). Since \( N \) is reduced, \( a\gamma a = 0 \) implies \( a = 0 \). By the zero symmetry of \( N \), \( a = 0 = a\gamma 0 = a\gamma a\gamma 0 = a\gamma a \in N_c \). Conversely, assume that \( a\gamma a = 0 \) for all \( \gamma \in \Gamma \). Then \( a\gamma a \in N_c \), hence \( a \in N_c \). Therefore \( a = a\gamma 0 = a\gamma 0\gamma a = a\gamma a \).

Proposition 2.4.9. Let \( N \) be a Γ'-near ring. Then we have the following properties.

(i) If \( N \) is s-weakly regular, then \( N \) is strongly reduced.

(ii) If \( N \) is C-regular, then \( N \) is strongly reduced. In particular, left or right strongly regular Γ'-near ring are strongly reduced.

(iii) Every integral Γ'-near ring \( N \) is strongly reduced.

Proof. (i). Suppose \( a \in N \) such that \( a^2 (= a\Gamma a) = 0 \). We have \( a = x\gamma a \) for some \( x \in a^2 \) and for all \( \gamma \in \Gamma \) so that \( a = 0 \). Thus \( a^2 = 0 \) implies \( a = 0 \) for every \( a \in N \). By Lemma 2.4.8, \( N \) is strongly reduced.

(ii) Suppose \( a \in a\gamma a \) for all \( a \in N \) and \( \gamma \in \Gamma \). If \( a\gamma a \in N_c \), then \( a \in a\gamma a \subseteq N_c \).

(iii) Let \( a \in N \) with \( a\gamma a \in N_c \) for all \( \gamma \in \Gamma \). Then \( (a - a\gamma a)\gamma a = a\gamma a - a\gamma a\gamma 0\gamma a = a\gamma a - a\gamma a = 0 \) and hence \( a = a\gamma a \in N_c \).
Proposition 2.4.10. Let $N$ be a strongly reduced $\Gamma$-near ring. Then we have the following properties:

(i) $N$ has the IFP;

(ii) If $N$ is simple then it is integral;

(iii) If $a\gamma b^n \in N_c$ for any positive integer $n$ and $\gamma \in \Gamma$, then

\[ \{a\Gamma b, b\Gamma a\} \cup a\Gamma N\Gamma b \cup b\Gamma N\Gamma a \subseteq N_c. \]

In particular, $a\Gamma b \in N_c$ implies $b\Gamma a \in N_c, a\Gamma N\Gamma b \subseteq N_c$ and $b\Gamma N\Gamma a \subseteq N_c$.

Proof. (i) If $a\gamma b = 0$ for $a, b \in N$ and $\gamma \in \Gamma$, then $(b\gamma a)^2 = (b\gamma a) \gamma (b\gamma a) = b\gamma 0$ since $N$ is zero symmetric. By Lemma 2.4.8, $b\gamma a = 0$. Now for $\gamma_1, \gamma_2 \in \Gamma$ and $x \in N, (a\gamma_1 x\gamma_2 b)^2 = (a\gamma_1 x\gamma_2 b) \gamma (a\gamma_1 x\gamma_2 b) = (a\gamma_1 x) \gamma_2 0 \gamma_1 x\gamma_2 b = a\gamma_1 x\gamma_2 0 = 0$. This implies that $a\gamma_1 x\gamma_2 b = 0$.

(ii) Assume that $N$ is simple. Let $a, b \in N$ such that $a\gamma b = 0$ for $\gamma \in \Gamma$. If $a = 0$, then we are done. Suppose $a \neq 0$. By (1), $(0 : b)_\gamma = \{x \in N/x\gamma b = 0\}$ is a two sided ideal. Now $0 \neq a \in (0 : b)_\gamma$. Since $N$ is simple, we have $(0 : b)_\gamma = N$ and consequently $b\gamma b = 0$. From Lemma 2.4.8, it follows that $b = 0$.

(iii) First, suppose $a\gamma b \in N_c$ for all $\gamma \in \Gamma$. Then $(b\gamma a)^2 = b\gamma a\gamma b\gamma a = b\gamma a\gamma b\gamma 0\gamma a = b\gamma a\gamma b\gamma 0 \in N_c$. Since $N$ is strongly reduced, we have $b\gamma a \in N_c$. Then $x\gamma b\gamma a \in N_c$ for each $x \in N$, hence
\[(a\gamma x\gamma b)^2 = a\gamma x\gamma b a\gamma x\gamma b = a\gamma x\gamma b a\gamma 0 \gamma x\gamma b = a\gamma x\gamma b a\gamma 0 = a\gamma x\gamma b a \in N_c.\]

Therefore \[a\gamma x\gamma b \in N_c\] for each \(x \in N\). Since \(b\gamma a \in N_c, b\Gamma N\gamma a \subseteq N_c.\]

Now suppose \(a\gamma b^n \in N_c.\) Then \((a\gamma b)^n \in N_c\) by the above argument. Since \(N\) is strongly reduced, \(a\gamma b \in N_c\). Hence by the first paragraph, the claim is proved. \(\blacksquare\)

**Definition 2.4.11.** A \(\Gamma\)-near ring \(N\) is called a weakly left (resp., right) duo if for every \(a \in N\) there is a positive integer \(n = n(a)\) such that \(N\Gamma a^n\) (resp., \(a^n\Gamma N\)) is an ideal of \(N\).

