CHAPTER 3

BAROTROPIC-BAROCLINIC INSTABILITY OF ZONAL FLOWS
ON A $\beta$-PLANE

3.1 Introduction

In this chapter we study the barotropic-baroclinic instability of zonal flows on a $\beta$-plane to infinitesimal normal mode disturbances. This problem, which we call Pedlosky's problem, is more complicated than the Kuo's problem studied in the previous chapter by the addition of (i) vertical variation of the zonal flow velocity and (ii) the vertical variation of the density of the fluid. Of course the normal mode disturbance variables also vary in the vertical direction. The major results known for the Pedlosky's problem are (i) the semicircle theorem which gives a bound on the phase velocity $c$ of unstable disturbances in terms of the extreme values of the flow velocity; (ii) an estimate for the growth rate of an unstable mode in terms of the maximum flow gradients. Both of these results due to Pedlosky (1964, 1979) are generalizations of the corresponding results of the Kuo's problem. (iii) A necessary condition for instability of zonal flows $U(y,z)$ satisfying the boundary conditions $\frac{\partial U}{\partial z} = 0$ at $z = 0, z_T$, is that the potential vorticity gradient $\frac{\partial \zeta}{\partial y}$
must vanish at least once in the flow domain (Pedlosky, 1979). This is a generalization of Kuo's instability condition. Unlike the results (i) and (ii) which are valid for any zonal flow, result (iii) is valid only for a particular class of zonal flows satisfying the additional boundary conditions. The lower boundary condition at \( z = 0 \) is satisfied when the surfaces of constant potential temperature are parallel to the horizontal lower boundary at \( z = 0 \). The upper boundary condition at \( z = z_T \) is satisfied, for example, when \( z_T + \infty \) and \( \frac{\partial U}{\partial z} (z_T = \infty) = 0 \) (see Pedlosky, 1979, Page 440).

Recently, Pedlosky's problem has been studied by Gnevyshev and Shrira (1990). These authors have obtained (i) an improvement of Pedlosky's semicircle theorem by extending a theorem of Miles (1964) for the baroclinic instability problem; (ii) a 'quasi-parabolic' bound for the unstable modes.

In this chapter, we consider the barotropic-baroclinic instability problem for zonal flows on the \( \beta \)-plane satisfying the boundary conditions \( \frac{\partial U}{\partial z} = 0 \) at \( z = 0, z_T \). First, we obtain a new estimate for the growth rate of any unstable normal mode. Then, we prove the boundedness of the wave velocities of non-singular neutral modes. Finally, we obtain parabolic instability regions, which improve
Pedlosky's theorem for two classes of flows. These results are generalizations of the corresponding results of Chapter 2 for the Kuo's problem. So these results are significant for the same reasons. In addition, our parabolic instability regions are better than the semicircle theorem obtained by Gnevyshev & Shrira (1990) because our regions depend on the potential vorticity gradient of the basic zonal flow. Also, in our result, the estimates for $c_i^2$ do not depend on $c_i$ itself as is the case with Gnevyshev and Shrira's quasi-parabolic bound.

3.2 Eigen Value Problem

The normal mode stability of barotropic-baroclinic zonal flows of an inviscid, incompressible stratified fluid on a $\beta$-plane is governed by the following Pedlosky's eigen value problem (Pedlosky, 1979):

\[
(U-c) \left[ \frac{1}{\rho} \left( \frac{\partial}{\partial z} \frac{\rho \partial}{\rho} \right) \right] + H \frac{\partial^2}{\partial y^2} - k^2 H + \Omega y H = 0, \quad (3.1)
\]

\[H = 0, \quad y = Y_1, Y_2, \quad (3.2)\]

and

\[
(U-c) H_z - U_z H = 0, \quad z = 0, z_T, \quad (3.3)
\]

