Chapter 2

Performance of the Estimators of Normal Mean with Known Coefficient of Variation

2.1 Introduction

Normal distribution is widely used in physical and biological sciences to model various outcomes of interest. Normal distribution belongs to two-parameter exponential family and admits a two dimensional minimal sufficient statistics \((\bar{X}, S^2)\) for the two dimensional parameter \(\mu\) and \(\sigma^2\), where \(\bar{X}\) denotes the sample mean and \(S^2\) denotes the sample variance. One of the common problems of interest is the estimation of the mean \(\mu\) when \(\sigma^2\) is unknown. The estimator \(\bar{X}\) is Uniformly Minimum Variance Unbiased (UMVU) estimator and is admissible under the squared error loss function (Lehmann, 1983). The other problem of interest is the estimation of \(\mu\) from the family \(N(\mu, c^2\mu^2)\) where \(c(>0)\) denotes the coefficient of variation \((c_v)\), and is assumed to be known. Such applications naturally arise in agricultural experiments and in the studies of pollutants where the mean pollutant level is directly related to the standard deviation. For other applications, we refer Gleser and Healy.
The $N(\mu, \sigma^2)$ family admits a two dimensional minimal sufficient statistic $(X, S^2)$ for real parameter $\mu$. The distribution of the minimal sufficient statistic is not complete (Casella and Berger, 1990, page 281) Thus UMVU estimator of $\mu$ does not exist. The estimation of $\mu$ from this family was the concern of many researchers in the past. Searls (1964) first proposed an estimator for the mean $\mu$ when $\sigma^2$ is known, which is biased. Khan (1968) proposed an estimator which has the smallest variance, uniformly in $\mu$, among all unbiased estimators that are linear in $\bar{X}$ and $S$, when the parameter space for $\mu$ is the positive part of the real line. Gleser and Healy (1976) extended the results of Khan and obtained an estimator having minimum mean squared error (MSE) uniformly for all $\mu$ among the class of all estimators, not necessarily unbiased. Although the maximum likelihood (ML) estimator of $\mu$ was first considered by Khan (1968), when $\mu$ is restricted to the positive part of the real line, Kunte (2000) derived the ML estimator of $\mu$, when the parameter space is the real line. This has motivated Guo and Pal (2003) to propose a new class of estimators, which includes the Searls estimator as a member. They proved that an estimator belonging to this class has smaller standardized mean squared error among the class of equivariance estimators under the group of scale and direction transformations. Soofi and Gokhale (1991) proposed the minimum discrimination information (MDI) estimator of $\mu$ for the general location scale family and compared this estimator with the ML estimator and other estimators through simulation. Their conclusion was that MDI estimator performs better for very small values of $\sigma^2$ and the ML estimator performs extremely well for moderate and large values of $\sigma^2$. There are other estimators like the Bayesian estimator proposed by Sinha (1983) and for a discussion of this estimator, we refer Soofi and Gokhale (1991) and Guo and Pal (2003).

Hodges and Lehmann (1970) introduced the concept of deficiency to measure the asymptotic performance of the estimators. For an estimator, the $r^{th}$ order deficiency refers to the coefficient of $1/n^r$ in the asymptotic expression for the MSE. When two estimators are to be compared, their asymptotic MSE to the order of $O(1/n^{r+1})$ should coincide and an estimator is
considered as efficient compared to the other if the \( r^{th} \) order deficiency factor of an estimator is smaller than the other. Deficiency refers to the rate of convergence of the estimator to the parameter value and is a very sensitive indicator to compare the asymptotic performance of the estimators. A good discussion is available in Lehmann (1983). The objectives of this chapter are

1. To derive asymptotic MSE to the order of \( O(\frac{1}{n^2}) \) of the eight estimators, description of which is given in section 2.2 and thereby compare the second order deficiency factor of those estimators for which the first order (to the order of \( O(\frac{1}{n}) \)) MSE term coincides

2. To compare the small sample performance of these estimators.

3. To derive the expression for the asymptotic MSE of these estimators to the order of \( O(\frac{1}{n}) \) using sampling moments from a general distribution and thereby to compare the asymptotic robustness of the estimators under violation of normality assumption, the conclusions of which are supplemented through a simulation.

A salient feature of this investigation is the derivation of the expression of asymptotic MSE to the order of \( O(\frac{1}{n^3}) \) under normality and for the general distribution, which are not available in the literature. While deriving the expressions for MSE of the estimators, we arrived at a new estimator which has the same MSE as that of ML estimator to the order of \( O(n^{-1}) \). This estimator is a perturbation of the estimator belonging to the class \( C_1 \) of Guo and Pal (2003). From the investigation it follows that, the MDI estimator has smaller second order deficiency than the ML and the new estimators. The bias corrected ML estimator has the minimum second order deficiency compared to the bias corrected MDI estimator. The bias corrected ML estimator is second order efficient (deficient estimator) (Rao (1963), Pfanzagl and Wefelmeyer (1978)) and the conclusion corroborates this result. The small sample comparison point out to the good performance of the ML estimator, when the parameter space for \( \mu \) is the real line. The MDI estimator performs well for very small
values of \( c_v \), while the new estimator has smaller MSE than the ML and the MDI estimators for small values of \( c_v \) when the sample size is moderate or large.

The simulation experiment using two parameter uniform, logistic, lognormal and gamma distributions was carried out to check the robustness of these estimators under violation of normality assumption. The results point out that, only for the uniform distribution, the ML, MDI, Khan, and Gleser Healy estimators are robust. The efficiency of these estimators decreases for large values of \( c_v \) when the underlying distributions are logistic, lognormal and gamma. The Searls, and equivariant estimators are robust for all the four distributions.

The organization of the chapter is as follows. Section 2.2 describes various estimators while the expression for the asymptotic MSE of above mentioned estimators are worked out in section 2.3 where the underlying distribution is normal. The results of the simulation experiment for the normal distribution are presented in section 2.4. The asymptotic MSE of the estimators for a general distribution and the small sample robustness of these estimators are presented in section 2.5. Real-life examples are discussed in section 2.6 and the chapter concludes in section 2.7.

### 2.2 Description of the Estimators

Given a random sample of size \( n \) from \( N(\mu, \sigma^2\mu^2) \) distribution. Let \( \bar{X} \) and \( S^2 \) denote the sample mean and variance respectively. In this section we briefly describe the estimators which are investigated in the thesis.

1. **Searls (1964)**

   Searls suggested the estimator for \( \mu \), which is biased, and is given by

   \[
   \hat{\mu}_S = \left[ \frac{n}{n + \sigma^2} \right] \bar{X} \tag{2.1}
   \]
2 Khan (1968)

When the parameter space is restricted to the positive part of the real line \( \mu \in \mathbb{R}^+ \), Khan proposed an estimator which has minimum variance, uniformly in \( \mu \), among all unbiased estimators that are linear in \( \bar{X} \) and \( S \). It is given by,

\[
\hat{\mu}_K = \frac{n\delta_n \bar{X} + c^2 \alpha_n S}{n\delta_n + c^2},
\]

where

\[
\delta_n = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\left[\Gamma\left(\frac{n}{2}\right)\right]^2} - 1,
\]

\[
\alpha_n = \frac{\Gamma\left(\frac{n-1}{2}\right)\sqrt{2}}{2c\Gamma\left(\frac{n}{2}\right)},
\]

and \( \Gamma() \) denotes the gamma function.

