CHAPTER – VII
GENERAL BULK SERVICE QUEUEING SYSTEM WITH MULTIPLE WORKING VACATION

INTRODUCTION

Tian, Li and Zhang (2009) provided a survey of the results of working vacation queues and demonstrated that the matrix analytic methods developed by Neuts (1995) are powerful tools for analyzing the WVQ’s. The survey also shows that neither bulk input nor bulk service WVQs are considered in the existing literature. Later Xu et al. (2009a) and Julia Rose Mary and Afthab Begum (2010) studied the bulk input Markovian $M^X / M / 1$ queue with working vacations and presented the PGF of the stationary queue length. In Chapter VI, we have analysed the bulk arrival Non-Markovian multiple and single working vacation queueing models $M^X / G / 1$, using supplementary variable technique for the first time and derived the results for $M / G / 1$ WV queues (Li et al. (2009)) and for $M^X / M / 1$ multiple and single working vacation queueing models (Julia Rose Mary and Afthab Begum (2010)) as particular cases. In the present chapter, we consider a Non-Markovian bulk service queueing model with multiple working vacation.

There are situations, particularly in transportation systems, where the service provided is such that a group (batch) of customers can be served simultaneously. The theory of batch service queues originated with the works of Bailey (1954). He considered a queue with Poisson arrival and fixed size service. Later many authors have investigated a variety of extensions of the basic model. The bulk service rule introduced by Neuts (1967) is the most general one and this has been further investigated by Medhi (1984), Borthakur et al. (1987). Julia Rose Mary and Afthab Begum (2009) have analysed the Markovian $M / M(a, b) / 1$ queueing model under multiple working vacation and derived the steady-state probability distribution and the mean queue length for the model.
In some practical situations such as production systems and distribution systems, the input to the queueing system may not be a Poisson process so that a more general arrival process should be used. Baba (2005) has analysed GI / M / 1 queueing system under multiple working vacation as an extension of M / M / 1 / WV model introduced by Servi and Finn (2002). The model being considered in this chapter generalizes the work of Baba (2005). It is assumed that the interarrival times form an independently identically distributed sequence of random variables having a general distribution function and the customers are served in batches following the General Bulk Service Rule (GBSR) introduced by Neuts (1967).

The results of (i) GI / M / 1 MWV queueing model (Baba, 2005) and (ii) M / M(a, b) / 1 MWV queueing model (Julia Rose Mary and Afthab Begum (2009)) are derived as special cases. It is also proved that when the arrival pattern is a Markov process, then the steady-state probability distribution at pre-arrival epoch and arbitrary epoch coincide.

7.1 MATHEMATICAL ANALYSIS OF GI / M(a,b) / 1 / MWV QUEUE

7.1.1 Model Description

In this chapter, we analyse the queueing system GI / M(a, b) / 1 under multiple working vacation.

Arrival Pattern

It is assumed that the interarrival times (A) form an independently identically distributed sequence of random variables having a general distribution function \( A(t) = \Pr(A \leq t) \).

Regular Busy Period

The server processes the customers in batches according to the GBSR introduced by Neuts (1967). According to this rule, the server starts service only when a minimum of \( a \) customers are present in the system. If the server after a service completion finds \( a \) or more but at most \( b \) customers present in the system, then he takes them all in a batch for service. If the server finds more than \( b \) customers, then he takes in the batch the first \( b \) customers for
service, while others wait. Thus each batch for service contains a minimum of ‘a’ units and a maximum of b units. This rule is referred as General Bulk Service Rule. The service time of batches of size x \((a \leq x \leq b)\) is assumed to be an independently, identically distributed random variable with exponential distribution of parameter \(\mu_b\).

**Working Vacation Period**

Whenever the server completes a service and finds less than ‘a’ customers in the queue then he begins a vacation which is an exponentially distributed random variable \(V\) with parameter \(\eta\). After completing a vacation, if the system length is still less than ‘a’, then he takes another vacation and the vacations are repeated until the server finds at least ‘a’ customers in the queue i.e., multiple vacation policy is adopted.

Suppose, during the vacation if the queue size becomes at least ‘a’ then the server starts a service under the GBSR mentioned earlier but with service rate \(\mu_v\) which is strictly less than the regular service rate \(\mu_b\). When the vacation ends, the server switches his service rate from \(\mu_v\) to \(\mu_b\). It is assumed that, the size of service batch that being served remains unchanged when the server enters into the regular busy period.

In either case (working vacation period or regular busy period), the service rates are assumed to be independent of the size of the batch in service \((a \leq x \leq b)\).

This queueing model is denoted by \(GI/M(a, b)/1/MWV\). The queueing system is formulated as an embedded two dimensional Markov chain by choosing arrival epochs as embedded points. The steady-state distribution for the number of customers in the queue at arrival epochs is derived by analyzing the embedded Markov chain defined. Using the theory of semi Markov process, the steady-state distribution for the number of customers in the queue at arbitrary epochs is also derived.
Embedded Markov Chain of Queue Length

Case 1: Pre-arrival Epoch

Let $t_n$, $n = 1, 2, \ldots$ ($t_0 = 0$) be the arrival epoch at which the $n^{th}$ customer arrives. The system is examined at time $(t_n - 0)$. The interarrival times $\{T_n, n \geq 1\}$ are independent and identically distributed with a general distribution function denoted by $A(t)$ with mean $1/\lambda$.

The LST of $A(t)$ is given by $A^{\ast}(\theta) = \int_0^\infty e^{-\theta t} dA(t)$.

The working vacation times, the service times during regular service period and the service times during working vacation are all exponentially distributed with rates $\eta, \mu_b$ and $\mu_v$ respectively.