where $U(y,z)$ is the basic zonal velocity, $c$ is the complex wave velocity of the disturbance, $k$ is the zonal wave number of the disturbance, $\rho(z)$ is the reference-state density,
\[ S = \frac{N^2 D^2}{f^2 L^2} \] is the stratification parameter and
\[ f_o^2 L \]
is the stratification parameter and
\[ Q_y = \beta - U_{yy} - \frac{1}{\rho_s} \left( \frac{s}{S} U_z \right)_z \] is the gradient of the
the stratification parameter and
\[ Q_y = \beta - U_{yy} - \frac{1}{\rho_s} \left( \frac{s}{S} U_z \right)_z \] is the gradient of the
potential vorticity. \[ N_s^2 = \frac{g}{\Theta_s} \frac{d \Theta}{dz} \] is the square of the
Brunt-Väisälä frequency with \( \Theta_s \), the reference-state
potential vorticity. \[ N_s^2 = \frac{g}{\Theta_s} \frac{d \Theta}{dz} \] is the square of the
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flows satisfying the boundary conditions (3.4):
(i) all pure barotropic flows whose basic velocity are functions of $y$ only;

(ii) $U(y,z) = V(y) \cos \left(\frac{n\pi z}{z_T}\right)$, $n = 1,2,\ldots$, where $V(y)$ is any twice continuously differentiable function in $(y_1, y_2)$;

(iii) $U(y,z) = V(y) \cos^n \left(\frac{n\pi z}{z_T}\right)$, $n = 1,2,\ldots$;

(iv) $U(y,z) = V(y) z^n (z-z_T)^n$, $n = 2,3,\ldots$;

(v) $U(y,z) = V(y) \left[z - \frac{z_T}{z^n} \sin \left(\frac{2\pi z}{z_T}\right)\right]^n$, $n = 1,2,\ldots$.

3.3 An Estimate for the Growth Rate

Theorem 1: An estimate for the growth rate of an unstable mode is given by

$$k_c \leq \frac{|Q_y|_{\text{max}}}{2(\alpha^2/k)},$$

(3.6)

where $\alpha^2 = k^2 + \frac{\gamma^2}{(y_2-y_1)^2}.$

Proof: For an unstable mode, multiplying (3.1) by $\frac{\bar{g}H}{\bar{U}-\bar{C}}$ (H$^*$ is the complex conjugate of H) and then integrating the resulting equation over the meridional cross-section using (3.2) and (3.5), we get
The real part of (3.7) yields

\[
\int_{y_1}^{y_2} \int_{y_1}^{z_T} \left\{ \rho_s [S^{-1} |H_z|^2 + |H_y|^2 + k^2|H|^2] \right\} \, dz \, dy \\
- \frac{\rho_s Q}{U-c} |H|^2 \, dz \, dy = 0. \quad (3.7)
\]

The real part of (3.7) yields

\[
\int_{y_1}^{y_2} \int_{y_1}^{z_T} \rho_s [S^{-1} |H_z|^2 + |H_y|^2 + k^2|H|^2] \, dz \, dy \\
\leq \int_{y_1}^{y_2} \int_{y_1}^{z_T} \frac{\rho_s |Q| |U-c| |H|^2}{|U-c|^2} \, dz \, dy. \quad (3.8)
\]

With the help of the inequalities (2.7) and (2.8), we get the following estimate from (3.8) after dropping the positive term \( \int_{y_1}^{y_2} \int_{y_1}^{z_T} \rho_s S^{-1} |H_z|^2 \, dz \, dy \):

\[
\alpha^2 \int_{y_1}^{y_2} \int_{y_1}^{z_T} \rho_s |H_z|^2 \, dz \, dy \leq \frac{|Q|_{\text{max}}}{2c} \int_{y_1}^{y_2} \int_{y_1}^{z_T} \rho_s |H|^2 \, dz \, dy. \quad (3.9)
\]

The bound (3.6) follows immediately from this inequality.