**Proposition 2.4.12.** Let \(N\) be a weakly left duo and strict left weakly \(\pi\)-regular. Then \(N\) is left strongly \(\pi\)-regular.

**Proof.** Let \(a \in N\). There exist positive integers \(m\) and \(n\) such that \(N\Gamma a^n = N\Gamma a^n\Gamma N\) and \(N\Gamma a^m = N\Gamma a^m\Gamma N\Gamma a^m.\) Observe that

\[N\Gamma a^{2n} = N\Gamma a^n \Gamma a^n = N\Gamma a^n \Gamma N \Gamma a^n = N\Gamma a^n \Gamma N \Gamma a^n \Gamma N = N\Gamma a^n \Gamma a^n \Gamma N = N\Gamma a^{2n} \Gamma N.\]

An induction argument yields \(N\Gamma a^{kn} = N\Gamma a^{kn} \Gamma N\) for any positive integer \(k.\) Also \(N\Gamma a^{2m} = (N\Gamma a^m) \Gamma a^m = (N\Gamma a^m \Gamma N \Gamma a^m) \Gamma a^m = N\Gamma a^m \Gamma a^{2m}.\) Again an induction argument yields \(N\Gamma a^{km} = N\Gamma a^m \Gamma N \Gamma a^{km}\) for any positive integer \(k.\)
Now using the above observations, we have that

\[ N\Gamma a^{mn}\Gamma N\Gamma a^{mn} = N\Gamma a^{mn}\Gamma a^{mn} = N\Gamma a^{2m}. \]

Also we have that

\[
\begin{align*}
N\Gamma a^{mn}\Gamma N\Gamma a^{mn} &= N\Gamma a^{mn}\Gamma (N\Gamma a^{m}\Gamma N\Gamma a^{mn}) \\
&= N\Gamma a^{mn}\Gamma a^{m}\Gamma N\Gamma a^{mn} \\
&= N\Gamma a^{mn+m}\Gamma N\Gamma a^{mn} \\
&= N\Gamma a^{mn+m}\Gamma (N\Gamma a^{m}\Gamma N\Gamma a^{mn}) \\
&= \cdots \\
&= N\Gamma a^{mn+2m}\Gamma N\Gamma a^{mn} \\
&= \cdots \\
&= N\Gamma a^{mn+mn}\Gamma N\Gamma a^{mn} \\
&= (N\Gamma a^{mn}\Gamma N\Gamma a^{mn})\Gamma N\Gamma a^{mn} \\
&= (N\Gamma a^{mn}\Gamma N\Gamma a^{mn}\Gamma N\Gamma a^{mn})\Gamma N\Gamma a^{mn} \\
&= (N\Gamma a^{mn}\Gamma N\Gamma a^{mn})\Gamma (N\Gamma a^{mn}\Gamma N\Gamma a^{mn}) \\
&= N\Gamma a^{2mn}\Gamma N\Gamma a^{2mn} \\
&= N\Gamma a^{4mn} \subseteq N\Gamma a^{2mn+1} \subseteq N\Gamma a^{2m}.
\end{align*}
\]

Hence \( N\Gamma a^{2mn} = N\Gamma a^{2mn+1} \). Therefore \( N \) is left strongly \( \pi \)-regular \( \Gamma \)-near ring.

Since left strongly regular \( \Gamma \)-near ring are strict left weakly \( \pi \)-regular, we have the following corollary.
Corollary 2.4.13. Let $N$ be weakly left duo $\Gamma$-near ring. Then the following statements are equivalent:

(i) $N$ is strict left weakly $\pi$-regular;

(ii) $N$ is left strongly $\pi$-regular.

Theorem 2.4.14. Let $N$ be a strongly reduced $\Gamma$-near ring. Then the following statements are equivalent:

(i) $N$ is left pseudo $\pi$-regular;

(ii) $N$ is weakly regular.

Proof.