Let $W(t)$ denote the number of customers in the queue at time $t$, $W_n = W(t_n - 0)$ and

$$J_n = \begin{cases} 0, & \text{if the } n^{th} \text{ arrival occurs during idle vacation period} \\ 1, & \text{if the } n^{th} \text{ arrival occurs during working vacation period} \\ 2, & \text{if the } n^{th} \text{ arrival occurs during regular busy period} \end{cases}$$

Since, the working vacation times, the service times during regular and working vacation are all exponentially distributed, the process $\{(W_n, J_n), n \geq 1\}$ is an embedded Markov chain with state space

$$\Omega = \{(n, j) ; n \geq 0, j = 1, 2\} \cup \{(n, 0) ; 0 \leq n \leq a - 1\}.$$

The steady-state queue size probabilities are defined by

$$R_n = \lim_{k \to \infty} \Pr(W(t_k - 0) = n, J_k = 0), \quad 0 \leq n \leq a - 1$$

$$Q_n = \lim_{k \to \infty} \Pr(W(t_k - 0) = n, J_k = 1), \quad n \geq 0$$

$$P_n = \lim_{k \to \infty} \Pr(W(t_k - 0) = n, J_k = 2), \quad n \geq 0.$$

Then $R_n, Q_n$ and $P_n$ respectively denote the probability that the queue contains $n$ customers and the server is idle in vacation state, is busy in vacation state and regular busy state at pre arrival epochs. During idle
vacation period, the number of customers in the system and queue are the same, whereas in working vacation period and in regular busy period n denotes the number of customers in the queue and the system will contain (n + k), (a ≤ k ≤ b) customers.

To derive the queue size equations satisfied by the steady-state probabilities, the following probabilities are introduced.

Let \( b_k \) denote the probability that \( k \) batches are served at regular service rate \( \mu_b \) during an interarrival time. Then

\[
b_k = \int_0^\infty e^{-\mu_b t} \frac{(\mu_b t)^k}{k!} dA(t), \quad k \geq 0
\]

(7.1)

and

\[
\sum_{k=0}^\infty b_k z^{kb} = B(z^b) = A^*(\mu_b (1-z^b))
\]

(7.2)

Let \( c_k \) denote the probability that, the working vacation time is greater than an interarrival time and \( k \) batches are served at rate \( \mu_v \) during an interarrival time.

Then

\[
c_k = \int_0^\infty e^{-\eta t} e^{-\mu_v t} \frac{(\mu_v t)^k}{k!} dA(t), \quad k \geq 0
\]

(7.3)

and

\[
\sum_{k=0}^\infty c_k z^{kb} = C(z^b) = A^*(\eta + \mu_v (1-z^b))
\]

(7.4)

Let \( d_k \) denote the probability that, the server returns from vacation in an interarrival time and \( k \) service completions occur in an interarrival time. Then

\[
d_k = \int_0^\infty \sum_{i=0}^k \left[ \int_0^t \eta e^{-\eta x} \frac{(\mu_v x)^i}{i!} e^{-\mu_v x} \frac{(\mu_b(t-x))^{k-i}}{(k-i)!} e^{-\mu_b(t-x)} dx \right] dA(t), \quad k \geq 0
\]

(7.5)

i.e., \( k \) services in an interarrival time can occur in such a way that, \( i \) (0 ≤ i ≤ k) service completions occur at rate \( \mu_v \) (till the server returns from vacation) and the remaining \( (k-i) \) service completions occur at rate \( \mu_b \)

and

\[
\sum_{k=0}^\infty d_k z^{kb} = D(z^b) = \frac{\eta [A^*(\mu_b (1-z^b)) - A^*(\eta + \mu_v (1-z^b))]}{\eta + (\mu_v - \mu_b) (1-z^b)}
\]

(7.6)
7.1.2 Steady-State Equations

The steady-state queue size equations at pre-arrival epochs, are obtained by noting the transitions between the states of the Markov chain and are given by:

**Working Vacation Period**

\[
Q_n = \sum_{k=0}^{\infty} Q_{k|b+n-1} c_k, \quad n \geq 1 \tag{7.7}
\]

\[
Q_0 = \sum_{k=1}^{b-1} \sum_{j=a-1}^{\infty} Q_{(k-1)j+b} c_k + R_{a-1} c_0 \tag{7.8}
\]

**Regular Busy Period**

\[
P_n = \sum_{k=0}^{\infty} P_{kb+n-1} b_k + \sum_{k=0}^{\infty} Q_{kb+n-1} d_k, \quad n \geq 1 \tag{7.9}
\]

\[
P_0 = \sum_{k=1}^{b-1} \sum_{j=a-1}^{\infty} P_{(k-1)j+b} b_k + \sum_{k=1}^{\infty} \sum_{j=a-1}^{b-1} Q_{(k-1)j+b} d_k + R_{a-1} d_0 \tag{7.10}
\]

**Idle Vacation Period**

\[
R_n = R_{n-1} + \sum_{k=0}^{\infty} Q_{kb+n-1} (1 - \sum_{i=0}^{k} (c_i + d_i)) + \sum_{k=0}^{\infty} P_{kb+n-1} (1 - \sum_{i=0}^{k} b_i),
\]

\[1 \leq n \leq a-1 \tag{7.11}\]

\[
R_0 = \sum_{k=1}^{b-1} \sum_{j=a-1}^{\infty} Q_{(k-1)j+b} (1 - \sum_{i=0}^{k} (c_i + d_i)) + \sum_{k=1}^{\infty} \sum_{j=a-1}^{b-1} P_{(k-1)j+b} (1 - \sum_{i=0}^{k} b_i) + R_{a-1} (1 - (d_0 + c_0)) \tag{7.12}
\]

7.1.3 Steady-State Solutions

Let \(E\) denote the forward displacement operator. Then \(E(P_n) = P_{n+1}\) and \(E(Q_n) = Q_{n+1}\). The equations (7.7) and (7.9) can be respectively written as:

\[
\left(E - \sum_{k=0}^{\infty} c_k E^{kb}\right) Q_n = 0, \quad n \geq 0 \tag{7.13}
\]

and

\[
\left(E - \sum_{k=0}^{\infty} b_k E^{kb}\right) P_n = \sum_{k=0}^{\infty} Q_{kb+n} d_k, \quad n \geq 0 \tag{7.14}
\]
The characteristic equation $z = C(z^b) = A^*(\eta + \mu_v (1-z^b))$, $\eta > 0$ of the homogeneous difference equation (7.13) has a unique root $r_1$ inside $(0, 1)$.