### 3.4 Boundedness of Real Eigen Values

The singular neutral modes that satisfy \( a \leq c_r \leq b \) where \( a = \min U(y,z) \) and \( b = \max U(y,z) \) are bounded by their very definition. Now, we establish the boundedness of wave velocities of non-singular neutral modes.
Theorem 2: Non-singular neutral modes of the eigen value problem governed by (3.1), (3.2) and (3.5) are bounded and the bounds are given by

\[ a - \frac{\|Q\|_{\text{max}} + |Q|_{\text{max}}}{2\alpha^2} \leq c_r \leq b + \frac{\|Q\|_{\text{max}} - |Q|_{\text{min}}}{2\alpha^2} \]  

(3.10)

**Proof:** Let \( c = c_r \) with \( U - c_r \neq 0 \) in \([y_1, y_2] \times [0, z_T] \) be a real eigen value with real eigen function \( H \). Using the Rayleigh - Ritz inequality in equation (3.7), we get

\[ Y_2 \int_{y_1}^{y_2} \int_{y_1}^{y_2} \rho_s S^{-1} |H_z|^2 \, dz \, dy + \int_{y_1}^{y_2} \int_{y_1}^{y_2} \rho_s \left[ \alpha^2 - \frac{Q}{U-c_r} \right] H^2 \, dz \, dy \leq 0. \]  

(3.11)

Since the term

\[ \int_{y_1}^{y_2} \int_{y_1}^{y_2} \rho_s S^{-1} (H_z)^2 \, dz \, dy \]  

is positive, the above inequality yields

\[ \int_{y_1}^{y_2} \int_{y_1}^{y_2} \rho_s \left[ \alpha^2 (U - c_r)^2 - (Q_y) (U - c_r) \right] H^2 \, dz \, dy \leq 0. \]  

(3.12)

Therefore, there exists \((y_8, z_8) \in [y_1, y_2] \times [0, z_T]\) such that

\[ \alpha^2 (U(y_8, z_8) - c_r)^2 - (U(y_8, z_8) - c_r) (Q_y) (y_8, z_8) \leq 0. \]  

(3.13)

The left hand side of (3.13) is a quadratic expression in \( c_r \),
having discriminant $(Q_y (y_B, z_B))^2$. So the roots are real and

$$2U(y_B, z_B) \alpha^2 - Q_y (y_B, z_B) - |(Q_y)(y_B, z_B)| \leq \frac{c_r}{2\alpha^2}$$

$$2U(y_B, z_B) \alpha^2 - Q_y (y_B, z_B) + |(Q_y)(y_B, z_B)| \leq \frac{2\alpha^2}{c_r}$$

(3.14)

The bounds given by (3.10) immediately follow from this.

**Theorem 3**: Non-singular neutral modes of the eigenvalue problem given by (3.1), (3.2) and (3.5) are bounded and the bounds are given by

$$a - \frac{(Q_y)_{\text{max}} + |Q_y|_{\text{max}}}{2\pi} \leq c_r \leq b + \frac{|Q_y|_{\text{max}} - (Q_y)_{\text{min}}}{2\pi}$$

$$\frac{2\pi}{(y_2 - y_1)^2} \leq \frac{2\pi}{(y_2 - y_1)^2}$$

(3.15)

**Proof**: For non-singular neutral modes, we have the inequality (3.12). If we drop the positive term

$$k^2 \int_{y_1}^{y_2} \int_{y_1}^{y_2} \rho_s |H|^2 \, dz \, dy$$

in (3.12) and then proceed as in theorem 2, we get the bound (3.15), independent of the wave number $k$.

**Example 1**: Let $U = 0$. Equation (3.1) then becomes

$$\frac{1}{\rho_s} \left( -\frac{\partial}{\partial z} H \right) + H_{yy} - (k^2 + \frac{\rho}{c^2})H = 0.$$

(3.16)
Now, the function $H = \sin [\mu(y-y_1)]$ where $\mu^2 = -(k^2 + \beta/c)$ satisfies (3.16), (3.2) and (3.5) whenever

$$c = - \frac{\beta}{k^2 + \frac{n^2 \pi^2}{(y_2 - y_1)^2}}, \quad n = 1, 2, \ldots \quad (3.17)$$

For $n = 1$, the eigen value given by (3.17) coincides with the lower bound given by (3.10). Thus the lower bound obtained is the best possible.

