CHAPTER 4

Strongly Prime Gamma - Near Rings

4.1 Introduction

Strongly prime rings were introduced by Handelmann and Lawrence [25] and in [24] Groenewald and Heyman investigated the upper radical determined by the class of all strongly prime rings. In [22], Groenewald introduced the concept of strongly prime to near-rings and in [18], G.L. Booth, N.J. Groenewald and S. Veldsman introduced the concept of equiprime near-rings.

In this chapter we extend the concepts of strongly prime and equiprime to $\Gamma$ - near rings. In the second section we give some
characterizations of strongly prime \( \Gamma \)- near rings. In the third section we show that the strongly prime radical \( \mathcal{P}_s(N) \) of \( N \) coincides with \( \mathcal{P}_s(L)^+ \) where \( \mathcal{P}_s(L) \) is strongly prime radical of the left operator near-ring \( L \) of \( N \). Finally in the last section we shall prove that the equiprime radical \( \mathcal{P}_e(N) \) of \( N \) coincides with \( \mathcal{P}_e(L)^+ \) where \( \mathcal{P}_e(L) \) is the equiprime radical of the left operator near-ring \( L \) of \( N \).

### 4.2 Strongly prime \( \Gamma \)- near rings

In this section we shall prove some equivalent conditions for strongly prime \( \Gamma \)- near rings.

**Definition 4.2.1.** Let \( N \) be a \( \Gamma \)- near ring, then the right \( \alpha \)- annihilator of a subset \( A \) of \( N \) is \( r_{\alpha}(A) = \{x \in N / A\alpha x = 0\} \).

**Definition 4.2.2.** A \( \Gamma \)- near ring \( N \) is said to be strongly prime if for each \( a \neq 0 \in N \), there exists a finite subset \( F \) of \( N \) such that \( r_{\alpha}(a\Gamma F) = 0 \ \forall \alpha \in \Gamma \). \( F \) is called an insulator for \( a \) in \( N \).

**Lemma 4.2.3.** If a \( \Gamma \)- near ring \( N \) is strongly prime, then \( N \) is prime.

**Proof.** Let \( 0 \neq A, B \triangleleft N \). We shall show that \( A\Gamma B \neq 0 \). Since \( A \neq 0 \) there exists a finite subset \( F \) of \( A \) such that \( r_{\alpha}(F) = 0 \), for each \( \alpha \in \Gamma \). Hence for each \( 0 \neq b \in B \) we have \( F\Gamma b \neq 0 \). Therefore \( A\Gamma B \neq 0 \). \( \blacksquare \)
Definition 4.2.4. A $\Gamma$-near ring $N$ is said to be left(right) weakly semiprime if $[x, \Gamma] \neq 0 ([\Gamma, x] \neq 0) \ \forall x \neq 0 \in N$.

$N$ is said to be weakly semiprime if it is both left and right weakly semiprime.

Proposition 4.2.5. If $N$ is strongly prime $\Gamma$-near ring, then $N$ is weakly semiprime $\Gamma$-near ring.

Proof. Suppose that $N$ is a strongly prime $\Gamma$-near ring. We shall prove that $N$ is a weakly semiprime $\Gamma$-near ring. Let $x \neq 0 \in N$. It is enough to prove that $[x, \Gamma] \neq 0$ and $[\Gamma, x] \neq 0$. Suppose that $[x, \Gamma] = 0$. Since $N$ is a strongly prime $\Gamma$-near ring, for every $\beta \in \Gamma$ there exists a finite subset $S_\beta(x)$ such that for $b \in N, \{x\beta c\alpha b/c \in S_\beta(x)\} = 0, \forall \alpha \in \Gamma$ implies that $b = 0$. Now $x\beta c\alpha x = [x, \beta] c\alpha x = 0c\alpha x = 0, \forall \beta, c \in \Gamma, c \in S_\beta(x)$. Hence $x = 0$, a contradiction. Thus $N$ is a weakly semiprime $\Gamma$-near ring.