For $\psi(z) = A^*(\eta + \mu_v (1-z^b))$, satisfies the inequality

$0 < \psi(0) = A^*(\eta + \mu_v) < \psi(1) = A^*(\eta) < 1$, and for $0 < z < 1$,

$\psi'(z) = b \mu_b \int_0^\infty t e^{-(\eta + \mu_b (1-z^b))t} dA(t) > 0$

$\psi''(z) = (b \mu_b)^2 \int_0^\infty t^2 e^{-(\eta + \mu_b (1-z^b))t} dA(t) > 0$

Therefore, $r_1 = A^*(\eta + \mu_v (1-r_1^b))$ with $0 < r_1 < 1$ (Baba (2005)) (7.15)

Hence the homogeneous difference equation (7.13) has solution

$Q_n = r_1^n Q_0, \quad n \geq 0$ (7.16)

The characteristic equation of the non-homogeneous difference equation (7.14) is $z = B(z^b) = A^*(\mu_v (1-z^b))$ and $B(z^b)$ is the pgf of $b_k$'s with $B(1) = 1$. Hence following the arguments of Gross and Harris (1998), the characteristic equation $z = B(z^b)$ has a unique root in $(0, 1)$, if $B'(1) > 1$, i.e.,

$\rho_b = \frac{\lambda}{b \mu_b} < 1$. Thus under the condition $\rho_b < 1$, the solution of the non-homogeneous equation (7.14) is given by

$P_n = \left[ A_d r^n + \left( \sum_{k=0}^\infty r_k^b d_k \right) \frac{r_1^n}{(r_1 - B(r_1^b))} \right] Q_0, \quad r_1 \neq r$ (7.17)

i.e., $P_n = (A_d r^n + B_d r_1^n) Q_0, \quad n \geq 0$ (7.18)

where $B_d = \frac{D(r_1^b)}{(r_1 - B(r_1^b))}$

Using equations (7.2), (7.6) and (7.15), it follows that

$B_d = \frac{-\eta}{\eta + (\mu_v - \mu_b)(1-r_1^b)}$ (7.19)

To find the remaining probabilities, the equations (7.11) and (7.12) are used.
Substituting for $Q_n$’s and $P_n$’s from equations (7.16) and (7.18), equation (7.11) implies,

$$R_n = R_{n-1} + \sum_{k=0}^{\infty} r_1^{kb+n-1} \left(1 - \sum_{i=0}^{k} (c_i + d_i)\right) + \sum_{k=0}^{\infty} \left(A_d r_1^{kb+n-1} + k r_1^{kb+n-1}\right) \left(1 - \sum_{i=0}^{k} b_i\right)$$

Using the identity $\sum_{k=0}^{\infty} x^{kb} \sum_{i=0}^{k} u_i = \frac{\sum_{k=0}^{\infty} u_k x^{kb}}{1-x^b}$, the right hand side of the above equation can be simplified as

$$R_n = R_{n-1} + r_1^{n-1} \left(1 - C(r_1^b) - D(r_1^b)\right) + k r_1^{n-1} \left(1 - B(r_1^b)\right) + A_d r_1^{n-1} \left(1 - B(r_1^b)\right)$$

Since $C(r_1^b) = r_1$, $B(r_1^b) = r$ and $D(r_1^b) = B_d(r_1 - B(r_1^b))$, $R_n$ can be further simplified as,

$$R_n = R_{n-1} + \left[A_d (1-r) r_1^{n-1} + \left(1 - r_1\right) (B_d + 1) r_1^{n-1}\right] Q_0, \quad 1 \leq n \leq a-1 \quad (7.20)$$

By adding equations (7.8), (7.10) and (7.12), we get

$$Q_0 + P_0 + R_0 = \sum_{k=1}^{a} \sum_{j=a-1}^{b-1} Q_{(k-1)b+j} \left(1 - \sum_{i=0}^{k-1} (c_i + d_i)\right) + \sum_{k=1}^{a} \sum_{j=a-1}^{b-1} P_{(k-1)b+j} \left(1 - \sum_{i=0}^{k-1} b_i\right) + R_{a-1}$$

Substituting for $Q_n$’s and $P_n$’s, it is found after simplification that

$$R_0 = R_{a-1} + \left[A_d \left(r_1^{a-1} - 1\right) + (B_d + 1) \left(r_1^{a-1} - 1\right)\right] Q_0 \quad (7.21)$$

Summing equations (7.20) and (7.21) over 0 to $n$,

$$R_n = R_{a-1} + \left[A_d \left(r_1^{a-1} - r_1^n\right) + (B_d + 1) \left(r_1^{a-1} - r_1^n\right)\right] Q_n, \quad 0 \leq n \leq a-1 \quad (7.22)$$

Equation (7.8) implies, $R_{a-1} c_0 = Q_0 - \sum_{k=1}^{\infty} \sum_{j=a-1}^{b-1} Q_{(k-1)b+j} c_k$, and substituting for $Q_n$’s from equation (7.16),

$$R_{a-1} = \left[\frac{r_1^b - r_1^a}{c_0 r_1^b (1-r_1)} + \frac{r_1^{a-1} - r_1^b}{r_1^b (1-r_1)}\right] Q_0 \quad (7.23)$$
Substituting for $R_{a-1}$, equation (7.22) is simplified as,

$$R_n = \left[ A_d \frac{r^{a-1} - r^n}{(1-r^n)} + (B_d + 1) \left( \frac{r^{a-1} - r^n}{1-r^n} \right) + \frac{1}{r_t^b (1-r_t)} \left( \frac{r_t^b - r_t^a}{c_0} + r_{t-1}^a - r_t^b \right) \right] Q_0$$

for $0 \leq n \leq a-1$ \hspace{1cm} (7.24)

Thus the steady-state queue size probabilities are expressed in terms of $Q_0$ and the constant $A_d$.