(i) $\Rightarrow$ (ii): Assume $N$ is left pseudo $\pi$ regular. Let $a \in N$. Then there exists $x \in <a>$ and a natural number $n$ such that $a^{n+1} = x\gamma a^{n+1}$ for all $\gamma \in \Gamma$. We shall show that $a = x\gamma a = a\gamma x$ for some $x \in <a>$ and for all $\gamma \in \Gamma$. If $n = 0$, then immediately $a = x\gamma a$. Now $(a - a\gamma x)\gamma a = a\gamma a - a\gamma x\gamma a = a\gamma a - a\gamma \gamma = 0 \in N_c$. Hence $(a - a\gamma x)^2 = (a - a\gamma x)\gamma (a - a\gamma x) = a\gamma (a - a\gamma x) - a\gamma x\gamma (a - a\gamma x) \in N_c$ by Property (iii) of Proposition 2.4.10, and so $(a - a\gamma x) \in N_c$. Therefore $a - a\gamma x = (a - a\gamma x)\gamma 0 = (a - a\gamma x)\gamma 0\gamma a = (a - a\gamma x)\gamma a = 0$.

If $n \geq 1$, $(a - x\gamma a)\gamma a^n = 0$. Hence $(a - x\gamma a)\gamma a \in N_c$ by Property (iii) of Proposition 2.4.10. Hence, $(a - x\gamma a)\gamma a = $
\[(a - x\gamma a) \gamma a\gamma 0\gamma a^{n-1} = (a - x\gamma a) \gamma a^n = 0\] and so \[(a - a\gamma x)^2 \in N_c\]
by Property (iii) of Proposition 2.4.10. Since \(N\) is strongly reduced, \(a - x\gamma a \in N_c\). Then \(a - x\gamma a = (a - x\gamma a) \gamma 0 = (a - x\gamma a) \gamma 0\gamma a = (a - x\gamma a) \gamma a = 0\), that is \(a = x\gamma a\). Obviously as above \(a = a\gamma x\) for some \(x \in \langle a \rangle\) . Hence \(N\) is weakly regular.

(ii) \(\Rightarrow\)(i): Clearly if \(N\) is left weakly regular, then \(N\) is left pseudo \(\pi\)-regular.

\textbf{Definition 2.4.15.} For any subset \(A\) of a \(\Gamma\)-near ring \(N\), we write
\[\sqrt{A} = \{a \in N/a^n (= a\Gamma a\Gamma \cdots a\Gamma a) \in A \text{ for some } n \geq 1\} .\] Clearly \(A \subseteq \sqrt{A}\).

Here we give some equivalent characterizations of left strongly regular \(\Gamma\)-near ring.

\textbf{Theorem 2.4.16.} Let \(N\) be a \(\Gamma\)-near ring. Then the following statements are equivalent:

(i) \(N\) is left strongly regular;

(ii) \(A = \sqrt{A}\) for every \(\Gamma\)-subgroup \(A\) of \(N\);

(iii) \(a = \langle a^2 \rangle_\Gamma\) (the \(\Gamma\)-subgroup generated by \(a^2 \in N\)) for every \(a \in N\);

(iv) \(N\) is strongly reduced and left strongly \(\pi\)-regular.
**Proof.** (i) ⇒(ii): Let \( A \) be a \( \Gamma \)- subgroup of \( N \). If \( a \in \sqrt{A} \), then \( a^n \in A \) for some \( n \geq 1 \). Since \( N \) is left strongly regular, \( a = x \gamma a^2 = x^2 \gamma a^2 = \cdots = x^{n-1} \gamma a^n \in A \) for all \( \gamma \in \Gamma \). Thus \( A = \sqrt{A} \).

(ii) ⇒(iii): Let \( 0 \neq a \in N \). Now \( a^3 \in N \Gamma a^2 \) so that \( a \in \sqrt{N \Gamma a^2} = N \Gamma a^2 \). Thus \( N \) is left strongly regular.

(i) ⇒(iii): Obvious.

(iii) ⇒(i): Suppose \( a \in < a^2 >_\Gamma \) for every \( a \in N \). Since \( a \in < a^2 >_\Gamma \), it is easy to see that \( < a >_\Gamma = < a^2 >_\Gamma = N \Gamma a = N \Gamma a^2 \). Hence \( a \in N \Gamma a^2 \) and hence \( N \) is left strongly regular.

(i) ⇒(iv): It is follows from the Property (ii) of Proposition 2.4.9.

(iv) ⇒(i): Assume that \( a^n = x \gamma a^{n+1} \) for some \( n \geq 1 \) and for all \( \gamma \in \Gamma \). By Theorem 2.4.14 it follows that \( a = x \gamma a^2 \). Thus \( N \) is left strongly regular.

**Definition 2.4.17.** A \( \Gamma \)- near ring \( N \) is said to be a \((P_0) \)- \( \Gamma \)- near ring if for each \( a \in N \), there exists an integer \( n > 1 \) such that \( a = a^n (= a \Gamma a \Gamma \cdots a \Gamma a) \). Obviously a \((P_0)\)-\( \Gamma \)- near ring is strongly reduced.

The following result is an immediate consequence of Theorem 2.4.16.

**Corollary 2.4.18.** Let \( N \) be a finite \( \Gamma \)-near ring. Then the following statements are equivalent: