INTRODUCTION

Most graph labeling methods trace their origin to one introduced by Rosa in 1967 or one given by Graham and Sloane in 1980. A vertex labeling of a graph G is an assignment of labels to the vertices of G that induces for each edge xy a label depending on the vertex labels f(x) and f(y). Similarly, an edge labeling of a graph G is an assignment f of labels to the edges of G that induces for each vertex v a label depending on the edge labels incident on it. Total labeling involves a function from the vertices and edges to some set of labels.

Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography, radar, astronomy, circuit design and communication network.

In 1990, Harary [4] introduced the notion of a sum graph. A graph G(V, E) is called a sum graph if there is a bijective labeling f from V to a set of positive integers S such that xy ∈ E if and only if f(x) + f(y) ∈ S. Since a vertex with the highest label in a sum graph can not be adjacent to any other vertex, every sum graph
must contain isolated vertices. Hence Harary [4] defined sum number of a graph as the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph. In 1994, Harary [5] generalized sum graphs by permitting S to be any set of integers and called them integral sum graphs. The minimum number of isolated vertices that must be added to G to make it an integral sum graph is called as the integral sum number.

In this thesis we study the edge analogue of sum graph, viz. edge sum graph and investigate their properties. We define edge sum number and find them for some standard graphs like triangular crocodile, fans and baskets. Also we find the edge sum number of \( K_n, K_{m,n} \) and \( K_{m,n}^+ \) and graphs with \( \Delta = n-1 \) and \( \delta \geq 2 \) where \( \Delta = \text{maximum of } \{\deg v : v \in V\} \) and \( \delta = \text{minimum of } \{\deg v : v \in V\} \).

The thesis consists of five chapters

1. Preliminaries

2. Edge sum graphs and edge sum numbers

3. Edge sum number of special graphs.

4. Edge sum number of \( K_{m,n} \) and \( K_{m,n}^+ \).
5. Edge sum number of graphs with $\Delta = n-1$

By a graph we mean a finite undirected graph without loops or multiple edges. In chapter 1, we collect the basic definitions, which are needed for the subsequent chapters. For graph theoretic terminology, we refer to the books of Harary [6], Y. Caro and L. Lesniak [3], Parthasarathy [8] and Bondy and Murty [2].

In chapter 2, we study the edge analogue of sum graph [9, 11, 12]. A bijection $f: E \rightarrow S$ where $S$ is a set of positive integers is called an edge function of the graph $G$. Define $F(v) = \sum \{f(e) : e$ is incident on $v\}$ on $V$. Then $F$ is called the edge sum function of the edge function $f$. $G$ is said to be an edge sum graph if there exists an edge function $f: E \rightarrow S$ such that $f$ and its edge sum function $F$ on $V$ satisfy the following conditions.

1. $F$ is into $S$. That is, $F(v) \in S$ for every $v \in V$.

2. If $e_1, e_2, \ldots, e_n \in E$ such that $f(e_1) + f(e_2) + \ldots + f(e_n) \in S$,

then $e_1, e_2, \ldots, e_n$ are incident on a vertex.
We prove the following theorems to study some intrinsic properties of edge sum graphs.

**Theorem:** $K_2$ is the only connected edge sum graph.

**Theorem:** Let $G(V, E)$ be a connected graph with edge function $f: E \to S$ and edge sum function $F$. If $v$ is the only vertex such that $F(v) \not\in S$, then $v$ is adjacent to a pendant vertex.

**Theorem:** Let $G(V, E)$ be an edge sum graph with edge function $f: E \to S$ and edge sum function $F$ of $f$. Let $w$ be a nonpendant vertex and $e = uv \in E$ be such that $F(w) = f(e)$. Then one of the following holds:

1. $\{u, v\}$ forms a $K_2$ component in $G$.
2. $\langle u, v, w \rangle$ is either $K_3$ or $P_2$ with one of $u, v$ as a pendant vertex in $G$.