Using equation (7.10), $A_d$ can be calculated as

$$A_d \left[ \frac{r^a - r^b}{r^b (1-r)} + b_0 \left( \frac{r^a - r^n}{(1-r^n)} \right) + B_d \left( \frac{r^a - r^n}{(1-r^n)} + b_0 \right) \right] + d_0 \frac{(r^b - r^a)}{c_0 r_t^b (1-r_t)} = 0$$

i.e., $A_d f(r) + B_d f(r_t) = \frac{d_0}{c_0} \left( \frac{(r_t^a - r_t^b)}{r_t^b (1-r_t)} \right)$ \hspace{1cm} (7.25)

where $f(x) = \frac{x^a - x^b}{x^b (1-x)} + b_0 \frac{(x^b - x^{a-1})}{x^b (1-x)}$

By using the normalizing condition $\sum_{n=0}^{\infty} P_n + \sum_{n=0}^{\infty} Q_n + \sum_{n=0}^{a-1} R_n = 1$, $Q_0$ can be calculated as,

$$Q_0^{-1} = A_d g(r) + (B_d + 1) g(r_t) + \frac{a}{r_t^b (1-r_t)} \left( \frac{r_t^b - r_t^a}{c_0} + (r_t^{a-1} - r_t^b) \right)$$

where $g(x) = \frac{1}{(1-x^b)} \left( \frac{x^a - x^b}{1-x} + ax^{a-1} \right)$ \hspace{1cm} (7.26)

Thus the steady-state queue size probabilities at arrival epochs are given by

$$Q_n = r_t^n Q_0, \hspace{1cm} n \geq 0$$

$$P_n = (A_d r_t^n + B_d r_t^n) Q_0, \hspace{1cm} n \geq 0$$

$$R_n = \left[ A_d h_n(r) + (B_d + 1) h_n(r_t) + \frac{1}{r_t^b (1-r_t)} \left( \frac{r_t^b - r_t^a}{c_0} + r_t^{a-1} - r_t^b \right) \right] Q_0,$$

for $0 \leq n \leq a-1$ \hspace{1cm} (7.28.3)
where \( h_n(x) = \frac{x^{a-1} - x^n}{1 - x} \), \( A_d f(r) + B_d f(r_1) = \frac{d_0}{c_0} \left( \frac{r_1^a - r_1^b}{r_1^b (1 - r_1)} \right) \) with

\[
f(x) = \frac{x^a - x^b}{x^b (1 - x)} + b_0 \frac{(x^b - x^{a-1})}{x^b (1 - x)} \quad \text{and} \quad B_d = \frac{-\eta}{\eta + (\mu_v - \mu_b) (1 - r_1^b)}
\]

and \( Q_0 \) is given by equation (7.26).

### 7.1.4 Mean Queue Length at Pre-arrival Epoch

The mean queue length \( L_q \) of the model, can be calculated by using equations (7.28.1) to (7.28.3).

\[
L_q = \sum_{n=0}^{\infty} n Q_n + \sum_{n=0}^{\infty} n P_n + \sum_{n=0}^{a-1} n R_n
\]

After simplification, it is found that

\[
L_q = \left[ A_d H(r) + (B_d + 1) H(r_1) + \frac{a(a-1)}{2r_1^b (1 - r_1)} \left( \frac{r_1^b - r_1^a}{c_0} + r_1^{a-1} - r_1^b \right) \right] Q_0
\]

where \( H(x) = \frac{x}{(1-x)^2} + \frac{1}{(1-x^b)} \left[ \frac{a(a-1)}{2} x^{a-1} + ax^a (1-x) - x (1-x^a) \right] \)

### 7.1.5 Performance Measures at Pre-Arrival Epochs

Let \( P_v \), \( P_I \) and \( P_{\text{busy}} \) denote that the server is busy in vacation, idle in vacation and in regular busy state respectively at pre-arrival epochs. Then,

\[
(P_v) = \sum_{n=0}^{a} r_1^n Q_0 = \frac{Q_0}{(1-r_1)}
\]

\[
(P_I) = \sum_{n=0}^{a-1} R_n
\]

\[
= \left[ \frac{A_d r_1^{a-1} a + (B_d + 1) r_1^{a-1} a}{(1-r_1^b)} + \frac{a}{(1-r_1^b) (1-r_1)} \left( \frac{r_1^b - r_1^a}{c_0} + r_1^{a-1} - r_1^b \right) \right] Q_0
\]

\[
- \frac{A_d (1-r_1^a)}{(1-r_1^b) (1-r)} - \frac{(B_d + 1) (1-r_1^a)}{(1-r_1^a) (1-r_1)} Q_0
\]

\[
(P_{\text{busy}}) = \sum_{n=0}^{\infty} (A_d r_1^n + B_d r_1^n) Q_0 = \left( \frac{A}{(1-r)} + \frac{B_d}{(1-r_1)} \right) Q_0
\]
Case 2 : Random Epoch

To obtain the limiting probabilities of queue size at random epochs, the system is examined at some time \( t \), preceding an arrival epoch. Then using the relation between the two sequences, \( (W(t), J(t)), (t_n \leq t < t_{n+1}) \) and \( (W_n, J_n), (n = 0, 1, 2, \ldots) \), the steady-state equations satisfied by the steady-state queue size probabilities

\[
\lim_{t \to \infty} \Pr(W(t) = n, J(t) = (0,1,2))
\]

are obtained:

**Working Vacation Period**

\[
Q^*_n = \sum_{k=0}^{\infty} Q_{kb+n-1} c_k^*, \quad n \geq 1 \tag{7.29}
\]

\[
Q^*_0 = \sum_{k=1}^{b-1} \sum_{j=a-1}^{b-1} Q_{(k-1)b+j} c_k^* + R_{a-1} c_0^* \tag{7.30}
\]