Proposition 4.2.6. If a $\Gamma$-near ring $N$ is strongly prime then, the left operator near-ring $L$ is strongly prime.

Proof. Let $\sum_i [x_i, \alpha_i] \neq 0 \in L$, then there exists $x \in N$ such that $\sum_i [x_i, \alpha_i] x \neq 0$, i.e., $\sum_i x_i \alpha_i x \neq 0$. Since $N$ is strongly prime, there exists a finite subset $F = \{a_1, a_2, \ldots, a_n\} (\text{say})$ such that for any
Fix $\alpha, \beta \in \Gamma$. Consider $G = \{[x_1, \alpha_1, \beta], \ldots, [x_n, \alpha_n, \beta]\}$. Our claim is that $G$ is an insulator for $\sum_i [x_i, \alpha_i]$. Let $\sum_j [y_j, \beta_j] \in L$ such that $\sum_i [x_i, \alpha_i] \sum_j [y_j, \beta_j] = 0$. We shall prove that $\sum_j [y_j, \beta_j] = 0$.

Now
\[ \sum_i [x_i, \alpha_i] G \sum_j [y_j, \beta_j] = 0 \]

implies
\[ \sum_i [x_i, \alpha_i] [x_\alpha k, \beta] \sum_j [y_j, \beta_j] = 0 \quad \forall k = 1, 2, \ldots, n. \]

Hence
\[ \left( \sum_i [x_i, \alpha_i] [x_\alpha k, \beta] \sum_j [y_j, \beta_j] \right) z = 0 \quad \forall z \in N; \quad k = 1, 2, \ldots, n. \]

This implies that
\[ \sum_i [x_i, \alpha_i] [x_\alpha k, \beta] \sum_j [y_j, \beta_j] z = 0 \quad \forall z \in N; \quad k = 1, 2, \ldots, n. \]

Hence
\[ \sum_i x_i \alpha_i x_\alpha k \beta \sum_j y_j \beta_j z = 0 \quad \forall z \in N; \quad k = 1, 2, \ldots, n. \]

By (4.2.1), $\sum_j y_j \beta_j z = 0 \quad \forall z \in N$. Therefore $\sum_j [y_j, \beta_j] = 0$. Thus $L$ is strongly prime.
Theorem 4.2.7. Let \( N \) be a left weakly semiprime \( \Gamma - \) near ring having no zero divisor, then \( N \) is strongly prime if and only if \( L \) is strongly prime.

Proof. Suppose that \( L \) is strongly prime. To prove \( N \) is strongly prime, let \( a \neq 0 \in N \). Since \( N \) is left weakly semiprime, \( [x, \Gamma] \neq 0 \) and since \( L \) is strongly prime, there exists a finite subset \( F = \left\{ \sum_{j=1}^{n} [y_{jk}, \beta_{jk}] / k = 1, 2, \ldots m \right\} \) (say) such that for any \( \sum_{\ell} [z_{\ell}, \delta_{\ell}] \in L \):

\[
[x, \Gamma] F \sum_{\ell} [z_{\ell}, \delta_{\ell}] = 0 \implies \sum_{\ell} [z_{\ell}, \delta_{\ell}] = 0 \quad (4.2.2)
\]

Consider \( F' = \left\{ y_{jk} \beta_{jk} x / j = 1, 2, \ldots, n; k = 1, 2, \ldots m \right\} \). Our claim is that \( F' \) is an insulator for \( x \). Let \( y \in N \) such that \( x \Gamma F' \Gamma y = 0 \). We shall prove that \( y = 0 \). Now \( x \Gamma F' \Gamma y = 0 \) implies \( x \alpha y_{jk} \beta_{jk} x \beta y = 0 \) \( \forall j = 1, 2, \ldots n; k = 1, 2, \ldots m \), for all \( \alpha, \beta \in \Gamma \). Therefore

\[
[x, \Gamma] [x \alpha y_{jk} \beta_{jk} x \beta y, \Gamma] = 0 \quad \forall j = 1, 2, \ldots n; k = 1, 2, \ldots m.
\]

Hence

\[
[x, \alpha] [y_{jk}, \beta_{jk}] [x \beta y, \Gamma] = 0 \quad \forall k = 1, 2, \ldots m.
\]

By (4.2.2), \( [x \beta y, \Gamma] = 0 \). Therefore \( x \beta y = 0 \). Since \( N \) is weakly semiprime and \( N \) has no zero divisor, \( y = 0 \) and consequently \( F' \) is an insulator for \( x \). Therefore \( N \) is strongly prime.

Converse part follows from Proposition 4.2.6. \( \blacksquare \)
We recall that for $X \subseteq N, \langle X \rangle$ is constructed by the following recursive rules

(i) $a \in \langle X \rangle \ \forall a \in X$.

(ii) If $b, c \in \langle X \rangle$, then $b + c \in \langle X \rangle$

(iii) If $b \in \langle X \rangle$ and $x, y \in N, \alpha \in \Gamma$, then $x \alpha (b + y) - x \alpha y \in \langle X \rangle$.

(iv) If $b \in \langle X \rangle$ and $x \in N, \alpha \in \Gamma$, then $b \alpha x \in \langle X \rangle$

(v) If $b \in \langle X \rangle$ and $x \in N$, then $x - b \in \langle X \rangle$

(vi) Nothing else is in $\langle X \rangle$.

Definition 4.2.8. Suppose $X \subseteq N$ and $d \in \langle X \rangle$. We call a sequence $s_1, s_2, \ldots, s_n$ of elements of $N$, a generating sequence of length $m$ for $d$ with respect to $X$. If $s_1 \in X$, $s_m = d, \alpha \in \Gamma$ and for each $i = 2, 3, \ldots m$, one of the following applies

\[ s_i \in X \]

\[ s_i = s_j + s_\ell, 1 \leq j, \ell < i \]

\[ s_i = s_j \alpha x, 1 \leq j < i \text{ and } x \in N \]

\[ s_i = x \alpha (s_j + y) - x \alpha y, 1 \leq j < i \text{ and } x, y \in N \]

\[ s_i = x + s_j - x, 1 \leq j < i \text{ and } x \in N \]

The complexity of $d$ with respect to $X$ denoted by $C_X (d)$, is the
Lemma 4.2.9. Let $N$ be a $\Gamma$–near ring. If $X \neq 0$ and $X \Gamma N = 0$, then $\langle X \rangle \Gamma N = 0$.

Proof. Let $X \Gamma N = 0$ and suppose $x \in \langle X \rangle$ arbitrary. We use induction on $C_X(x)$. If $C_X(x) = 1$, then $x \in X$ and from our assumption we have $X \Gamma N = 0$. Suppose $y \Gamma N = 0 \forall y \in \langle X \rangle$ such that $C_X(y) < n$ and let $C_X(x) = n$. We have the following possibilities:

(i) $x = a + b$ where $a, b \in \langle X \rangle$ and $C_X(a), C_X(b) < n$. Hence

$$x \Gamma N = (a + b) \Gamma N$$

$$= a \Gamma N + b \Gamma N$$

$$= 0$$

(ii) $x = a \alpha n$ where $a \in \langle X \rangle$, $n \in N$, $\alpha \in \Gamma$ and $C_X(a) < n$. Hence

$$x \Gamma N = (a \alpha n) \Gamma N$$

$$\subseteq a \Gamma N$$

$$= 0$$

(iii) $x = a \alpha (d + b) - a \alpha b$ where $d \in \langle X \rangle$ and $a, b \in N$, $\alpha \beta \in \Gamma$ with $C_X(d) < n$. If $m$ is arbitrary element of $N$, then

$$x \beta m = (a \alpha (d + b) - a \alpha b) \beta m$$

length of a generating sequence of least length for $d$ with respect to $X$. 

Lemma 4.2.9. Let $N$ be a $\Gamma$–near ring. If $X \neq 0$ and $X \Gamma N = 0$, then $\langle X \rangle \Gamma N = 0$.
\[ = a\alpha (d\beta m + b\beta m) - (a\alpha b) \beta m = a\alpha b\beta m - a\alpha b\beta m = 0. \]

Hence \( x\Gamma N = 0 \).

(iv) If \( x = a + b - a \) where \( b \in \langle X \rangle, a \in N, \alpha \in \Gamma \) and \( C_X (b) < n \).

Let \( m \in N \), then

\[ x\alpha m = (a + b - a) \alpha m = a\alpha m + b\alpha m - a\alpha m = 0. \]

This completes the proof. \( \blacksquare \)

**Corollary 4.2.10.** If every non zero ideal of a \( \Gamma \)-near ring \( N \) contains a subset \( F \) with \( r_\alpha (F) = 0, \forall \alpha \in \Gamma \), then for each \( a \in N, a \neq 0, \beta \in \Gamma \), there is a \( y \in N \) with \( a\beta y \neq 0 \).

**Proof.** Let \( a \neq 0 \in N \) and suppose \( F \) is a subset of \( \langle a \rangle \) such that \( r_\alpha (F) = 0, \forall \alpha \in \Gamma \). For every \( n \neq 0 \in N \), we have \( F\Gamma n \neq 0 \) and therefore \( \langle a \rangle \Gamma N \neq 0 \). From Lemma 4.2.9, there exists \( y \neq 0 \in N \) such that \( a\beta y \neq 0 \), for all \( \beta \in \Gamma \). \( \blacksquare \)

**Theorem 4.2.11.** Let \( N \) be a \( \Gamma \)-near ring, then \( N \) is strongly prime if and only if every non zero ideal of \( N \) contains a finite subset \( F \) with \( r_\alpha (F) = 0, \forall \alpha \in \Gamma \).
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**Proof.** Let $I \neq 0$ be an ideal in $N$ and $a \neq 0 \in I$. Since $N$ is strongly prime, there exists a finite subset $F \subseteq N$ such that $r_\alpha (a \Gamma F) = 0$, $\forall \alpha \in \Gamma$. Put $F_1 = a \Gamma F$. Hence $F_1$ is a finite subset subset of $I$ with $r_\alpha (F_1) = 0$, $\forall \alpha \in \Gamma$.

Conversely, let $a \neq 0 \in N$, then $\langle a \rangle \neq 0$. From our assumption, there exists a finite subset $F$ of $\langle a \rangle$ such that $r_\alpha (F) = 0$, $\forall \alpha \in \Gamma$. It follows from the Corollary 4.2.10 that there exists $y \in N$ with $a \beta y \neq 0$ for all $\beta \in \Gamma$. Again we use our assumption, we can find a finite subset $G_1 = \{g_1, g_2, \ldots, g_n\} \subseteq \langle a \beta y \rangle$ with $r_\alpha (G) = 0$, $\forall \alpha, \beta \in \Gamma$. For each $i$, let $s_{i_1}, s_{i_2}, \ldots, s_{i_m}$ be the corresponding generating sequence of $g_i$. Each of these sequence involve a finite number of terms of the form $a \beta y$ or $(a \beta y) \gamma t_k, t_k \in N$, $\forall \alpha, \beta, \gamma \in \Gamma$. Let $G_1 = \{a \beta y, (a \beta y) \gamma t_k / these occur in the generating sequence of an element of G\}$. Clearly $G_1$ is finite and $r_\alpha (G_1) \subseteq r_\alpha (G) = 0$, $\forall \alpha \in \Gamma$. Take $H = \{x/a \beta x \in G_1, \forall \beta \in \Gamma\}$. Our claim is that $H$ is an insulator for $a$. Now $r_\alpha (G_1) = 0$ implies that for any $n \in N, G_1 \alpha n = 0$, $\forall \alpha \in \Gamma$ implies $n = 0$. Since $a \Gamma H \subseteq G_1$, we have $H$ is an insulator for $a$ and consequently $N$ is strongly prime. ■

**Proposition 4.2.12.** Let $N$ be zero symmetric $\Gamma$—near ring then the following are equivalent.

(i) $N$ is strongly prime $\Gamma$—near ring.
(ii) Every non zero right $\Gamma-$ subgroup of $N$ contains a finite subset $F$ such that $r_\alpha (F) = 0, \ \forall \alpha \in \Gamma$.

(iii) Every non zero right ideal of $N$ contains a finite subset $F$ such that $r_\alpha (F) = 0, \ \forall \alpha \in \Gamma$.

(iv) Every non zero ideal of $N$ contains a finite subset $F$ such that $r_\alpha (F) = 0, \ \forall \alpha \in \Gamma$.

Proof. $(i) \Rightarrow (ii)$ : Let $I \neq 0$ be a right $\Gamma-$ subgroup of $N$ and let $a \neq 0 \in I$. Since $N$ is strongly prime, $a$ has an insulator $F$ such that $r_\alpha (a \Gamma F) = 0, \ \forall \alpha \in \Gamma$. Let $G = a \Gamma F$. Then $G \subseteq I$ and $r_\alpha (G) = 0, \ \forall \alpha \in \Gamma$.

$(ii) \Rightarrow (iii) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (i)$ follows from Theorem 4.2.11.

Proposition 4.2.13. Let $N$ be a zero symmetric $\Gamma-$ near ring with d.c.c. on right annihilators, then $N$ is 3-prime if and only if $N$ is strongly prime.

Proof. Suppose $N$ is strongly prime. To prove $N$ is 3-prime, let $a, b \in N$ such that $a \neq 0$ and $b \neq 0$. Since $N$ is strongly prime, there exists a finite subset $F$ of $N$ such that $a \Gamma F \Gamma b \neq 0$. Hence $a \Gamma N \Gamma b \neq 0$. Conversely, let $I \neq 0$ be an ideal in $N$ and for each $\alpha \in \Gamma$, consider
the collection of right $\alpha$— annihilators $\{r_\alpha (F)\}$ where $F$ runs over all finite subset of $I$. From our hypothesis, there exists a minimal element $M = r_\alpha (F_0)$. If $M \neq 0$, let $m \neq 0 \in M$ and $a \neq 0 \in I$. Since $N$ is 3-prime, there exists $n \neq 0 \in N$ such that $a\beta n \gamma m \neq 0$ for all $\beta, \gamma \in \Gamma$. Hence $a\gamma n \neq 0$. Let $S_\alpha = r_\alpha (F_0 \cup \{a\gamma n\}) \ \forall \alpha \in F$. Now $m \in M$ but $m \notin S_\alpha$ implies that $S_\alpha$ is smaller than $M$, a contraction. This forces that $M = (0)$. Hence for every non zero ideal $I$ of $N$, there exists a finite subset $F$ such that $r_\alpha (F) = 0 \ \forall \alpha \in \Gamma$ and consequently $N$ is strongly prime.

4.3 Radicals of strongly prime $\Gamma$— near rings.

In this section we shall prove that the strongly prime radical $\mathcal{P}_s (N)$ of $N$ coincides with $\mathcal{P}_s (L)^+$ where $\mathcal{P}_s (L)$ is the strongly prime radical of the left operator near - ring $L$ of $N$.

Definition 4.3.1. An ideal $I$ of a $\Gamma$— near ring $N$ is said to be strongly prime if for each $a \notin I$, there exists a finite subset $F$ such that for any $b \in N$, $a\Gamma F \Gamma b \subseteq I$ implies that $b \in I$. $F$ is called an insulator for $a$.

Proposition 4.3.2. Let $N$ be a $\Gamma$— near ring. If $P$ is a strongly prime ideal of $N$, then $P^{+\prime} = \{l \in L/\ell x \in P \ \forall x \in N\}$ is a strongly prime ideal of $L$. 


Proof. Suppose that $P$ is a strongly prime ideal of $N$. We shall prove that $P^{+\prime}$ is a strongly prime ideal of $L$. Let $\sum_i [x_i, \alpha_i] \notin P^{+\prime}$, then there exists $x \in N$ such that $\sum_i [x_i, \alpha_i] x \notin P$, that is $\sum_i x_i \alpha_i x \notin P$. Since $P$ is strongly prime in $N$, there exists a finite subset $F = \{f_1, f_2, \cdots, f_n\}$ of $N$ such that for any $b \in N$,

$$\sum_i x_i \alpha_i x \Gamma F b \subseteq P \text{ implies } b \in P.$$  

(4.3.1)

Fix $\alpha, \beta \in \Gamma$.

Consider the collection $F' = \{[x\alpha f_1, \beta], \cdots, [x\alpha f_n, \beta]\}$. Our claim is that $F'$ is an insulator for $\sum_i [x_i, \alpha_i]$. Let $\sum_j [y_j, \beta_j] \in L$ such that $\sum_i [x_i, \alpha_i] F' \sum_j [y_j, \beta_j] \subseteq P^{+\prime}$. To prove $\sum_j [y_i, \beta_i] \in P^{+\prime}$. Now

$$\sum_i [x_i, \alpha_i] F' \sum_j [y_j, \beta_j] \subseteq P^{+\prime}$$

implies

$$\sum_i [x_i, \alpha_i] [x\alpha f_k, \beta] \sum_j [y_j, \beta_j] \in P^{+\prime} \quad \forall \; k = 1, 2, \cdots, n,$$

i.e.,

$$\left( \sum_i [x_i, \alpha_i] [x\alpha f_k, \beta] \sum_j [y_j, \beta_j] \right) z \in P \quad \forall \; z \in N; \; k = 1, 2, \cdots, n.$$ 

Hence

$$\sum_i x_i \alpha_i x \Gamma F \sum_j y_j \beta_j z \subseteq P \quad \forall \; z \in N.$$

By (4.3.1), $\sum_j y_j \beta_j z \in P$ \quad $\forall \; z \in N$. i.e., $\sum_j [y_j, \beta_j] z \in P$ \quad $\forall z \in N$.

Hence $\sum_j [y_j, \beta_j] \in P^{+\prime}$ and therefore $F'$ is an insulator for $\sum_i [x_i, \alpha_i]$ and consequently $P^{+\prime}$ is a strongly prime ideal of $L$.  

$\blacksquare$
Proposition 4.3.3. Let $N$ be a distributive strongly 2-primal $\Gamma$-near ring with strong left unity. If $Q$ is a strongly prime ideal of $L$, then $Q^+ = \{x \in N/ [x, \alpha] \in Q \quad \forall \alpha \in \Gamma\}$ is a strongly prime ideal of $N$.

Proof. Suppose $Q$ is a strongly prime ideal of $L$. We shall prove that $Q^+$ is a strongly prime ideal of $N$. Let $x \notin Q^+$, then there exists $\alpha \in \Gamma$ such that $[x, \alpha] \notin Q$. Since $Q$ is a strongly prime ideal of $L$, there exists a finite subset $F = \left\{ \sum_{j=1}^{n} [y_{jk}, \beta_{jk}] / k = 1, 2, \ldots, m \right\}$ (say) such that for any $\sum_{\ell} [z_{\ell}, \delta_{\ell}] \subseteq L$,

$$[x, \alpha] F \sum_{\ell} [z_{\ell}, \delta_{\ell}] \subseteq Q \text{ implies that } \sum_{\ell} [z_{\ell}, \delta_{\ell}] \in Q.$$ (4.3.2)