3 Gleser and Healy (1976)

Gleser and Healy came out with an estimator which has minimum MSE and linear in \( \bar{X} \) and \( S \) and is given by,

\[
\hat{\mu}_{GH} = \hat{\mu}_K \left(1 + \frac{c^2 \delta_n}{c^2 + n\delta_n}\right)^{-1},
\]

where \( \hat{\mu}_K \) and \( \delta_n \) are given in (6 3).

4 Maximum Likelihood (Kunte (2000))

The ML estimator of \( \mu \) is given by,

\[
\hat{\mu}_{ML} = \left\{ \begin{array}{ll}
\frac{-\bar{X} + \sqrt{(1+4c^2)\bar{X}^2 + 4c^2S^2}}{2c^2} & \text{if } \bar{X} > 0 \\
\frac{-\bar{X} - \sqrt{(1+4c^2)\bar{X}^2 + 4c^2S^2}}{2c^2} & \text{if } \bar{X} < 0
\end{array} \right.
\]

(2 4)
5 Minimum Discrimination Information (MDI) Estimator

Soofi and Gokhale (1991) proposed MDI estimator which is asymptotically unbiased and efficient and is given by,

$$
\hat{\mu}_{MDI} = \frac{\bar{X} + \sqrt{\bar{X}^2 + 4S^2 (c^2 + 1)}}{2 (c^2 + 1)}
$$

(2.5)


Guo and Pal (2003) derived a class of equivariant estimators $C_q$, which is given by,

$$
C_q = \left\{ \hat{\mu}_q = \bar{X} \left( \sum_{i=0}^{q} k_{q_i} \left[ \frac{\bar{X}^2}{c^2S^2} \right]^i \right) \bigg| k_{q_0}, k_{q_1}, \ldots, k_{qq} \text{ are constants} \right\}
$$

(2.6)

When $q=0$, the class of estimators is $C_0 = \{ \hat{\mu}_0 = k_{00} \bar{X} | k_{00} \text{ is a constant} \}$. They found the optimum value of $k_{00}$ by minimizing the standardized mean squared error (SMSE) of $\hat{\mu}_0$ and the optimum value of $k_{00}$ is $k_{00}^* = n/(n + c^2)$. The estimator $\hat{\mu}_0$ after substituting $k_{00}^*$ is the same as the Searls estimator.

When $q=1$, the class of estimators is

$$
C_1 = \left\{ \hat{\mu}_1 = \left[ k_{10} + k_{11} \frac{\bar{X}^2}{c^2S^2} \right] \bar{X} \mid k_{10}, k_{11} \text{ are constants} \right\}
$$

By minimizing the SMSE of $(\hat{\mu}_1)$ with respect to $k_{10}$ and $k_{11}$, they obtained the optimal values of these constants, which satisfy the following equations

$$
1 = k_{10}^* \left[ 1 + \frac{c^2}{n} \right] + k_{11}^* \left( \frac{n}{c^2} \right) \left\{ 1 + \frac{6(c^2/n) + 3(c^2/n)^2}{c^2(n - 3)} \right\},
$$

$$
1 + 3 \left( \frac{c^2}{n} \right) = k_{10}^* \left[ 1 + 6 \left( \frac{c^2}{n} \right) + 3 \left( \frac{c^2}{n} \right)^2 \right] + k_{11}^* \left( \frac{n}{c^2} \right) \left\{ 1 + 15 \left( \frac{c^2}{n} \right) + 45 \left( \frac{c^2}{n} \right)^2 + 15 \left( \frac{c^2}{n} \right)^3 \right\}
$$

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2.3 Mean Squared Error of the Estimators to the Order of $O(n^{-2})$

In this section we derive the expression for the asymptotic bias (to the order of $O(n^{-1})$) and MSE (to the order of $O(n^{-2})$) for the various estimators using Taylor series expansion. The expressions are given below. As indicated in section 2.1, one of the ways to compare the asymptotic efficiency of various estimators is through deficiency. We have made an attempt to compute the second order deficiency of the estimators. For the comparison of the estimators it is important that the expression for the MSE of the estimators agrees to the order of $O(n^{-1})$. The algebra is quite lengthy and to save space, details have been omitted. We have sketched the derivation in Appendix A.2. The expressions for the bias and MSE for the various estimators are given below.

1 Sample Mean

\[
\begin{align*}
\text{Bias} & = 0, \\
\text{MSE} & = \frac{c^2 \mu^2}{n}
\end{align*}
\]  \hfill (27)

2 ML Estimator

\[
\begin{align*}
\text{Bias} & = \frac{\mu c^4}{n (2 c^2 + 1)^2} + O(n^{-2}), \\
\text{MSE} & = \frac{c^2 \mu^2}{n (2 c^2 + 1)} - \frac{c^8 \mu^2}{n^2 (2 c^2 + 1)^4} + O(n^{-3})
\end{align*}
\]  \hfill (28)

3 Bias Corrected ML Estimator

The bias corrected ML estimator is given by,

\[
\hat{\mu}_{BCML} = \hat{\mu}_{ML} \left[ 1 - \frac{c^2}{n (2 c^2 + 1)} \right]
\]
The bias and MSE is given by,

\[
\text{Bias} = 0 + O(n^{-2}), \\
\text{MSE} = \frac{c^2 \mu^2}{n (2c^2 + 1)} - \frac{2\mu^2 c^\theta (3c^2 + 1)}{n^2 (2c^2 + 1)^4} + O(n^{-3}) \tag{2.9}
\]

4 MDI Estimator

\[
\text{Bias} = -\frac{\mu c^4}{n (2c^2 + 1)^2} + O(n^{-2}), \\
\text{MSE} = \frac{c^2 \mu^2}{n (2c^2 + 1)} - \frac{\mu^2 c^\theta (c^2 + 4)}{n^2 (2c^2 + 1)^4} + O(n^{-3}) \tag{2.10}
\]

5 Bias Corrected MDI Estimator

The bias corrected MDI estimator is given by,

\[
\hat{\mu}_{\text{BCMDI}} = \hat{\mu}_{\text{MDI}} \left[1 + \frac{c^2}{n (2c^2 + 1)} \right]
\]

The bias and MSE is given by,

\[
\text{Bias} = 0 + O(n^{-2}), \\
\text{MSE} = \frac{c^2 \mu^2}{n (2c^2 + 1)} + \frac{2\mu^2 c^\theta (c^2 - 1)}{n^2 (1 + 2c^2)^4} + O(n^{-3}). \tag{2.11}
\]

For a distribution having first four moments, Soofi and Gokhale (1991) derived the asymptotic bias and MSE of ML and MDI estimators to the order of \(O(n^{-1})\). Our expression to this order agrees with their expression when the underlying distribution is \(N(\mu, c^2 \mu^2)\)
6 Khan Estimator

\[
\text{Bias} = \frac{\mu c^2}{n\delta_n + c^2} \left( c\sqrt{n}\alpha_n - 1 \right) - \frac{\alpha_n}{4\sqrt{n}} + O(n^{-2}) ,
\]

\[
\text{MSE} = \frac{c^2 \mu^2 \left[ c^2 + n \left( c^4 \alpha_n^2 + \delta_n^2 \right) \right]}{n\delta_n + c^2} - \frac{c^5 \mu^2 \alpha_n (32n^2 - 8n + 1)}{16n^{3/2} [n\delta_n + c^2]^{2}} + O(n^{-3}) \tag{2.12}
\]

Note: The Khan estimator is unbiased. However, (2.12) is the expression for the asymptotic bias to the order of $O(n^{-1})$