**Regular Busy Period**

\[
P^*_n = \sum_{k=0}^{\infty} P_{kb+n-1} b_k^* + \sum_{k=0}^{\infty} Q_{kb+n-1} d_k^*, \quad n \geq 1 \tag{7.31}
\]

\[
P^*_0 = \sum_{k=1}^{b-1} \sum_{j=a-1}^{b-1} P_{(k-1)b+j} b_k^* + \sum_{k=1}^{b-1} \sum_{j=a-1}^{b-1} Q_{(k-1)b+j} d_k^* + R_{a-1} d_0^* \tag{7.32}
\]

**Idle Vacation Period**

\[
R^*_n = R_{n-1} + \sum_{k=0}^{\infty} Q_{kb+n-1} \left( 1 - \sum_{i=0}^{k} (c_i^* + d_i^*) \right)
+ \sum_{k=0}^{\infty} P_{kb+n-1} \left( 1 - \sum_{i=0}^{k} b_i^* \right), \quad 1 \leq n \leq a-1 \tag{7.33}
\]

\[
R^*_0 = \sum_{k=1}^{b-1} \sum_{j=a-1}^{b-1} Q_{(k-1)b+j} \left( 1 - \sum_{i=0}^{k} b_i^* \right)
+ \sum_{k=1}^{b-1} \sum_{j=a-1}^{b-1} P_{(k-1)b+j} \left( 1 - \sum_{i=0}^{k} b_i^* \right) + R_{a-1} (1 - c_0^* - d_0^*) \tag{7.34}
\]

where \( b_k^*, c_k^* \) and \( d_k^* \) are the corresponding quantities for \( b_k, c_k \) and \( d_k \) respectively, where the interarrival time \( A \) is replaced by \( T \) – the period of time between a random epoch and the preceding arrival epoch.
The distribution function \( F_T(t) \) and the density function \( f_T(t) \) of \( T \) are given by

\[ F_T(t) = \lambda \int_0^t (1 - A(x)) \, dx, \quad f_T(t) = \lambda (1 - A(t)) \, dt \]

Thus we have,

\[ b_k^* = \int_0^\infty \frac{(\mu_b \, t)^k}{k!} e^{-\mu_b \, t} \lambda (1 - A(t)) \, dt, \quad c_k^* = \int_0^\infty e^{-\eta \, t} \frac{(\mu_v \, t)^k}{k!} e^{-\mu_v \, t} \lambda (1 - A(t)) \, dt \]

\[ d_k^* = \int_0^\infty \sum_{r=0}^k \left( \int_0^t \eta \frac{e^{-\eta x} (\mu_v \, x)^r}{r!} e^{-\mu_v \, x} \frac{(\mu_b \, (t-x))^{k-r}}{(k-r)!} e^{-\mu_b \, (1-x)} \, dx \right) \lambda (1 - A(t)) \, dt \]

i.e., \( b_k^* \) denotes the probability that \( k \) customers are served at regular service rate \( \mu_b \) in an interval of time \( T \) and similarly \( c_k^* \) and \( d_k^* \) can be interpreted.

And it is easy to prove that,

\[ \sum_{k=0}^\infty b_k^* \, z^{kb} = \frac{\lambda (1 - A^*(\mu_b(1-z^b)))}{\mu_b(1-z^b)}, \quad b_0^* = \frac{\lambda (1 - A^*(\mu_b))}{\mu_b} \]

\[ \sum_{k=0}^\infty c_k^* \, z^{kb} = \frac{\lambda (1 - A^*(\eta + \mu_v(1-z^b)))}{\eta + \mu_v(1-z^b)}, \quad c_0^* = \frac{\lambda (1 - A^*(\eta + \mu_v))}{\eta + \mu_v} \]

\[ \sum_{k=0}^\infty d_k^* \, z^{kb} = \frac{\eta \lambda}{\eta + (\mu_v - \mu_b)(1-z^b)} \left[ \frac{1 - A^*(\mu_b(1-z^b))}{\mu_b(1-z^b)} - \frac{1 - A^*(\eta + \mu_v(1-z^b))}{\eta + \mu_v(1-z^b)} \right] \]

and \( d_0^* = \frac{\eta \lambda}{\eta + (\mu_v - \mu_b)} \left[ \frac{1 - A^*(\mu_b)}{\mu_b} - \frac{1 - A^*(\eta + \mu_v)}{\eta + \mu_v} \right] \)

Substituting for the steady-state probabilities at pre-arrival epochs from equations (7.28.1) to (7.28.3), equations (7.29) to (7.34) give the limiting probability distribution at arbitrary epochs.

Equation (7.29) implies

\[ Q_n^* = \sum_{k=0}^\infty r_t^{kb+n-1} c_k^* Q_0 \]

\[ Q_n^* = (r_t^{n-1} Q_0) \left( \frac{\lambda}{\eta + \mu_v (1-r_t^b)} \right) \left( 1 - A^*(\eta + \mu_v (1-r_t^b)) \right) \]

\[ Q_n^* = \frac{\lambda (1-r_1) r_t^{n-1} Q_0}{\eta + \mu_v (1-r_t^b)}, \quad n \geq 1 \quad (7.35) \]
\( P^*_n \) can be calculated from equation (7.31) using equations (7.28.1) and (7.28.2) as,

\[
P^*_n = \left[ \sum_{k=0}^{\infty} \left( A_d r^k b + B_d r^k b + \sum_{k=0}^{\infty} r^k b k^* \right) \right] Q_0
\]

Similarly, equations (7.30), (7.32) and (7.34) give the values for \( B^*_n \).