Consider $F' = \{y_{jk} \beta_{jk} x / j = 1, 2, \ldots, n; k = 1, 2, \ldots, m\}$. Our claim is that $F'$ is an insulator for $x$. Let $a \in N$ such that $x \Gamma F' \Gamma a \subseteq Q^+$. To prove $a \in Q^+$. Now $x \Gamma F' \Gamma a \subseteq Q^+$ implies

$$\left[ x \Gamma F' \Gamma a, \Gamma \right] \subseteq Q,$$

i.e., $[x \alpha y_{jk} \beta_{jk} x \beta a, \gamma] \subseteq Q$,

$\forall j = 1, 2, \ldots, n; k = 1, 2, \ldots, m$ and $\forall \alpha, \beta, \gamma \in \Gamma$. This implies that

$$[x, \alpha] F [x \beta a, \gamma] \subseteq Q.$$ (4.3.3)

By (4.3.2) $[x \beta a, \gamma] \subseteq Q$. Now since $Q$ is strongly prime in $L$, $Q$ is prime in $L$. By Proposition 1.1.26, $Q^+$ is prime ideal of $N$. Since $N$
is strongly 2-primal, \( Q^+ \) is completely prime in \( N \). Hence \( x\gamma a \in Q^+ \) and \( x \notin Q^+ \) implies \( a \in Q^+ \). Thus \( Q^+ \) is strongly prime in \( N \).

**Proposition 4.3.4.** Let \( N \) be a distributive strongly 2-primal \( \Gamma \)-near ring with strong left unity and \( L \), a left operator near-ring of \( N \). Then \( \mathcal{P}_s(N) = \mathcal{P}_s(L)^+ \).

**Proof.** Let \( P \) be a strongly prime ideal of \( L \). Then by Proposition 4.3.3, \( P^+ \) is a strongly prime ideal of \( N \). Moreover \( (P^+)^+ = P \) by Theorem 1.1.27. Suppose \( Q \) is a strongly prime ideal in \( A^+ \), then by Proposition 4.3.2, \( Q^+ \) is strongly prime in \( L \) and \( (Q^+)^+ = Q \) by Theorem 1.1.27. Thus the mapping \( P \mapsto P^+ \) defines a 1-1 correspondence between the set of strongly prime ideals of \( L \) and \( N \).

Hence \( \mathcal{P}_s(L)^+ = (\cap P)^+ = \cap P^+ = \mathcal{P}_s(N) \).

### 4.4 Equiprime radicals of \( \Gamma \)-near rings

In this section we shall prove that the equiprime radical \( \mathcal{P}_e(N) \) of \( N \) coincides with \( \mathcal{P}_e(L)^+ \) where \( \mathcal{P}_e(L) \) is the equiprime radical of left operator near-ring \( L \) of \( N \).

**Definition 4.4.1.** Let \( N \) be a \( \Gamma \)-near ring, and \( P \) be an ideal in \( N \). Then \( P \) is said to be **equiprime** if \( a, x, y \in N, a \notin P, a\alpha\beta x - a\alpha\beta y \in P \) \( \forall n \in N, \alpha, \beta \in \Gamma \) implies \( x - y \in P \).
Proposition 4.4.2. Let $N$ be a $\Gamma$-near ring. If $P$ is an equiprime ideal of $N$, then $P^+ = \{ \ell \in L/\ell x \in P \ \forall x \in N \}$ is an equiprime ideal of $L$.