3.5 Instability Regions

For an unstable mode, the transformation $H = (U-c)^{1/2}G$ reduces the system of equations (3.1), (3.2) and (3.5) to the system of equations

$$\frac{1}{\rho_R} \left[ - \frac{\partial}{\partial z} \left( U-c \right) G \right]_z + \left[ (U-c) \frac{\partial}{\partial y} \right]_y$$
$$- k^2 (U-c) G - \frac{1}{4 (U-c)} \left( S^{-1} \left( \frac{\partial U}{\partial z} \right)^2 + \left( \frac{\partial U}{\partial y} \right)^2 \right) G$$
$$- \left( \frac{1}{2 \rho_R} \left( - \frac{\partial}{\partial z} U \right) \frac{\partial}{\partial z} + \frac{1}{2} \left( \frac{\partial U}{\partial y} \right)^2 - \beta \right) G = 0, \quad (3.18)$$
$$G = 0, \quad y = y_1, y_2, \quad (3.19)$$

and

$$G_z = 0, \quad z = 0, z_T. \quad (3.20)$$

It should be remarked here that without loss of generality we can assume $U > 0$ throughout the flow domain.
(see Drazin & Howard, 1966). That is we can take
\[
\frac{\beta}{2\pi^2 \left( Y_2 - Y_1 \right)^2}
\]
without loss of generality.

**Theorem 4**: If \((c,G)\) is a solution of (3.18) - (3.20) and
\[
f(y,z) = \frac{1}{\rho^8} \left( -\frac{\rho^8}{G} U_z z + U_{yy} - \beta - 2a \alpha^2 < 0 \right)
\]
then
\[
\forall \quad (y,z) \in \left[ y_1, y_2 \right] \times \left[ 0, z_T \right],
\]
then
\[
c^2 \leq \lambda \left( c_r + \frac{b}{m-1} \right),
\]
where
\[
\lambda = \frac{(m-1)}{2} \left[ \frac{S^{-1}(U) - \left( U_0 \right)^2}{|f|} \right]_{\text{max}} \quad \text{and}
\]
\[
m = \frac{b}{a - \frac{\beta}{2\alpha^2}} > 1.
\]

**Proof**: Multiplying the equation (3.18) by \(\rho^8 G^*\) \((G^*\) is the complex conjugate of \(G)\) and then integrating the resulting equation over the meridional cross-section using (3.19) and (3.20), we get
\[
\int_{Y_1}^{Y_2} \int_{0}^{z_T} \rho^8 (U-c) \left[ S^{-1} |G_x|^2 + |G_y|^2 + k^2 |G|^2 \right] dy dz = \frac{\rho^8 \left( S^{-1}(U_0)^2 + (U_0)^2 \right)}{4(U-c)}
\]

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\[ + \rho_s \left[ \frac{1}{2\rho_s} \left( -\frac{\rho_s}{s} U_z \right)_z + \frac{1}{2} U_{yy} - \beta \right] |G|^2 \] dzdy = 0.

(3.22)

Multiplying the imaginary part of (3.22) by \(-mc_r\) and then adding the resulting equation to the real part of (3.22), we get

\[
\int_{y_1}^{y_2} \int_{z_1}^{z_T} \left[ \rho_s \left[ U-(m+1)c_r \right] \left[ S^{-1} |G_z|^2 + |G_y|^2 + k^2 |G|^2 \right] \right.
\]
\[
\left. + \rho_s \left[ \frac{1}{2\rho_s} \left( -\frac{\rho_s}{s} U_z \right)_z + \frac{1}{2} U_{yy} - \beta \right] |G|^2 \right] dzdy = 0.
\]

(3.23)

Since \([U-(m+1)c_r]_{\text{max}} = -(a - \frac{\beta}{2\alpha^2}) < 0\), equation (3.23) gives after some manipulations,

\[
\int_{y_1}^{y_2} \int_{z_1}^{z_T} \rho_s \left[ 2f(y,z)c_i^2 + [U+(m-1)c_r] \left[ S^{-1} (U_z)^2 + (U_y)^2 \right] \right]
\]
\[
\left. - |G|^2 \right] dzdy \geq 0. \quad (3.24)
\]