Substituting the value of \( R^*_n \) in equation (7.33),

\[
R^*_n = R_{n-1} + \frac{A_d Q_0 r^{n-1}}{(1-r^b)} \left( 1 - \frac{\lambda (1-r)}{\mu_b (1-r^b)} + \frac{(B_d + 1) r^{n-1} Q_0}{(1-r^b)} \right)
\]

Substituting the value of \( R_n \) in the above equation it is found after algebraic manipulation that,

\[
R^*_n = \left[ A_d h^n_* (r, \mu_b (1-r^b)) + (B_d + 1) h^n_* (r, \eta + \mu_v (1-r^b)) \right]
\]

\[
\frac{1}{r^b (1-r^b)} + \frac{1}{r^b (1-r^b)} \left[ \frac{r^b - r^b}{c_0} + r^b - r^b \right]
\]

where \( h^n_* (x, y) = \frac{1}{(1-x^b)} \left( x^{n-1} - \frac{\lambda (1-x) x^{n-1}}{y} \right), \quad 1 \leq n \leq a-1 \)

Similarly, equations (7.30), (7.32) and (7.34) give the values for \( Q^*_0, P^*_0 \) and \( R^*_0 \) as

\[
Q^*_0 = \left[ \frac{r^b (1-r^b)}{r^b (1-r^b)} \left( \frac{\lambda (1-r^b)}{\mu_b (1-r^b)} + \frac{r^b - r^b}{c_0} \right) \right] Q_0
\]

\[
P^*_0 = \frac{Q_0 A_d (r^a - r^b)}{(1-r^b) r^b} \left[ \frac{\lambda (1-r)}{\mu_b (1-r^b)} - b^* \right] + \frac{B_d Q_0 (r^a - r^b)}{(1-r^b) r^b} \left[ \frac{\lambda (1-r^b)}{\eta + \mu_v (1-r^b)} - b^* \right] + \frac{d^*_0}{c_0} \left[ \frac{r^b - r^b}{c_0} \right]
\]
\[ R_0^* = \left[ A_d h_0^* (r, \mu_b (1 - r_b)) + (B_d + 1) h_0^* (r_1, \eta + \mu_v (1 - r_1^b)) \right. \]
\[ \left. + \frac{1}{r_1^b (1 - r_1)} \left( \frac{r_b^b - r_1^b}{c_0} (1 - c_0^* - d_0^*) + (r_1^{a-1} - r_1^b) (1 - b_0^*) \right) \right] Q_0 \]  
(7.38.3)

where \( h_0^*(x, y) = \frac{x^{a-1} - x^b}{(1 - x) x^b} - \frac{\lambda (1 - x)}{(1 - x^b) y} - b_0^* \)

and the other queue size probabilities at arbitrary epochs are

\[ Q_n^* = \frac{\lambda (1 - r_1) r_1^{n-1} Q_0}{\eta + \mu_v (1 - r_1^b)}, \quad n \geq 1 \]  
(7.38.4)

\[ P_n^* = \left[ A_d r_1^{n-1} \frac{\lambda (1 - r)}{\mu_b (1 - r_b)} + B_d r_1^{n-1} \frac{\lambda (1 - r_1)}{\eta + \mu_v (1 - r_1^b)} \right] Q_0, \quad n \geq 1 \]  
(7.38.5)

\[ R_n^* = \left[ A_d h_n^* (r, \mu_b (1 - r_b)) + (B_d + 1) h_n^* (r_1, \eta + \mu_v (1 - r_1^b)) \right. \]
\[ \left. + \frac{1}{r_1^b (1 - r_1)} \left( \frac{r_b^b - r_1^b}{c_0} + r_1^{a-1} - r_1^b \right) \right] Q_0, \quad 1 \leq n \leq a - 1 \]  
(7.38.6)

### 7.1.6 Mean Queue Length at Arbitrary Epoch

The mean queue length \( L_q^* \) of the model at arbitrary epoch can be calculated by using equations (7.38.4 to 7.38.6).

\[ L_q^* = \sum_{n=0}^{\infty} n Q_n^* + \sum_{n=0}^{\infty} n P_n^* + \sum_{n=0}^{a-1} n R_n^* \quad \text{implies} \]

\[ L_q^* = \left[ A_d H(r, \mu_b (1 - r_b)) + (B_d + 1) H(r_1, \eta + \mu_v (1 - r_1^b)) \right. \]
\[ \left. + \left( \frac{r_b^b - r_1^b}{c_0} + (r_1^{a-1} - r_1^b) \right) \frac{a(a-1)}{2r_1^b(1 - r_1)} \right] Q_0 \]

where \( H(x, y) = \frac{\lambda}{(1 - x) y} + \frac{1}{(1 - x^b)} \left( \frac{x^{a-1} a(a-1)}{2} + \frac{\lambda (a x^{a-1} (1 - x) - (1 - x^a))}{y (1 - x)} \right) \)

### 7.2 PARTICULAR CASES

**I.** \( M / M(a, b) / 1 / MWV \)

It is verified that if the interarrival time follows exponential distribution, then the limiting probabilities at arbitrary epochs ((7.38.1) to (7.38.6)) and at
pre-arrival epochs ((7.28.1) to (7.28.3)) coincide and give the steady-state queue size probabilities of the Markovian M / M(a, b) / 1 / MWV. To prove this, it is enough to note the following identifications.

When $A(t) = (1 - e^{-\lambda t})$, $A^*(0) = \frac{\lambda}{\lambda + \theta}$, and $r$ and $r_1$ respectively satisfy the equations

$\frac{\lambda}{\mu_b (1-r^b)} = \frac{r}{1-r}$ \quad and \quad $\frac{\lambda}{\eta + \mu_v (1-r_1^b)} = \frac{r_1}{1-r_1}$

And also for every $k$, $b_k^* = b_k = \left(\frac{\lambda}{\lambda + \mu_b}\right) \left(\frac{\mu_b}{\mu_b + \lambda}\right)^k$

$c_k^* = c_k = \left(\frac{\lambda}{\lambda + \mu_v + \eta}\right) \left(\frac{\mu_v}{\lambda + \mu_v + \eta}\right)^k$ and $d_0^* = d_0 = \left(\frac{\lambda \eta}{(\lambda + \mu_b)(\lambda + \eta + \mu_v)}\right)$.

