Chapter 3

Small Sample Performance of Tests for Coefficient of Variation

3.1 Introduction

Coefficient of variation (c.v.) is a widely used measure of dispersion in scientific investigation, especially in the applied research areas. In the past, several papers have appeared on the comparison of c.v.'s of 'k' normal populations (Gupta and Ma (1996), Bennett (1976)). However, little attention was paid to the tests and confidence intervals of population c.v. from a single normal distribution. The practical instances of this application include the confidence interval for c.v. for the rain fall of a region, yield etc. Another instance of the application is the estimation of a normal mean with known c.v. Several estimators are proposed in the past for this. However, if the partial knowledge regarding c.v. is available, it becomes difficult to use these estimators and tests are required to ascertain the partial knowledge. This has motivated us to derive the explicit test statistics for testing the specified value of the population c.v. when the underlying distribution is normal. These tests are the likelihood ratio (LR), Wald and score tests. Under the null hypothesis, all the three test statistics have an asymptotic standard normal distribution. Simulations are carried out to
compare the type I error rates of these three tests and the results indicate that the LR and score tests maintain the type I error rates for all values of $c_v$ and the Wald test maintains the type I error rate for small values of $c_v (< 1.0)$ and the test becomes liberal as $c_v$ increases. The small sample power comparison indicate that, when the alternatives are two sided, for the left alternatives, the LR test performs better than the Wald and score tests while for the right alternatives, the latter two tests perform better. The investigation is also carried out to find the small sample robustness. The results indicate that the score test maintains the type I error rate for the uniform, logistic, lognormal and gamma distributions whereas the Wald test maintains the type I error rate for lognormal and gamma distributions.

The contents of this chapter are presented in 6 sections. Bartlett-adjusted likelihood ratio (BALR) test statistic is also derived and Section 3.2 presents the expression for the four test statistics. The results for the small sample comparison of type I error rates are given in section 3.3 while section 3.4 deals with the power comparison. Section 3.5 presents the results concerning the small sample robustness when the underlying distributions are uniform, logistic, lognormal and gamma. Conclusions are presented in section 3.6.

### 3.2 Tests for Coefficient of Variation

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) normal random variables with mean $\mu$ and variance $\sigma^2$. Let $c = \sigma/\mu$, denote the $c_v$. Let $C = S/\bar{X}$ denote the sample $c_v$ where $S$ and $\bar{X}$ denote the sample standard deviation and sample mean respectively. The problem of interest is testing $H_0: c = c_0$ vs $H_1: c = c_1$ ($\neq c_0$).

#### 3.2.1 Likelihood Ratio Test

Let $L(\theta, y) = \prod_{i=1}^n f(y_i, \theta)$ denote the likelihood based on ‘$n$’ i.i.d random variables, where $\theta$ is a real parameter. For testing the $H_0: \theta = \theta_0$, the LR test ($\lambda$) is the ratio of the $L(\theta_0, y)$ to $L(\hat{\theta}, y)$, where $\hat{\theta}$ denotes ML estimator of $\theta$. For testing $H_0$, the LR test rejects for small
values of $\lambda$ or equivalently for large values of $-2 \ln \lambda$

Let $\theta$ be vector valued and be partitioned into $(\psi, \tau)$. Consider the hypothesis for testing $H_0: \psi = \psi_0$. Let $\hat{\psi}$ and $\hat{\tau}$ be the unrestricted ML estimator of $\psi$ and $\tau$ and $\hat{\psi}_r$ denote the ML estimator when $\psi = \psi_0$, i.e., the restricted ML estimator of $\tau$ under $H_0$. Then the LR test statistic is given by,

$$-2 \ln \lambda = 2 \left[ \ln L(\hat{\psi}, \hat{\tau}) - \ln L(\psi_0, \hat{\tau}) \right]$$

Under certain regularity conditions, it can be proved that under $H_0$, $-2 \ln \lambda$ is asymptotically distributed as chi-square with degrees of freedom $(d_f) = \text{dim}(\Theta) - \text{dim}(\Theta_{H_0})$, where $d_f$ is the difference between the dimensions of the entire parameter space and the parameter space induced by $H_0$.

Therefore the LR test for testing $H_0: \psi = \psi_0$ is given by,

$$-2 \ln \lambda = 2 \left[ \ln L(\hat{\mu}, \hat{\sigma}^2) - \ln L(\hat{\mu}_{ML}, \hat{\sigma}^2) \right]$$

where $\hat{\mu}$ and $\hat{\sigma}^2$ denote the sample mean $\bar{X}$ and the sample variance $S^2$ respectively, $\hat{\mu}_{ML}$ the restricted ML estimator is obtained by substituting $c = \psi_0$ in (2.4). After simplification, the test statistic is,

$$W_L = n \left\{ \ln \left( \frac{\sigma^2_{\hat{\mu}}}{S^2} \right) + \frac{S^2 + (\bar{X} - \hat{\mu}_{ML})^2}{\sigma^2_{\hat{\mu}} \hat{\sigma}^2} - 1 \right\}$$

Under $H_0$, $\sqrt{W_L}$ is asymptotically distributed as standard normal variate. $H_0$ is rejected if $\sqrt{W_L} > Z_{\alpha/2}$.
3.2.2 Bartlett-adjusted Likelihood Ratio Test

The asymptotic null distribution of the LR test statistic (3.1) is central chi-square with 1 d.f. To improve the chi-square approximation of the LR test statistic the Bartlett correction has been proposed in the literature (Cox and Hinkley (1974)). The correction is obtained so that the first moment of the corrected statistic agrees with the first moment of the chi-square distribution to the order of $O(n^{-2})$

The expected value of the LR test statistic is given by

$$E(-2 \ln \lambda) = 1 + \frac{b}{n} + O(n^{-2})$$

Then the adjusted LR test statistic $\frac{-2 \ln \lambda}{1 + O(\delta/n)}$ has a chi-square distribution to the order of $O(n^{-2})$. Thus, BALR test statistic for testing $H_0 \ c = c_0$ vs $H_1 \ c \neq c_0$ is given by,

$$W_B = \frac{W_L}{1 + \frac{b}{n}}$$

(3.2)

where $b = \left[ \frac{24 c_0^6 + 33 c_0^4 + 12 c_0^2 + 2}{6 (1 + 2 c_0^2)^3} \right]$ and $W_L$ is given in (3.1). Under $H_0$, $\sqrt{W_B}$ is asymptotically distributed as standard normal variate

3.2.3 Wald Test

Let the expected value and variance of sample c v be given by,

$$E(C) = c + O(n^{-1})$$

$$V(C) = \frac{c^2 (1 + 2 c^2)}{2n} + O(n^{-2})$$
Then the Wald test statistic is given by,

\[ W_w = \frac{\sqrt{n}(C - c_0)}{\sqrt{\frac{\bar{c}^2(1 + 2\bar{c}^2)}{2}}} \]  

(3.3)

Under \( H_0 \), \( W_w \) is asymptotically distributed as standard normal variate \( H_0 \) is rejected if \( |W_w| > Z_{\alpha/2} \) and \( Z_{\alpha/2} \) refers to the upper \((\alpha/2)\)th percentile of the standard normal distribution.

### 3.2.4 Score Test

Let \( \theta = (\theta_1, \theta_2, \ldots, \theta_k) \) is vector valued and partitioned into \((\psi, \tau)\) Then, the score test statistic for testing \( \psi = \psi_0 \) is given by,

\[ W_S = U'(\psi_0, \hat{\tau}) \cdot I^{-1}(\psi_0, \hat{\tau}) \cdot U(\psi_0, \hat{\tau}), \]  

(3.4)

where \( U_i(\cdot) = \frac{\partial \log L(\cdot)}{\partial \theta_i} \) is the \( i \)th component of the score vector \( U(\theta) \) and

\[ I(\theta) = I(\theta_1, \theta_2) = E \left( -\frac{\partial^2 \log L(\cdot)}{\partial \theta_i \partial \theta_j} \right) \]

is the Fisher information matrix. Under \( H_0 \), \( W_S \) is asymptotically distributed as central chi-square with the same d.f. as the LR test statistic.