7 Gleser Healy Estimator

\[
\text{Bias} = \frac{\mu c^2}{(n+c^2)\delta_n + c^2} + \left( c\sqrt{n}\alpha_n - \delta_n - 1 \right) - \frac{\alpha_n}{4\sqrt{n}} + O(n^{-2}) ,
\]

\[
\text{MSE} = \frac{c^2 \mu^2 \left[ c^2 (1 + \delta_n)^2 + n \left( c^4 \alpha_n^2 + \delta_n^2 \right) \right]}{n\delta_n + c^2 (1 + \delta_n)} - \frac{c^5 \mu^2 \alpha_n (1 + \delta_n) (32n^2 - 8n + 1)}{16n^{3/2} [n\delta_n + c^2 (1 + \delta_n)]} + O(n^{-3}) \tag{2.13}
\]

8 Searls Estimator

\[
\text{Bias} = \mu \left( \frac{n}{n+c^2} - 1 \right) ,
\]

\[
\text{MSE} = \frac{c^2 \mu^2}{n+c^2} \tag{2.14}
\]
9 Equivariant Estimator: Class $C_1$

\[
\text{Bias} = \mu \left( k_{10} + \frac{k_{11}}{c^4} - 1 \right) + \frac{k_{11} \mu (3c^2 + 2)}{nc^4} + O(n^{-2}),
\]
\[
\text{MSE} = \mu^2 \left( k_{10} + \frac{k_{11}}{c^4} - 1 \right)^2
+ \frac{\mu^2}{nc^8} \left\{ c^{10} k_{10}^2 + 4 k_{11} k_{10} c^4 \left( 1 + 3c^2 \right) + k_{11} \left[ 3k_{11} \left( 5c^2 + 2 \right) - 2c^4 \left( 3c^2 + 2 \right) \right] \right\}
+ \frac{k_{11} \mu^2}{c^8 n^2} \left\{ 2k_{10} c^4 \left( 3c^4 + 12c^2 + 4 \right) + k_{11} \left( 45c^4 + 90c^2 + 28 \right) \right\}
- 4c^4 \left( 3c^2 + 2 \right) \right) + O(n^{-3})
\]  
(2.15)

10 Equivariant Estimator: Class $C_q$

\[
\text{Bias} = \mu \left\{ \sum_{i=0}^{q} \frac{k_{qi}}{c^i} - 1 \right\} + \frac{1}{n} \sum_{i=0}^{q} \frac{1}{n} \sum_{i=1}^{q} \left\{ c^2(2t + 1) \right\} + O(n^{-2}),
\]
\[
\text{MSE} = \mu^2 \left[ \sum_{i=0}^{q} \frac{k_{qi}}{c^i} \right]^2
+ \frac{\mu^2}{n} \left\{ \sum_{i=0}^{q} \frac{k_{qi}}{c^i} \left[ k_{qi}(2t + 1) \right] c^2(2t + 1) \left( t + 1 \right) \right\}
+ \sum_{i=0}^{q} \frac{1}{n} \sum_{j=0}^{q} \frac{k_{qi} k_{qj}}{c^{i+j}} \left[ c^2(2j + 1)(2t + 2j + 1) + 2j(t + j + 1) \right]\}
\]
\[
\frac{\mu^2}{n^2} \sum_{i=0}^{q} \frac{1}{c^i} \left[ \frac{k_{qi}(1 + 2t)}{3} \right] \left\{ 3c^4(16t^2 - 1) + 6c^2(8t^2 + 6t + 1) + 2(6t^2 + 7t + 1) \right\}
- \frac{c^{4t}}{6} \left\{ 3c^2(2t + 1)(2t - 1)(2t - 2) + 12c^2(2t + 1)(t + 1) + 2(t + 1)(t + 2)(3s + 1) \right\}
+ \frac{1}{n} \sum_{i=0}^{q} \frac{k_{qi} k_{qj}}{c^{i+j}} \left[ c^4(j + 1)(2j + 1) + c^2(j + 1)(2j + 1)(3s + j + 2) \right] + O(n^{-3})
\]  
(2.16)
From (2.16) it is clear that, the class of equivariant estimators is not unbiased to the order of $O(1)$. In order to compare these estimators with the other estimators we propose a bias corrected estimator for the class $C_1$ and the result is summarized in the following theorem.

**Theorem 2.1:** Among the class of estimators proposed by Guo and Pal (2003), the bias corrected estimator for the class $C_1 = \{ \hat{\mu}_1 = \left[ k_{10} + k_{11} \frac{X^2}{S^2} \right] \bar{X} \mid k_{10}, k_{11} \text{ are constants} \}$ is given by,

$$C_1(BC) = \left\{ \hat{\mu}_1(BC) = \left[ \frac{k_{11}}{c^2} \left( \frac{X^2}{S^2} - \frac{1}{c^2} \right) + 1 \right] \bar{X} \mid k_{11} \text{ is a constant} \right\} \quad (2.17)$$

The bias and MSE of this estimator is given by,

$$\text{Bias} = \frac{1}{n} \left[ \frac{k_{11} \mu (2 + 3c^2)}{c^4} \right] + O(n^{-2})$$

$$\text{MSE} = \frac{\mu^2}{n} \left\{ c^2 \left[ 1 + \frac{2k_{11}}{c^4} \right]^2 + \frac{2k_{11}^2}{c^8} \right\} + \frac{\mu^2 k_{11}}{n^2 c^8} \left\{ k_{11} (39c^4 + 66c^2 + 20) + 6c^6(c^2 + 2) \right\} + O(n^{-3}) \quad (2.18)$$

**Proof:** Generally the bias corrected estimator is obtained by using the same estimator for $\mu$ in the bias expression. However, when the same estimator is used for $\mu$ (i.e., $\hat{\mu}_1$) in $\mu (k_{10} + \frac{k_{11}}{c^4} - 1)$, we observe that the bias corrected estimator is not free from bias to the order of $O(n^{-1})$. We have used the sample mean $\bar{X}$ as an estimator of $\mu$ in bias expression and the bias corrected estimator is given by,

$$\hat{\mu}_1(BC) = \bar{X} \left[ k_{10} + k_{11} \frac{\bar{X}^2}{S^2} - \bar{X} \left( k_{10} + \frac{k_{11}}{c^4} - 1 \right) \right] = \left[ \frac{k_{11}}{c^2} \left( \frac{\bar{X}^2}{S^2} - \frac{1}{c^2} \right) + 1 \right] \bar{X}$$
This estimator is unbiased to the order of $O(n^{-1})$. Using Taylor series expansion we can derive the asymptotic bias and MSE of this estimator as given in (2.18).

**Theorem 2.2:** The estimator

$$\hat{\mu}_{BR} = \bar{X} \left[1 - \frac{c^4}{(1 + 2c^2)} \left(\frac{\bar{X}^2}{s^2} - \frac{1}{c^2}\right)\right], \quad (2.19)$$

has the same first order deficiency as that of ML estimator. The asymptotic bias (to the order of $O(n^{-2})$) and MSE (to the order of $O(n^{-3})$) of this estimator is given by,

$$\text{Bias} = -\frac{c^2 \mu (3c^2 + 2)}{n(1 + 2c^2)} + O(n^{-2})$$

$$\text{MSE} = \frac{c^2 \mu^2}{n(1 + 2c^2)} + \frac{c^4 \mu^2 (27c^4 + 36c^2 + 8)}{n^2 (1 + 2c^2)^2} + O(n^{-3}) \quad (2.20)$$

**Proof:** Comparing the asymptotic MSEs for the ML estimator and the bias corrected estimator for class $C_1$ to the order of $O(n^{-1})$ and solving for $k_{11}$ we get $k_{11} = -c^6/(1 + 2c^2)$ Substituting the constant $k_{11}$ in (2.19) and (2.20), we get the required result.

In the subsequent discussion, this estimator is referred to as Bhat and Rao estimator.

The focus of this chapter is to compare the deficiency of the ML estimator with the other estimators. From the above expressions it follows that MDI, and Bhat and Rao estimators have the same first order deficiency as that of ML estimator. The following theorem gives the deficiency of the ML estimator with respect to the MDI, and Bhat and Rao estimators and also deficiency of bias corrected ML estimator with bias corrected MDI estimator.

**Theorem 2.3:**

1. The second order deficiency of the ML estimator with respect to the MDI estimator is

$$d(\hat{\mu}_{ML}, \hat{\mu}_{MDI}) = \frac{4c^4}{(1 + 2c^2)^3} > 0 \quad (2.21)$$
2 The second order deficiency of the ML estimator with respect to the Bhat and Rao estimator is

\[
d(\hat{\mu}_{ML}, \hat{\mu}_{RB}) = -\frac{4c^2(3c^2 + 2)(1 + c^2)(1 + 3c^2)}{(1 + 2c^2)^3} < 0
\]  

(2.22)

3 The second order deficiency of the bias corrected ML estimator with respect to the bias corrected MDI estimator is

\[
d(\hat{\mu}_{BCML}, \hat{\mu}_{BCMDI}) = -\frac{8c^4}{(1 + 2c^2)^3} < 0
\]  

(2.23)

**Proof:** Let \( T_1 \) and \( T_2 \) are two estimators of a real parameter \( \theta \) and thus the asymptotic MSE be expressible as,

\[
MSE(T_1) = \frac{a}{n^r} + \frac{b_1}{n^{r+1}} + O\left(\frac{1}{n^{r+2}}\right) \quad \text{and} \quad MSE(T_2) = \frac{a}{n^r} + \frac{b_2}{n^{r+1}} + O\left(\frac{1}{n^{r+2}}\right)
\]  

(2.24)

respectively. Then the deficiency of \( T_2 \) with respect to \( T_1 \) and is given by,

\[
d(T_2, T_1) = \frac{b_2 - b_1}{ar}
\]  

(2.25)

(Hodges and Lehmann (1970)) The result follows from the expressions of MSEs given in (2.24) and (2.25).

From the expressions (2.21), (2.22) and (2.23), it follows that the number of additional observation required by the ML estimator to obtain the same MSE as that of the MDI estimator is \( \frac{4c^2}{(1 + 2c^2)^3} \) and the number of additional observations required by the Bhat and Rao and bias corrected MDI estimators so as to obtain the same MSEs as that of ML and bias corrected ML estimators are \( \frac{4c^2(3c^2 + 2)(1 + c^2)(1 + 3c^2)}{(1 + 2c^2)^3} \) and \( \frac{8c^4}{(1 + 2c^2)^3} \) respectively. Thus the ML estimator is not second order deficient and the bias corrected ML estimator is second order efficient as well as deficient estimator.
The expressions for MSE of all other estimators do not match with the ML estimator to the order of $O(n^{-1})$, it is not possible to compare the deficiency of these estimators with the ML estimator.

Rao (1963) introduced the concept of the second order efficiency and showed that the bias corrected ML estimator is second order efficient in the multinomial setup. Pfanzagl and Wefelmeyer (1978) proved that the bias corrected ML estimator is second order efficient estimator. The bias corrected ML estimator is second order efficient as well as deficient and the expression for MSE of bias corrected ML and bias corrected MDI estimators are given in (2.9) and (2.11) respectively.

2.4 Small Sample Performance of the Estimators

2.4.1 Description of Simulation Experiment

From the previous discussion it is clear that it is not possible to compare the various estimators theoretically. We have conducted a simulation experiment to compare the small sample performance of the nine estimators namely sample mean, ML, bias corrected ML, MDI, Khan, Gleser Healy, Searls, equivariant ($C_1$), and Bhat and Rao estimators. The description of the simulation experiment is given below.

Soofi and Gokhale (1991) pointed out that the ratio of the respective MSEs of any two estimators is invariant under change of scale. Thus the relative efficiencies are invariant for the choice of $\mu$ and we have used the value $\mu = 100$ as used by Soofi and Gokhale (1991) in the present experiment. The values of $c_v$'s used in the simulations are 0.01, 0.05, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5, 0.75, 1.0, 1.5, 2.0, 2.5 and 3.0. Sample sizes are $n = 20, 40$ and 80. The number of simulation runs carried out for each experiment is 50,000. Unlike Soofi and Gokhale (1991), we have not considered very small sample sizes as they are unlikely to occur.
2.4.2 Results and Discussion

Table 2.1 gives the estimated bias for the various estimators when the sample size is 20. From the table, it is clear that when the coefficient of variation (c.v.) is small, all the estimators are almost unbiased, the bias being smaller than $1 \times 10^{-3}$. When the c.v. is moderate ($c.v. = 0.5$), the bias of the various estimators are small compared to the value of $\mu = 100$. When the c.v. increases to 1.5, there is no remarkable increase in the bias compared to the case of $c.v. = 0.5$. When the c.v. increases to 3.0, the bias of the Searls, equivariant, and Bhat and Rao estimators increases to 31.19, 30.83, and 77.02 respectively. The explanation for the high value of the bias for the Bhat and Rao estimator is the substantial contribution of the $(1/n)$ term in the bias expression while for the other two are biased estimators. The other estimators namely, ML, bias corrected ML, MDI, Khan, and Gleser Healy are unbiased estimators, thus the bias of these estimators is near to zero in all the cases. The pattern is the same for the sample sizes 40 and 80. It may be observed that the bias of the Bhat and Rao estimator drastically decreases with the increase in sample size.

Figure 2.1 and Figure 2.2 represent the MSE of the various estimators when the sample sizes are 20 and 80 respectively. It is clear from the figures that the ML estimator and the bias corrected ML estimator are having the lowest MSE for all the configurations followed by the Gleser Healy, MDI, and Khan estimators. When the c.v. is small, the performance of the Searls, equivariant, and Bhat and Rao estimators are comparable to the above estimators. However, when the c.v. increases ($> 1.0$), the MSE of these estimators increases faster compared to the other estimators. It can also be observed that for large sample sizes and for moderate values of c.v. ($\leq 0.25$), the Bhat and Rao estimator performs well compared to all other estimators including the ML estimator.
2.5 Robustness of the Estimators under Violation of Normality Assumption

2.5.1 Mean Square Error of the Estimators to the Order of $O(n^{-2})$ for a General Distribution.

In applied sciences, most often, the scientists use the estimators/tests derived under the normality assumption to any data set. Thus it is pertinent to investigate how robust are these estimators. For this purpose, we have derived the bias and MSE of eight estimators namely, ML, bias corrected ML, MDI, bias corrected MDI, Khan, Gleser Healy, Searls, equivariant $C_1$ class, and Bhat and Rao estimators using sampling moments from a general distribution having first six finite moments. Since the sample mean and Searls estimators are independent of sample variance, the expression for asymptotic bias and MSE of these estimators are the same as mentioned in 2.7 and 2.14 respectively. For the other estimators, the expressions are given below. The algebra is tedious and the first four sampling cross moments of $X$ and $S^2$ are given in the Appendix A1.