Thus the steady-state queue size probabilities of M / M(a, b) / 1 / MWV queueing model are given by

$Q_n = r_1^n Q_0$, \quad $n \geq 0$

$P_n = (A r^n + B r_1^n) Q_0$, \quad $n \geq 0$

where

$A = \frac{(1-r)}{\mu_b (1-r^a)} \left[ \frac{\eta}{(1-r)} - \frac{B \mu_b (1-r_1^a)}{1-r_1} \right]$

$B = \frac{\eta r_1}{\mu_b r_1 (1-r^b) + \lambda (r_1 - 1)}$ \quad and \quad $R_n = \left[ \frac{\mu_b}{\lambda} \left(\frac{A (1-r_1^{n+1})}{(1-r)} + \frac{B (1-r_1^{n+1})}{(1-r_1)} \right) + \frac{\mu_v}{\lambda} \left(\frac{1-r_1^{n+1}}{(1-r_1)} \right) \right] Q_0$, \quad $0 \leq n \leq a-1$

$Q_0^{-1} = F(r_1, \mu_v) + A F(r, \mu_b) + B F(r_1, \mu_b)$

where $F(x, y) = \frac{1}{(1-x)} \left[ 1 + \frac{y}{\lambda} \left(\frac{a - x (1-x^a)}{1-x} \right) \right]$.

The expected queue length is given by

$L_q = [A H(r, \mu_b) + B H (r_1, \mu_b) + H (r_1, \mu_v)] Q_0$

where

$H(x, y) = \frac{x}{(1-x)^2} + \frac{y}{\lambda (1-x)} \left[ \frac{a(a-1)}{2} + \frac{ax^{a+1} (1-x) - x^2 (1-x^a)}{(1-x)^2} \right]$.

These results coincide with the corresponding results of Julia Rose Mary and Afthab Begum (2009).
II. GI / M / 1 / MWV

When \( a = b = 1 \), the steady-state queue size probabilities coincide with the corresponding results of Baba (2005).

It is found from equations (7.19), (7.25) and (7.26) that
\[
\frac{A}{r} = -\frac{B_d}{r_1}, \quad -B_d = \frac{\eta}{\eta - (\mu_b - \mu_v)(1-r_1)} = \alpha \quad \text{and} \quad \frac{Q_0}{r_1} = (1-r) \beta
\]
where \( \beta = \frac{\eta - (\mu_b - \mu_v)(1-r_1)}{\eta - (\mu_b - \mu_v)(1-r)} \)

Using these relations, in equations (7.21.1) to (7.21.3) and equations (7.38.1) to (7.38.6), the steady-state queue size probabilities at pre-arrival epochs are given by
\[
Q_n = (1-r) \beta r_1^{n+1}, \quad n \geq 0
\]
\[
P_n = (1-r) \alpha \beta (r_1^{n+1} - r_1^{n+1}), \quad n \geq 0
\]

and the steady-state queue size probabilities at arbitrary epochs are given by
\[
Q_n^* = \frac{\lambda}{\mu_b} Q_0 r_1^{n-1} \left( \frac{1-r_1}{\eta + \mu_v (1-r_1)} \right), \quad n \geq 0
\]
\[
P_n^* = \frac{\lambda}{\mu_b} Q_0 \left[ \frac{A r_1^{n-1}}{\mu_b} + \frac{B_d r_1^{n-1} (1-r_1)}{\eta + \mu_v (1-r_1)} \right], \quad n \geq 0
\]
and
\[
R_0^* = 1 - \frac{\alpha \lambda \beta}{\mu_b} \frac{(1-\alpha) \lambda (1-r) \beta}{\eta + \mu_v (1-r_1)}
\]

The expected queue size is given by
\[
L_q^* = (1-r) \beta \frac{\lambda}{\mu_b} \left[ \frac{\alpha r}{(1-r) (\eta + \mu_v (1-r_1))} + \frac{(1-\alpha) r_1}{(1-r_1)(\eta + \mu_v (1-r_1))} \right]
\]

and the expected system size is obtained as
\[
L_s^* = L_q^* + (1 - R_0^*)
\]
\[
= (1-r) \beta \frac{\lambda}{\mu_b} \left[ \frac{\alpha}{(1-r)^2} + \frac{1-\alpha}{(1-r_1)(\eta + \mu_v (1-r_1))} \right]
\]
7.3 NUMERICAL ANALYSIS

To demonstrate the influence of the system parameters on (i) the waiting line ($L_q$), (ii) steady state queue size probabilities and (iii) the average number of customers waiting in the queue, when the system is in different states, we consider different distributions for the interarrival time.

The two basic characteristic equations $z = A^*(\eta + \mu_v (1-z^b))$ and $z = A^*(\mu_b (1-z^b))$ having unique roots $r_1$ and $r$ lying inside $(0, 1)$ respectively correspond to $z = \left(\frac{k \lambda}{k \lambda + \eta + \mu_v (1-z^b)}\right)^k$ and $z = \left(\frac{k \lambda}{k \lambda + \mu_b (1-z^b)}\right)^k$ for Erlang-k type ($E_k$) distribution and

$z = e^{-\frac{(\eta + \mu_v (1-z^b))}{\lambda}}$ and $z = e^{-\frac{\mu_b (1-z^b)}{\lambda}}$ for deterministic (D) distribution.

In Table 7.1, the mean queue size both at arbitrary epochs ($L_{arb}$) and at pre-arrival epochs ($L_{arrival}$) are presented for different values of the vacation parameter ($\eta$) and vacation service rate ($\mu_v$) corresponding to different interarrival time distributions (Erlang-k = 1, 3, 5, 10 and deterministic) to know how the expected queue length changes with the parameters. It is shown that

(i) both $L_{arb}$ and $L_{arrival}$ decrease as $\mu_v$ or $\eta$ increases.
(ii) The performance measures at arbitrary epoch and pre-arrival epochs coincide for Markovian interarrival time distribution.
(iii) The smaller values of $\eta$ significantly affect the queue size.
(iv) When $\mu_v = \mu_b$, the queue length of working vacation model and classical non-vacation model coincide.