For the present case, the score vector and the Fisher information matrix is given by,

\[
\begin{bmatrix}
\frac{s^2 + (X - \mu)^2}{\sigma^2} - \frac{1}{\mu} \\
\frac{s^2 + (X - \mu)^2}{\sigma^2} - \frac{1}{c}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\frac{2s^2 + 1}{\sigma^2 c^2} & \frac{2}{c \mu} \\
\frac{2}{c \mu} & \frac{2}{c}
\end{bmatrix}
\]

(3.5)
By substituting (3.5) in (3.4) and simplifying we get the score test statistic as,

\[ W_S = \frac{n}{2} \left\{ 1 + \left[ \frac{S^2 + (\bar{X} - \hat{\mu}_{ML})^2}{c_0^2 \hat{\mu}_{ML}^2} \right] - \frac{2S^2}{c_0^2 \hat{\mu}_{ML}^2} \right\} \tag{3.6} \]

Under \( H_0 \), \( \sqrt{W_S} \) asymptotically distributed as standard normal variate.

The Wald test for the c v was first proposed by Rao and Bhatta (1989) while the LR and score tests for testing the equality of population c v for k normal populations were considered by Gupta and Ma (1996).

### 3.3 Small Sample Comparison of Type I Error Rates

In order to check the adequacy of the normal approximation to the null distribution of the four test statistics, a replicated experiment was carried out. 50000 replicated samples of size ‘n’ are generated from a normal distribution with mean \( \mu = 100 \) and variance \( \sigma^2 \) where, \( \sigma^2 = c_0^2 \mu^2 \). The test is carried out for testing various values of c v and they correspond to \( c_0 = 0.01, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.4, 0.5, 0.75, 1.0, 1.5, 2.0, 2.5, 3.0 \). The Type I error rates were estimated by counting the number of times the null hypothesis is rejected for nominal level of significance \( \alpha = 0.05 \). The sample sizes considered are n=20, 40 and 80.

Table 3.1 gives the estimated type I error rates for n=20 and 80 when \( \alpha = 0.05 \). From the table, it follows that the LR and BALR test maintain type I error rates for all values of \( c_0 \). The type I error rates cluster around the nominal level \( \alpha = 0.05 \) for n=40 and 80. The score test is slightly conservative when n=20 and the estimated type I error rates range from 0.0375 to 0.0474. However, the estimated type I error rates converge to a value of 0.05 when \( c_0 \) increases or the sample size increases. In contrast to the above three tests, the results indicate that, although the Wald test maintains the type I error rates for values of c v in the interval \( 0 < c_0 < 1 \), it is stringent for small values of \( c_0 \) and it becomes liberal as \( c_0 \) increases. The estimated type I error rates range from 0.0266 to 0.0742 for the values of c v.
$0.01 \leq \theta_0 \leq 1.0$ and increases to 0.1327 when $c = 2.5$ for $n=20$. For $n=40$ and 80, the same conclusion follows.

### 3.4 Power Comparisons

In order to compare these tests in terms of power, we restrict our attention to the two-sided alternatives. The power of these tests were estimated using samples from a normal distribution with mean $\mu = 100$ and for various values of $c$, lying in the two sides of the interval around $\theta_0$, a specified value under the null hypothesis. The simulation experiment is analogous to the case of the null hypothesis described in the previous section. Figure 31 gives the power functions of the four tests when $n=20$ and $\alpha = 0.05$. The following conclusions emerge:

1. For small and moderate values of $c$ (i.e. $c_0 \leq 0.5$), the four tests perform well when the sample size is moderate to large. The power functions of these tests overlap.

2. For small sample sizes or large values of $c_0$, when the alternatives are to the left side of the null hypothesis, the LR and BALR tests have better power compared to the Wald and score tests even for values of $c$ which are at a moderate distance from $c_0$. For example, when $n=20$, $c_0 = 0.5$ and $c_1$ (the value of $c$ under the alternative hypothesis) = 0.29, the powers of the LR, BALR, Wald and score tests are 0.8640, 0.8585, 0.6837 and 0.6008 respectively.

3. On the contrary, the Wald and score tests have better power for the right alternatives. In this case, the power of the Wald test is higher compared to the score test when $c_0$ is greater than or equal to 0.5. This conclusion is true for $n=40$ and 80.
3.5 Robustness of the Tests

In addition to the power comparison reported in the previous section, we have also investigated the small sample robustness of the four tests derived under normality assumption. The distributions considered for investigation are the same which we have used in chapter 2, namely, the two-parameter uniform, logistic, lognormal and gamma distributions. The parameters of the distributions are chosen such that they coincide with the values of the mean and coefficient of variation for the normal distribution described in section 3.3.

The estimated type I error rates for these distributions are presented in Figure 3.2 for n=20 and \( \alpha=0.05 \). From the figure, it can be concluded that, for the uniform distribution, the LR, BALR and score tests maintain the type I error rates for all values of \( c_0 \) while the Wald test becomes liberal for large values of \( c_0 (> 1.5) \). For the logistic distribution, the Wald test is liberal for all values of \( c_0 \) and the score test maintains the type I error rate. The LR and BALR tests are liberal for small values of \( c_0 (< 0.5) \), as \( c_0 \) increases these tests maintain the type I error rate. For the lognormal distribution, all tests maintain the type I error rates for \( c_0 \leq 1.0 \). As \( c_0 \) increases, the LR, BALR and score tests become liberal. On the contrary, the Wald test becomes stringent for large values of \( c_0 \). For the gamma distribution, the type I error rate decreases as \( c_0 \) increases for all the tests. The type I error rates are maintained in these tests for small values of \( c_0 (< 0.5) \) and thereafter they become stringent.

3.6 Conclusion

In this chapter we have compared the small sample performance of the LR, BALR, Wald and score tests for the c.v. of a normal distribution. The estimated type I error rates indicate that, the Wald test is liberal when \( c_0 \) is large, while the score test is slightly conservative for small values of \( c_0 \). The estimated type I error rates for the LR and BALR tests are very...
close to the nominal level of $\alpha = 0.05$. It clearly shows that the normal approximation for
the null distribution of the test statistic is quite accurate for these test statistics. The power
comparisons indicate that, for the alternatives to the left of the null hypothesis, the LR and
BALR tests are better while for the alternatives to the right of the null hypothesis, the score
and Wald tests perform better.

The sensitivity analysis indicates that, among all the four tests, score test maintains
the type I error rates for the symmetric distributions and is robust. For the right skewed
distributions it is robust for small values of $c_0$. One of the possible explanation for this
phenomenon may be that the score test is conservative when the underlying distribution is
normal. The Wald test is robust for the right skewed distributions while for the LR and
BALR tests no systematic conclusion emerges regarding robustness, although the tests are
robust for the gamma and uniform distributions.
Table 3.1 Estimated type I error rates for the various tests for \( n=20, 80 \) when \( \alpha=0.05 \)

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>( n=20 )</th>
<th>( n=80 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Wald LR</td>
<td>Bartlett LR</td>
</tr>
<tr>
<td></td>
<td>-adjusted LR</td>
<td>-adjusted LR</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0269 0.0630</td>
<td>0.0609 0.0386</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0268 0.0596</td>
<td>0.0579 0.0384</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0266 0.0624</td>
<td>0.0602 0.0375</td>
</tr>
<tr>
<td>0.15</td>
<td>0.0277 0.0615</td>
<td>0.0592 0.0388</td>
</tr>
<tr>
<td>0.20</td>
<td>0.0295 0.0617</td>
<td>0.0598 0.0397</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0306 0.0609</td>
<td>0.0587 0.0397</td>
</tr>
<tr>
<td>0.30</td>
<td>0.0316 0.0611</td>
<td>0.0588 0.0398</td>
</tr>
<tr>
<td>0.40</td>
<td>0.0374 0.0619</td>
<td>0.0596 0.0420</td>
</tr>
<tr>
<td>0.50</td>
<td>0.0402 0.0589</td>
<td>0.0561 0.0405</td>
</tr>
<tr>
<td>0.75</td>
<td>0.0553 0.0597</td>
<td>0.0571 0.0422</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0742 0.0577</td>
<td>0.0550 0.0439</td>
</tr>
<tr>
<td>1.50</td>
<td>0.1093 0.0602</td>
<td>0.0571 0.0474</td>
</tr>
<tr>
<td>2.00</td>
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<td>0.0566 0.0466</td>
</tr>
<tr>
<td>2.50</td>
<td>0.1327 0.0602</td>
<td>0.0567 0.0469</td>
</tr>
<tr>
<td>3.00</td>
<td>0.1256 0.0601</td>
<td>0.0572 0.0470</td>
</tr>
</tbody>
</table>
Figure 3.1 Estimated power functions for the various tests when $n=20$, $c_0=0.1, 0.5, 1.0$ and $2.0$ and $\alpha=0.05$. 
Figure 3.2  Estimated type I error rates for the different tests for various distributions when $n=20$ and $\alpha=0.05$