1 ML Estimator

\[
\text{Bias} = \frac{(5c^2 + 1)c^4 \mu^4 - (4c^2 + 1)\mu\mu_3 - c^2 \mu_4}{n\mu^3 (2c^2 + 1)^3} + O(n^{-2}),
\]

\[
\text{MSE} = \frac{\mu_4 + 2\mu_3 \mu + \mu^4 c^2 (1 - c^2)}{n\mu^2 (2c^2 + 1)^3} + \frac{1}{n^2 \mu^6 (2c^2 + 1)^5} \left\{ -2\mu^2 \mu_3 c^2 (2c^2 + 1)^2 \\
-2\mu_3^3 (5c^2 + 1) (2c^2 + 1) + 15\mu_2^2 c^4 + 12\mu_4 \mu_3 c^2 (9c^2 + 2) + 4c^2 + 12c^2 + 1 \right\} + O(n^{-3})
\]

(2.26)
2 Bias corrected ML Estimator

\[
\text{Bias} = \frac{3 \mu^4 c^8 - (4 c^2 + 1) \mu \mu_3 - c^2 \mu_4}{n \mu^8 (2 c^2 + 1)^3} + O(n^{-2}),
\]

\[
MSE = \frac{\mu_4 + 2 \mu_3 \mu + \mu^4 c^2 (1 - c^2)}{n \mu^2 (2 c^2 + 1)^2} + \frac{1}{n^2 \mu^6 (2 c^2 + 1)^6} \left\{ -2 \mu^2 \mu_6 c^2 (2 c^2 + 1)^2 \right. \\
-2 \mu^3 \mu_5 (5 c^2 + 1) (2 c^2 + 1)^2 + 15 \mu_4^2 c^4 + 12 \mu_4 \mu_3 c^2 (9 c^2 + 2) \\
-\mu^4 \mu_4 (2 c^8 - 92 c^6 - 10 c^4 + 11 c^2 + 2) + 6 \mu^2 \mu_3 \mu_3 (30 c^4 + 12 c^2 + 1) \\
-2 c^2 \mu_5^3 (38 c^6 - 144 c^4 - 79 c^2 - 10) - \mu^8 c^8 (c^8 + 124 c^6 \\
-34 c^4 - 47 c^2 - 8) \} + O(n^{-3}) \quad (2.27)
\]

3 MDI Estimator

\[
\text{Bias} = \frac{\mu^4 c^4 (2 + c^2) - \mu_3 \mu - \mu_4 (1 + c^2)}{(1 + 2 c^2)^2 \mu^3 n} + O(n^{-2}),
\]

\[
MSE = \frac{\mu_4 + 2 \mu_3 \mu + \mu^4 c^2 (1 - c^2)}{n \mu^2 (2 c^2 + 1)^2} + \frac{1}{n^2 \mu^6 (2 c^2 + 1)^6} \left\{ -2 \mu^2 \mu_6 (c^2 + 1) (2 c^2 + 1)^2 \right. \\
-2 \mu^3 \mu_5 (c^2 + 2) (2 c^2 + 1)^2 + 15 \mu_4^2 (c^2 + 1)^2 + 12 \mu_4 \mu_3 (c^2 + 3) (c^2 + 1) \\
-\mu^4 \mu_4 (6 c^8 + 10 c^6 - 9 c^4 + 2) + 6 \mu^2 \mu_3 \mu_3 (8 c^6 + 14 c^4 + 10 c^2 + 5) \\
+4 c^2 \mu_5^3 (c^2 + 2) (9 c^4 + 5 c^2 + 4) + \mu^8 c^8 (31 c^8 + 44 c^6 + 46 c^4 \\
+15 c^2 + 8) \} + O(n^{-3}) \quad (2.28)
\]
4 Bias corrected MDI Estimator

\[ \text{Bias} = \frac{3 \mu^4 c^4 (1 + c^2) - \mu_3 \mu - \mu_4 (1 + c^2)}{(1 + 2 c^2)^3 \mu^3 n} + o(n^{-2}), \]
\[ \text{MSE} = \frac{\mu_4 + 2 \mu_3 \mu + \mu^4 c^2 (1 - c^2)}{n \mu^2 (2 c^2 + 1)^2} + \frac{1}{n^2 \mu^6 (2 c^2 + 1)^6} \left( -2 \mu^2 \mu_6 (c^2 + 1) (2 c^2 + 1)^2 \right. \]
\[ -2 \mu^3 \mu_5 (c^2 + 2) (2 c^2 + 1)^2 + 15 \mu_4^2 (c^2 + 1)^2 + 12 \mu \mu_4 \mu_3 (c^2 + 3) (c^2 + 1) \]
\[ -\mu^4 \mu_4 (2 c^8 + 8 c^6 - 9 c^2 + 2) + 6 \mu^2 \mu_3 (8 c^6 + 14 c^4 + 10 c^2 + 5) \]
\[ + 2 c^2 \mu_5 \mu_3 (26 c^6 + 52 c^4 + 29 c^2 + 16) + \mu^4 c^4 (31 c^8 + 58 c^6 \]
\[ + 57 c^4 + 17 c^2 + 8) \right) + o(n^{-3}) \] (2.29)

5 Khan Estimator

\[ \text{Bias} = \frac{8 \mu^4 c^3 \sqrt{n} (c \alpha_n \sqrt{n} - 1) + \alpha_n (c^4 \mu^4 - \mu_4)}{8 \sqrt{n c^3 (n \delta_n + c^2)}} + o(n^{-2}), \]
\[ \text{MSE} = \frac{1}{64 c^3 \mu^6 n^{3/2} (n \delta_n + c^2)^2} \left( 64 \mu^8 c^5 (\delta_n^2 + c^4 \alpha_n^2) n^{5/2} - 64 c^4 \mu^5 \alpha_n \right. \]
\[ (2 \mu^3 c^4 - \mu_3 \delta_n) n^2 + 64 \mu_4^2 c^7 n^{3/2} - 8 \mu \alpha_n [2 \mu^2 \delta_n c^2 \mu_5 - \]
\[ (2 \mu^3 c^4 + 3 \mu_3 \delta_n) \mu_4 - 9 \mu^4 \mu_3 \delta_n c^4 + 2 \mu \alpha_n [8 c^2 \mu_5 \mu_0 \]
\[ + 3 \mu_4 (5 \mu_4 - 2 \mu^4 c^4) + 48 c^2 \mu^2 \mu_3^2 + 31 c^2 \mu^2 ] \right) + o(n^{-3}) \] (2.30)
6 Gleser Healy Estimator

\[ \text{Bias} = \frac{\mu^2 c^3 \sqrt{n} (c_\alpha \sqrt{n} - 1 - \delta_n) + c_n (\mu^4 c^4 - \mu_4) + O(n^{-2})}{8 \sqrt{n} \mu^3 c [n \delta_n + c^2 (1 + \delta_n)]}, \]

\[ \text{MSE} = \frac{1}{64 \mu^6 c^6 n^{3/2} [n \delta_n + c^2 (1 + \delta_n)]^2} \{64 \mu^8 c^8 (c^4 \alpha_n^2 + \delta_n^2) n^{5/2} - 64 \alpha_n c^2 (1 + \delta_n) - \mu_3 \delta_n \} n^2 + 64 c^7 \mu^8 (1 + \delta_n)^2 n^{3/2} - 8 \alpha_n [2 \mu^2 c^2 \delta_n \mu_5 - 2 c^4 \mu_4 (\mu_3 + \mu^3 \delta_n) - 3 \delta_n (\mu_4 + 3 \mu^4 c^4) \mu_3 + 2 \mu^7 \delta (\delta_n + 1)] n - \alpha_n [8 c^2 \mu^2 \mu_6 + 3 \mu_4 (5 \mu_4 - 2 \mu^4 c^4) + 48 c^2 \mu^2 \mu_5^2 + 31 c^8 \beta^2] (1 + \delta_n)] + O(n^{-3}) \]  