Proof. Let $\ell \notin P^+$ and $\ell', \ell'' \in N$ such that $\ell' - \ell'' \notin P^+$. From definition of $P^+$, there exist $a, b \in N$ such that $\ell a \notin P$ and $(\ell' - \ell'') b \notin P$, that is $\ell a \notin P$ and $\ell' b - \ell'' b \notin P$. From the hypothesis, there exists $c \in N$ such that

\[
(\ell a) \alpha \beta (\ell' b) - (\ell a) \alpha \beta (\ell'' b) \notin P, \ \forall \alpha, \beta \in \Gamma
\]

i.e., $[\ell a, \alpha] [c, \beta] \ell' b - [\ell a, \alpha] [c, \beta] \ell'' b \notin P, \ \forall \alpha, \beta \in \Gamma$

i.e., $\ell [a, \alpha] [c, \beta] \ell' b - \ell [a, \alpha] [c, \beta] \ell'' b \notin P, \ \forall \alpha, \beta \in \Gamma$.

Hence

\[
(\ell [a \alpha, \beta] \ell' - \ell [a \alpha, \beta] \ell'') b \notin P, \ \forall \alpha, \beta \in \Gamma.
\]

This proves that

\[
\ell [a \alpha, \beta] \ell' - \ell [a \alpha, \beta] \ell'' \notin P^+, \ \forall \alpha, \beta \in \Gamma
\]

and consequently $P^+$ is an equiprime ideal of $L$.

Proposition 4.4.3. Let $N$ be a $\Gamma$-near ring. If $Q$ is an equiprime ideal of $L$, then $Q^+ = \{ x \in N/ [x, \alpha] \in Q \ \forall \alpha \in \Gamma \}$ is an equiprime ideal of $N$. 

Proof. Let \( x \notin Q^+ \) and \( a, b \in N \) such that \( a - b \notin Q^+ \). We claim that \( x \Gamma N \Gamma a - x \Gamma N \Gamma b \notin Q^+ \). Since \( x \notin Q^+ \) and \( a - b \notin Q^+ \), then there exist \( \alpha, \beta \in \Gamma \) such that \([x, \alpha] \notin Q \) and \([a - b, \beta] \notin Q \) implies that \([x, \alpha] \notin Q \) and \([a, \beta] - [b, \beta] \notin Q \). Since \( Q \) is a equiprime ideal in \( L \), there exists \( \ell = \sum_i [y_i, \beta_i] \in L \) such that \([x, \alpha] \ell [a, \beta] - [x, \alpha] \ell [b, \beta] \notin Q \). Hence \([x \alpha a - x \alpha b, \beta] \notin Q \). This implies that \( x \alpha a - x \alpha b \notin Q^+ \).

\[
i.e., \quad x \alpha \sum_i [y_i, \beta_i] a - x \alpha \sum_i [y_i, \beta_i] b \notin Q^+
\]
i.e., \( x \alpha \sum_i y_i \beta_i a - x \alpha \sum_i y_i \beta_i b \notin Q^+ \).

But clearly \( x \alpha \sum_i y_i \beta_i a - x \alpha \sum_i y_i \beta_i b \in x \Gamma N \Gamma a - x \Gamma N \Gamma b \). Thus \( x \Gamma N \Gamma a - x \Gamma N \Gamma b \notin Q^+ \) and consequently \( Q^+ \) is an equiprime ideal of \( N \).

\[\textbf{Theorem 4.4.4.} \text{Let } N \text{ be a } \Gamma-\text{near ring with left operator near-ring } L, \text{ then } \mathcal{P}_e(L)^+ = \mathcal{P}_e(N).\]

Proof. Let \( P \) be an equiprime ideal of \( L \). Then by Proposition 4.4.3, \( P^+ \) is an equiprime ideal of \( N \). Moreover \( (P^+)^{++} = P \) by Theorem 1.1.27. Suppose \( Q \) is an equiprime ideal in \( N \), then by Proposition 4.4.2, \( Q^{++} \) is an equiprime ideal in \( L \) and \( (Q^{++})^+ = Q \) by Theorem 1.1.27. Thus the mapping \( P \to P^+ \) defines a 1-1 correspondence between the set of equiprime ideals of \( L \) and \( N \).

Hence \( \mathcal{P}_e(L)^+ = (\cap P)^+ = \cap P^+ = \mathcal{P}_e(N) \).