Therefore, there exists \((y_s,z_s) \in [y_1, y_2] \times [0,z_T]\) such that

\[
2 f(y_s,z_s) c_i^2 + [U(y_s,z_s) + (m-1)c_r] P
\]
\[
\left[ S^{-1}(U_z(y_s,z_s))^2 + (U_y(y_s,z_s))^2 \right] \geq 0. \quad (3.25)
\]

This implies that
Theorem 5: Under the conditions of theorem 4, the parabola given by
\[ c_i^2 = \lambda \left( c_r + \frac{b}{m-1} \right), \tag{3.26} \]
intersects the semicircle given by
\[ (c_r - \frac{a+b}{2})^2 + c_i^2 \leq \left( \frac{b-a}{2} \right)^2 \tag{3.27} \]
if
\[ \lambda < \left( \frac{m+1}{m-1} b + a \right) - \sqrt{\left( \frac{m+1}{m-1} b + a \right)^2 - (b-a)^2}. \tag{3.28} \]

Proof: Proof follows by proceeding exactly along the same lines as in the proof of the theorem 5 of Chapter 2 of this thesis.

Theorem 6: If \((c, G)\) is a solution of (3.18) - (3.20) and
\[ g(y, z) = \frac{1}{\rho^g} \left( \frac{\rho^g}{s} U_z \right) s U_y + U_y \beta + 2ba^2 > 0 \]
\[ \forall (y, z) \in [y_1, y_2] \times [0, z_T] \]
then
\[ c_i^2 \leq \bar{\lambda} \left( c_r - \frac{a}{m+1} \right), \tag{3.29} \]
where
\[ \bar{\lambda} = \frac{(m+1)}{2} \frac{S^{-1}(U_y)^2 + (U_y)^2}{g} \]
Proof: Multiplying the imaginary part of (3.22) by \(mc_r\) and then adding the resulting equation to the real part of (3.22), and then proceeding as in theorem 4, we get the result.

Theorem 7: Under the conditions of theorem 6, the parabola given by

\[
\bar{c}_i^2 = \bar{x} (c_r - \frac{a}{m+1}),
\]

intersects the semicircle (3.27), if

\[
\bar{x} < (\frac{m-1}{m+1} a + b) - \sqrt{(\frac{m-1}{m+1} a + b)^2 - (b-a)^2}.
\]

Proof: Proceeding exactly as in theorem 5 of Chapter 2, we get the result.

Theorem 8: For flows satisfying either the conditions of theorems 4 and 5 or the conditions of theorems 6 and 7, the instability region given by Pedlosky's semicircle is reduced.

Proof: The result follows by proceeding as in theorem 8 of Chapter 2.

3.6 Concluding Remarks

In this chapter, we have analysed the normal mode stability of a class of barotropic-baroclinic zonal flows satisfying the conditions \(U_z = 0\) at \(z = 0, z_T\). The
condition $U_z(z=0) = 0$ is satisfied when the surfaces of constant potential temperature are parallel to the horizontal lower boundary at $z = 0$. The condition $U_z = 0$ at $z = z_T$ is satisfied, for example, when $z_T \to \infty$ and $U_z(z_T = \infty) = 0$. Also, a class of flows satisfying these conditions have been given in section 3.2 of this chapter. First, we have obtained a new estimate for the growth rate of an unstable mode. Then, we have shown that the wave velocities of non-singular neutral modes are bounded. The lower bound we have obtained is the best possible in the sense that it coincides with an eigen value for a particular zonal flow. Also, we have obtained new parabolic instability regions which in conjunction with the Pedlosky's semicircle region reduce the known instability region for two particular classes of zonal flows. In addition, our parabolic instability regions are better than the semicircle theorem obtained by Gnevyshev & Shrira (1990) because our regions depend on the potential vorticity gradient of the basic zonal flow. Also, in our result, the estimates for $c_\perp^2$ do not depend on $c_\perp$ itself as is the case with Gnevyshev and Shrira's quasi-parabolic bound.