(2.31)

7 Equivariant Estimator: Class C1

\[ \text{Bias} = \mu \left( k_{10} + \frac{k_{11}}{c^4} - 1 \right) + \frac{k_{11}}{\mu^2 c^2} \left[ \mu_4 - 3 \mu_3 \mu c^2 + \mu^3 c^2 (3 c^2 - 1) \right] + O(n^{-2}), \]

\[ \text{MSE} = \mu^2 \left( k_{10} + \frac{k_{11}}{c^4} - 1 \right)^2 + \frac{1}{\mu^2 c^2} \left[ c^2 \mu^4 k_{10}^2 + 2 c^4 k_{11} k_{10} [\mu_4 - 4 \mu_3 \mu c^2 + \mu^4 c^2 (5 c^2 - 1)] \right. \]

\[ + \mu^6 c^4 (6 c^2 - 1)] - k_{11} [3 k_{11} \left( \mu_4 - 4 \mu_3 \mu c^2 + \mu^4 c^2 (5 c^2 - 1) \right) + \frac{k_{11}}{n^2 \mu^3 c^6} \{2 c^4 k_{10} [c^2 \mu_6^2 - c^2 \mu_6] + 4 c^4 \mu_3 \mu_5 + 3 \mu_4 c^2 (c^2 \mu^4 + 4 \mu_3 \mu) + 6 c^2 \mu^2 \mu_3^2 (1 + 2 c^2) \}
\]

\[ - 4 c^4 \mu_3 \mu_5 (2 c^2 + 3) + 3 c^2 \mu^8 (c^4 + 4 c^2 + 1)] - k_{11} [4 \mu_3^2 c^2 \mu_6 - 18 \mu^3 \mu_4 c^4
\]

\[ - 15 \mu_4^2 + 3 c^2 \mu \{24 \mu_3 + \mu^3 c^2 (6 - 5 c^2) \} \mu_4 - 6 \mu^2 c^2 (15 c^2 + 4) \mu_3^2
\]

\[ + 4 \mu^5 \mu (9 + 25 c^2) \mu_3 - \mu^8 c^8 (45 c^4 + 45 c^2 + 13)]
\]

\[ + 2 c^4 \left[ \mu^3 c^2 \mu_5 - 3 \mu^3 \mu_4 - 3 \mu_4^2 + 3 c^2 \mu_4 (\mu_3^2 - 3 \mu_3) - 6 \mu^2 c^2 (c^2 + 1) \right] + O(n^{-3}) \]  

(2.32)
8 Bhat and Rao Estimator

\[ \text{Bias} = \frac{\{c^4\mu^4(1 - 3c^2) + 3\mu_3\mu_2^2 - \mu_4\}}{n\mu^2c^2(2c^2 + 1)} + O(n^{-2}), \]

\[ \text{MSE} = \frac{\mu_4 + 2\mu_3\mu + \mu_1^2c^2(1 - c^2)}{n\mu^2(2c^2 + 1)^2} + \frac{1}{n^2\mu^6(2c^2 + 1)^6}\{ -2c^2\mu_5 \mu_6 \\
+ 2c^2\mu_5\mu_8(3c^2 - 1) + 9\mu_4^2 + 3\mu_4\mu_5\left[c^4\mu_3^3(5c^2 - 4) - 2\mu_3(6c^2 - 1)\right] \\
+ 42\mu^2c^4\mu_8^2 - 6c^4\mu^5\mu_3(10c^4 - 2c^2 - 1) + c^8\mu^8(27c^4 - 9c^2 - 5)\} + O(n^{-3}) \]

(2.33)

These expressions reduce and agree with the expression in the section (2.3) for the normal distribution.

2.5.2 Small Sample Robustness of the Estimators

From the asymptotic expressions for the bias and MSE of the eight estimators given in section (2.5), it is difficult to judge whether the estimators are robust. Thus a simulation study is carried out to compare the robustness of these estimators. The ratios of the efficiencies (reciprocal of the MSEs) of the estimators under the general distribution to that of normal distribution are worked out. These ratios (expressed in percentages) indicate how far the estimators are robust under violation of normality assumption. If the ratios are around 100, the conclusion is that the MSE does not change even when the true distribution is not normal and the estimators lead to the same conclusions as in the case of normal distribution. However, if the ratios are less than 60, it can be concluded that the estimators are less precise when the underlying distribution is not normal. In the sequel these ratios are referred as indices of robustness. The distributions chosen for the simulation are uniform, logistic, lognormal and gamma. It may be noted that uniform distribution is thin tailed while gamma distribution is heavy tailed. Further two of the distributions, namely, uniform and logistic are symmetric while the other two are right skewed. The simulation experiment is similar to
the one described under normality (section 2.3). The parameters of these distributions have been chosen such that the mean and \(c_v\) of these distributions coincide with the mean and \(c_v\) of the normal distribution. The index of robustness is obtained and is given in Figure 2.3. From the figure, the following conclusion emerges:

1. For the uniform distribution when \(n=40\), the index of robustness of ML, bias corrected ML, MDI, Khan, and Gleser Healy estimators increases as \(c_v\) increases and cluster around the value 122 when \(c_v = 0.5\) and raises to the value around 200 - 210 when \(c_v\) is 2 indicating that these estimators become more precise than the normal distribution although the estimators are derived from the normality assumption. On the contrary, for the logistic distribution the indices for these estimators cluster around 100 for small and moderate value of \(c_v\) \((\leq 0.5)\). As \(c_v\) increases the index decreases and assumes value around 70 when \(c_v = 1.5\). For the lognormal and gamma distributions, the pattern is similar to that of logistic distribution. However, the decrease in the index is more drastic compared to the logistic distribution as \(c_v\) increases. For instance, when \(c_v = 1.0\), the index of robustness for lognormal and gamma distributions cluster around 11 and 25 respectively while for logistic distribution it is around 75. The same conclusion follows when the sample sizes are 20 and 80.

2. A salient feature to be observed is that, for all the four distributions, the sample mean, Searls, and equivariant \((C_1)\) estimators are robust whatever may be the value of \(c_v\) and sample size (the index cluster around 100, irrespective of the value of \(c_v\) and sample size).

3. The performance of Bhat and Rao estimator for all the four distributions is same as that of ML and other estimators for small and moderate values of \(c_v\) \((\leq 1.0)\). As \(c_v\) increases the index converges to the value 100 indicating that for any distribution, the performance of this estimator is same as that of the normal distribution.
By comparing the estimated MSEs of all the estimators for different distributions, we conclude that for the uniform and the logistic distributions, the bias corrected ML estimator and ML estimator have got minimum MSE. For the lognormal and the gamma distributions, Searls, and equivariant estimators have got minimum MSE. Thus for the symmetric distributions ML and bias corrected ML estimators perform well compared to the other estimators.

2.6 Real Life Examples

2.6.1 Example 1: Crop cutting experiment

The first example relates to crop cutting experiment conducted by the district statistical office of Dakshina Kannada District, Karnataka, India. The data relates to the yield of rice for the years 2005-2006. The number of agricultural holdings on which the experiments were conducted is 77. In each holding, the data was obtained from two plots. The records on yield of rice were also available for the crop cutting experiment for the year 2004-2005, although there were some missing entries. The number of holdings for which the past data were available is equal to 63.