The graphical representation of the effect of $\mu_v$ and $\eta$ on the mean queue length ($L_{arb}$) for Deterministic and Erlang-3 type interarrival type can be seen in Figures 7.1a and 7.1b respectively.
Figure 7.1a D/M(a, b)/1/MWV

Figure 7.1b E_3/M(a, b)/1/MWV
In Table 7.2, the values of the expected queue size at arbitrary and prearrival epochs are presented for different values of arrival rate $\lambda$ and for different vacation service rates $\mu_v$ for two different values of regular service rate $\mu_b$ for Erlang-3 interarrival distribution. The values show that mean queue size increases with the arrival rate $\lambda$ and decreases as the service rate increases.

**Table 7.2 Mean queue size with respect to $\lambda$ and $\mu_v$ for $\mu_b = 1$ and $\mu_b = 1.5$**

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\mu_v$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$\mu_b = 1$</td>
<td>34.508</td>
<td>24.641</td>
<td>17.238</td>
<td>12.381</td>
<td>9.390</td>
</tr>
<tr>
<td></td>
<td>$\mu_b = 1.5$</td>
<td>34.425</td>
<td>24.478</td>
<td>16.989</td>
<td>12.063</td>
<td>9.033</td>
</tr>
<tr>
<td>6</td>
<td>$\mu_b = 1$</td>
<td>43.918</td>
<td>33.017</td>
<td>24.044</td>
<td>17.459</td>
<td>13.098</td>
</tr>
<tr>
<td></td>
<td>$\mu_b = 1.5$</td>
<td>43.663</td>
<td>32.697</td>
<td>23.638</td>
<td>16.967</td>
<td>12.541</td>
</tr>
<tr>
<td>7</td>
<td>$\mu_b = 1$</td>
<td>53.727</td>
<td>42.085</td>
<td>31.916</td>
<td>23.758</td>
<td>17.862</td>
</tr>
<tr>
<td></td>
<td>$\mu_b = 1.5$</td>
<td>53.096</td>
<td>41.420</td>
<td>31.187</td>
<td>22.950</td>
<td>16.978</td>
</tr>
</tbody>
</table>

The influence of $\lambda$ and $\mu_v$ on the mean queue length at arbitrary epochs is graphically represented in Figures 7.3a and 7.3b for Deterministic and Erlang–3 interarrival time distribution. Table 7.3 gives the data for the graphs.

**Table 7.3 Mean queue size with respect to $\lambda$ and $\mu_v$**

$(\mu_b, \eta, a, b) = (0.9, 0.1, 5, 15)$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\mu_b$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.15</td>
<td>2.106</td>
<td>2.088</td>
<td>2.062</td>
<td>2.047</td>
</tr>
<tr>
<td></td>
<td>9.936</td>
<td>4.462</td>
<td>3.032</td>
<td>2.507</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.22</td>
<td>3.981</td>
<td>3.543</td>
<td>3.252</td>
<td>2.789</td>
</tr>
<tr>
<td></td>
<td>17.411</td>
<td>7.654</td>
<td>4.661</td>
<td>3.485</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.29</td>
<td>4.916</td>
<td>4.573</td>
<td>4.211</td>
<td>4.101</td>
</tr>
<tr>
<td></td>
<td>25.872</td>
<td>12.033</td>
<td>6.910</td>
<td>4.893</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.37</td>
<td>7.881</td>
<td>7.526</td>
<td>7.380</td>
<td>7.063</td>
</tr>
<tr>
<td></td>
<td>34.950</td>
<td>17.713</td>
<td>9.873</td>
<td>6.736</td>
<td></td>
</tr>
</tbody>
</table>
In Table 7.4, we present the computations of the system size probabilities $P_v$, $P_{\text{busy}}$, $P_I$ and the expected queue size $L$ both at prearrival epoch and at departure epoch, for Erlangian–3 interarrival time distribution. The table values show the effect of the parameters on the performance measures. The parameters chosen are $(\eta, \lambda, \mu_b, \mu_v, a, b) = (0.1, 7, 0.9, 0.1, 5, 15)$.

### Table 7.4 Probabilities and Mean queue size for different parametric values

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$P_I$</th>
<th>$P_B$</th>
<th>$P_V$</th>
<th>$L$</th>
<th>$r_1$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$ 0.1</td>
<td>0.011</td>
<td>0.504</td>
<td>0.485</td>
<td>108.391</td>
<td>0.91</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td>0.010</td>
<td>0.504</td>
<td>0.487</td>
<td>108.719</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.034</td>
<td>0.596</td>
<td>0.371</td>
<td>28.348</td>
<td>0.91</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>0.030</td>
<td>0.596</td>
<td>0.375</td>
<td>28.660</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda$ 6</td>
<td>0.028</td>
<td>0.467</td>
<td>0.505</td>
<td>44.219</td>
<td>0.88</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>0.024</td>
<td>0.467</td>
<td>0.508</td>
<td>44.535</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.015</td>
<td>0.613</td>
<td>0.373</td>
<td>65.296</td>
<td>0.93</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
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<td>0.613</td>
<td>0.375</td>
<td>65.620</td>
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</tr>
<tr>
<td>$\mu_v$ 0.3</td>
<td>0.031</td>
<td>0.452</td>
<td>0.517</td>
<td>32.550</td>
<td>0.91</td>
<td>0.97</td>
</tr>
<tr>
<td></td>
<td>0.027</td>
<td>0.452</td>
<td>0.521</td>
<td>32.864</td>
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<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.089</td>
<td>0.317</td>
<td>0.595</td>
<td>11.943</td>
<td>0.93</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>0.079</td>
<td>0.317</td>
<td>0.605</td>
<td>12.223</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_b$ 0.8</td>
<td>0.017</td>
<td>0.606</td>
<td>0.377</td>
<td>55.684</td>
<td>0.93</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>0.015</td>
<td>0.606</td>
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</tr>
<tr>
<td>1.2</td>
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<td>0.568</td>
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<td>0.86</td>
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<tr>
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<td>0.405</td>
<td>0.57</td>
<td>53.550</td>
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<td></td>
</tr>
</tbody>
</table>