**Theorem:** Let $G(V, E)$ be an edge sum graph with edge function $f: E \to S$ and edge sum function $F$ of $f$. Let $\ell_1, \ell_2, \ldots, \ell_m$ where $m > 1$ is a collection of edges incident on a vertex $w$ (say).
Let \( w_1 = \ell_1 \) for \( 1 \leq i \leq m \). If there exists an edge \( e = uv \) such that
\[
f(\ell_1) + f(\ell_2) + \ldots + f(\ell_m) = f(e),
\]
then one of the following holds:

1. \( \{u, v\} \) forms a \( K_2 \) component in \( G \).

2. \( \langle u, v, w \rangle \) is \( K_3 \) or \( P_2 \) or \( P_1 \) with one of \( u, v \) as a pendant vertex in \( G \).

Also, we define **edge sum number** \( \sigma_E (G) \) of a graph \( G \) as the smallest number \( r \) such that \( G \cup rK_2 \) is an edge sum graph. We call an edge function \( f \) which makes \( G \cup rK_2 \) an edge sum graph an **optimal edge function** and its corresponding edge sum function an **optimal edge sum function**. If \( E_1 \) and \( E_2 \) are respectively the edge sets of \( G \) and \( rK_2 \) and \( V \) is the vertex set of \( G \cup rK_2 \), we call the optimal edge sum function \( F \) an **outer edge sum function** if \( F(V) \subseteq f(E_2) \) and **inner edge sum function** if \( F(V) \cap f(E_1) \) is not empty.

We find the edge sum numbers of some standard graphs and these are enumerated in the following theorems in this chapter.
Theorem: If a graph has no pendant vertex and is also triangle free, then any optimal edge sum function is necessarily an outer edge sum function.

In chapter 3 we find the edge sum number of some special graphs such as $M_{d,p}$ graph, fans, baskets and $K_{1,n:m}$ graphs.

Theorem: $[10] \sigma_e(K_{1,n:m}) = 3$ if $m + 1$ divides $mn$ and $2$ if $(m+1)$ does not divide $mn$.

Theorem: If $F_q$ is a fan for $q \geq 1$ then $\sigma_e(F_q) = 1$ for $q$ is odd and $q \geq 3$.

Theorem: If $B_q$ is a basket then $\sigma_e(B_q) = 1$ for $q \geq 4$.

Theorem: If $M_{d,p}$ is a graph for every odd positive integer $d$ and non-negative integers $p$. Then $\sigma_e(M_{d,p}) = 1$.

In chapter 4, we find the edge sum number of $K_{m,n}$ and $K_{m,n+1}$ where $x$ is an edge.

Theorem: Let $K_{m,n}$ be the complete bipartite graph with the partitions having $m$ and $n$ vertices. Then $\sigma_e(K_{n,n}) = 1$ for $n \geq 3$;
\( \sigma_e(K_{2,2n}) = \sigma_e(K_{3,3(2n-1)}) = 3 \) for \( n \geq 2 \) and \( \sigma_e(K_{m,n}) = 2 \) in all the other cases.

**Result 1:**

Let \( G_1 = K_{m,n} \) where \( n+2 \geq m \). Let \( V = V_1 \cup V_2 \) where \( V_1 = \{u_1, u_2, \ldots, u_m\} \) and \( V_2 = \{v_1, v_2, \ldots, v_n\} \) where \( V_1 \) and \( V_2 \) are the partitions of the vertex set \( V \) of \( G_1 \). Then, \( G_1 + x \) where \( x \) is an edge joining any two vertices of \( V_1 \) (say) \( u_1 \) and \( u_2 \) is of edge sum number 1.

In chapter 5, we find the edge sum number of graphs with \( \Delta = n-1 \) where \( n \) is the number of vertices of the graph.

**Theorem:** Let \( K_n \) be the complete graph with \( n \) vertices. Then \( \sigma_e(K_3) = 2, \sigma_e(K_n) = 1 \) for \( n \geq 4 \). We also find the edge sum number of graphs \( G \) with \( \Delta = n-1 \) and \( \delta \geq 2 \). We divide into three cases according as there exists no pair, only one pair and more than one pair of vertices \( u, v \) with \( \deg u = \deg v = 2 \) and are adjacent. In the first two cases the edge sum number is 1 and in the third case it is 2.