This is a situation where one can make use of the past data to have a knowledge regarding the $cv$. The $cv$ for the past data is 0.28. The QQ plot of the yield for the year 2005-2006 is given in Figure 2.4, which indicates that the observations are normally distributed. This conclusion is further supported by the Lilliefors Kolmogorov-Smirnov test with $p$-value of 0.2. The estimates of the mean, bias, and MSE for the ten estimators are presented in Table 2.2. From the table, it is clear that for the plot I, the ML, bias corrected ML, MDI, and bias corrected MDI have the lowest MSE (0.1135). This is followed by the Gleser Healy (0.1136), Khan (0.1141), Bhat and Rao (0.1144), Searls estimator (0.1286) and equivariant estimator (0.1288). The relative efficiency of the ML estimator is 113%, indicating that sample mean requires 13% additional observation to have the same efficiency as that of ML estimator.
This clearly indicates the advantage of using the knowledge of c.v. for estimating the mean. For the plot II, the c.v. for the past data is 0.24. From the Table 2.2, it is clear that, the MSE for Bhat and Rao estimator is least followed by the ML and other estimators. This corroborates the conclusion arrived at section (2.4.2).

2.6.2 Example 2: Estimation of fish catch

This example relates to the estimation of fish catches in the Nethravathi river basin in Coastal Karnataka, India. The primary objective of the study was the estimation of the fish stock in the Nethravathi river basin. For this purpose, a sample of 45 fishermen was randomly selected and the fish catches were noted down on a daily basis. To investigate the performance of the different estimators, we have used the monthly fish catches for December, 2005, and January, 2006. The objective was to estimate the fish catches per boat for the January, 2006 using the information on the c.v. for the December, 2005. The QQ plot and the histogram of the fish catches for the January, 2006 is reported in Figure 2.5. The QQ plot indicates that the data is not normal. We have tried various transformations to see if the observations follow normality. In all the cases, the conclusions were that even after the transformation, the data does not follow normality.

To save space, the results are not reported here. The histogram indicates that the underlying distribution is right skewed. For this data, the coefficient of skewness (\(\sqrt{\beta_1}\)) is 2.35, and the coefficient of kurtosis (\(\beta_2\)) is 7.8. The c.v. of fish catches for the month of December, 2005, is 1.06, and for the January, 2006, is 1.07.

The bias and MSE of the various estimators are presented in Table 2.3. From the table, it is clear that, for this data set which is not normal and right skewed, the MSE of the Searls estimator is the lowest (4.4302) followed by the equivariant \(C_1\) estimator (4.4359) and sample mean (4.7704). The estimator which has the maximum MSE is Bhat and Rao estimator (8.48). The performance of ML, MDI, Khan, and Gleser Healy estimators are poor for this data set. This conclusion corroborates the Searls, equivariant, and the sample mean.
are robust for skewed distribution as observed in section (2.5.2)

2.7 Conclusions

This chapter focuses on the performance of the ML estimator compared to other estimators proposed in the literature along with a new estimator. We were able to demonstrate that ML estimator is not the second order deficient estimator. However, the small sample performance of the ML estimator is remarkably superior to the other estimator, especially when the CV is large. The new estimator performs well for small values of CV and when the sample size is moderate to large. The other estimators that performed well are MDI, and Gleser Healy estimator. For small values of CV (≤ 0.2), all estimators perform equally well irrespective of sample size.

To see the concentration of the estimators around the true value, histograms of the various estimators were also plotted. To save space, we have reported the histograms when CV is 0.01 and 3.0 and n=40 in Figure 2.6. The concentration around the true value is more for the ML, bias corrected ML, and MDI estimator compared to the other estimators for moderate to large value of CV. This indicates superior performance of the ML, bias corrected ML, and the MDI estimators.

The estimators proposed by Khan (1968) and Gleser and Healy (1976) are efficient in the class of linear estimators when the parameter space is restricted to the positive side of the real line. However, the simulation configuration includes the case where the sample mean is negative. This can be seen from the plot of the probability density function of $N(\mu, c^2\mu^2)$ family presented in Figure 2.7. These estimators perform well even in the case where the parameter space is the real line.

The estimator proposed by Searls, and Guo and Pal (2003) possesses optimal theoretical properties in the class of equivariant estimators. However, based on small sample performance, we do not recommend the use of this estimator for the practitioners and scientists in an allied
discipline The estimator proposed by Bhat and Rao performs well when the sample size is moderate to large and for small and moderate values of $c_v$. Our recommendation is to use the ML estimator for small samples and moderate to large values of $c_v$ and the estimator of Bhat and Rao for small values of $c_v$ when the sample size is moderate to large.

In the present investigation we have also considered the small sample performance of these estimators for the violation of normality assumption. The distributions used for this investigation are the two-parameter uniform, logistic, lognormal and gamma. The first two distributions are symmetric while the latter two are right skewed. The investigation indicates that for small values of $c_v$ ($\leq 0.5$), all estimators are robust. Therefore our recommendation for the practitioners is to use ML or MDI estimators when the samples are from symmetric distribution and to use Searls, and equivariant $C_1$ class of estimators when the samples are from skewed distribution. Real life examples confirm this conclusion.
| cv  | Sample | ML    | BCML  | MDI  | Khan | Gleser | Seals | Equiva- | Bhat & |
|-----|--------|-------|-------|------|------|--------|-------| rent C_1 | Rao   |
| 0.01| 0.0018 | 0.0013| 0.0013| 0.0013| 0.0018| 0.0013| 0.0013| 0.0013  | 0.0006 |
| 0.05| -0.0084| -0.0213| -0.0213| -0.0214| -0.0089| -0.0213| -0.0209| -0.0209| -0.0383 |
| 0.10| -0.0233| -0.0733| -0.0738| -0.0742| -0.0246| -0.0736| -0.0733| -0.0732| -0.1417 |
| 0.15| 0.0265 | -0.0797| -0.0821| -0.0843| 0.0255 | -0.0824| -0.1279| -0.0865| -0.2752 |
| 0.20| -0.0207| -0.1944| -0.2013| -0.2078| -0.0158| -0.2014| -0.1891| -0.2263| -0.4518 |
| 0.30| 0.0158 | -0.3351| -0.3641| -0.3925| 0.0167 | -0.3669| -0.4320| -0.4196| -1.0182 |
| 0.40| -0.0179| -0.5365| -0.6095| -0.6803| -0.0008| -0.6123| -0.7924| -0.7974| -1.7651 |
| 0.50| 0.0268 | -0.7013| -0.8392| -0.9767| -0.0055| -0.8492| -1.2544| -1.1943| -2.6923 |
| 0.75| 0.0023 | -0.9755| -1.3224| -1.6813| -0.0042| -1.3542| -2.6694| -2.7101| -5.5561 |
| 1.00| 0.0907 | -1.0593| -1.6090| -2.1774| 0.0606 | -1.6492| -4.5974| -4.6456| -9.4296 |
| 1.50| 0.2590 | -1.1158| -1.9432| -2.8196| 0.0962 | -2.0151| -10.2833| -9.8467| -20.1010 |
| 2.00| -0.2536| -1.1832| -2.1592| -3.1978| 0.0531 | -2.2459| -16.5141| -16.8495| -34.9397 |
| 2.50| 0.2288 | -1.0933| -2.1533| -3.2972| 0.1389 | -2.2614| -23.6317| -23.6148| -54.6400 |
| 3.00| 0.2801 | -1.2354| -2.3434| -3.5450| -0.0070| -2.4613| -31.1889| -30.8279| -77.0157 |
Table 2.2  Estimated population mean, bias and MSE of various estimators for the yield

<table>
<thead>
<tr>
<th>Estimators</th>
<th>Plot-I (c.v.=0.2819)</th>
<th></th>
<th></th>
<th>Plot-II (c.v.=0.2374)</th>
<th></th>
<th></th>
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<tbody>
<tr>
<td></td>
<td>Est Mean (in kg)</td>
<td>Bias</td>
<td>MSE</td>
<td>Est Mean (in kg)</td>
<td>Bias</td>
<td>MSE</td>
</tr>
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<td>0.0000</td>
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<td>11 2458</td>
<td>0.0004</td>
<td>0.0832</td>
</tr>
<tr>
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<td>11 2454</td>
<td>0.0000</td>
<td>0.0832</td>
</tr>
<tr>
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<td>0.0829</td>
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<td>0.0000</td>
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<td>11 1840</td>
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<td>0.0825</td>
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<tr>
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<td>11 1766</td>
<td>0.0000</td>
<td>0.0822</td>
</tr>
<tr>
<td>Equivalent (C₁)</td>
<td>11 1744</td>
<td>-0.0114</td>
<td>0.1286</td>
<td>10 7746</td>
<td>-0.0078</td>
<td>0.0847</td>
</tr>
<tr>
<td>Searls</td>
<td>11 1737</td>
<td>-0.0115</td>
<td>0.1288</td>
<td>10 7722</td>
<td>-0.0079</td>
<td>0.0849</td>
</tr>
<tr>
<td>Bhat and Rao</td>
<td>11 2758</td>
<td>-0.0225</td>
<td>0.1144</td>
<td>11 0317</td>
<td>-0.0157</td>
<td>0.0806</td>
</tr>
</tbody>
</table>
Table 2.3  *Estimated population mean, bias and MSE of various estimators for the data on fish catches for the month of January, 2006*

<table>
<thead>
<tr>
<th>Estimators</th>
<th>Est Mean (in kg)</th>
<th>Bias</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Mean</td>
<td>13 8222</td>
<td>0.0000</td>
<td>4.7704</td>
</tr>
<tr>
<td>ML</td>
<td>13 8822</td>
<td>-0.1749</td>
<td>6.9043</td>
</tr>
<tr>
<td>BCML</td>
<td>13 8452</td>
<td>-0.2143</td>
<td>6.9363</td>
</tr>
<tr>
<td>MDI</td>
<td>13 8821</td>
<td>-0.2159</td>
<td>7.0171</td>
</tr>
<tr>
<td>BCMDI</td>
<td>13 9190</td>
<td>-0.1763</td>
<td>6.9809</td>
</tr>
<tr>
<td>Khan</td>
<td>14 0443</td>
<td>-0.0510</td>
<td>6.8949</td>
</tr>
<tr>
<td>Gleser Healy</td>
<td>13 9351</td>
<td>-0.1679</td>
<td>6.9824</td>
</tr>
<tr>
<td>Equivalent (Ci)</td>
<td>13 4855</td>
<td>-0.3285</td>
<td>4.4302</td>
</tr>
<tr>
<td>Searls</td>
<td>13 4876</td>
<td>-0.3278</td>
<td>4.4359</td>
</tr>
<tr>
<td>Bhat and Rao</td>
<td>13 8815</td>
<td>-0.5141</td>
<td>8.4841</td>
</tr>
</tbody>
</table>
Figure 2.1  Estimated MSE of various estimators for different values of $c_v$ when $n=20$

Figure 2.2  Estimated MSE of various estimators for different values of $c_v$ when $n=80$

Note Lines overlap and thus all the lines are not visible separately
Figure 2.3  Estimated index of sensitivity for various estimators and distributions for different values of CV when n=40
Figure 2.4  QQ plot for the yield data for the year 2005-2006 - plot I

Figure 2.5  QQ plot and Histogram of fish catches for the January, 2006
Figure 2.6 Histograms based on 50000 replicated samples for various estimators when n=40 and cv =0.01 and 0.0.

Figure 2.7 Probability density function of $N(\mu, \sigma^2)$, for $\mu=100$ and cv =0.01 and 0.0.
Appendix A.1

Product Moments of Sample Mean and Sample Variance

Let $X_1, X_2, \ldots, X_n$ be random variables from a general distribution with probability density function $f_{\mu, \sigma^2}(x)$, admitting first six moments. Let $\bar{X}$ and $S^2$ denote the sample mean and variance respectively. Then $\rho_{ij} = E((\bar{X} - \mu)^i(S^2 - \sigma^2)^j)$ for $i, j = 1, 2, 4$ and $i + j \leq 4$ denote the $(i+j)^{th}$ order cross product moments of sample mean and variance and is given by,

\[
\begin{align*}
\rho_{10} &= 0 \\
\rho_{01} &= 0 \\
\rho_{20} &= \frac{\sigma^2}{n} \\
\rho_{11} &= \frac{\mu_3}{n} \\
\rho_{02} &= \frac{\mu_4 - \sigma^4}{n} + \frac{2\sigma^4}{n^2} \\
\rho_{30} &= \frac{\mu_3}{n^2} \\
\rho_{21} &= \frac{\mu_4 - 3\sigma^4}{n^2}
\end{align*}
\]
\[
\begin{align*}
\rho_{12} &= \frac{\mu_5 - 6\sigma^2 \mu_3}{n^2} + O(n^{-3}) \\
\rho_{03} &= \frac{\mu_6 - 3\sigma^2 \mu_4 - 6\mu_3^2 + 2\sigma^6}{n^2} + O(n^{-3}) \\
\rho_{40} &= \frac{3\sigma^4}{n^2} + O(n^{-3}) \\
\rho_{31} &= \frac{3\sigma^2 \mu_3}{n^2} + O(n^{-3}) \\
\rho_{22} &= \frac{\sigma^2 (\mu_4 - \sigma^4) + 2\mu_3^2}{n^2} + O(n^{-3}) \\
\rho_{13} &= \frac{3\mu_3 (\mu_4 - \sigma^4)}{n^2} + O(n^{-3}) \\
\rho_{04} &= \frac{3(\mu_4 - \sigma^4)^2}{n^2} + O(n^{-3})
\end{align*}
\]

where \( \mu_i = E(X - \mu)^i \), for \( i = 1 \) to \( 6 \), the \( i^{th} \) central moment of the population mean.

These expressions agree with the corresponding expressions for the normal distribution when the appropriate values for the moments of the normal distribution are substituted in the above expression.
Appendix A.2

Derivation of Asymptotic Mean Squared Error for an Estimator

Let $X_1, X_2, \ldots, X_n$ be random variables from a general distribution with probability density function $f_{\mu, \sigma^2}(x)$, admitting first six moments. Let $\bar{X}$ and $S^2$ denote the sample mean and variance respectively. Consider $\hat{\mu}$ (which is a function of $\bar{X}$ and $S^2$) be an estimator of the parameter $\mu$.

Let $\rho_{ij} = E((\bar{X} - \mu)^i (S^2 - \sigma^2)^j)$ denote the cross product moments of sample mean and variance and $D_{ij} = \frac{\partial^{i+j} \hat{\mu}}{\partial \bar{X}^i \partial S^2^j}$ is the $(i+j)^{th}$ order partial derivative of an estimator $\hat{\mu}$ at $\bar{X} = \mu$ and $S^2 = \sigma^2$ for $i, j = 1, 2, 3, 4$ and $i + j \leq 4$.

Then the Taylor series expansion for $\hat{\mu}$ around $\bar{X} = \mu$ and $S^2 = \sigma^2$ is,

$$
\hat{\mu} = \hat{\mu} |_{\bar{X} = \mu, S^2 = \sigma^2} + \left[(\bar{X} - \mu) D_{10} + (S^2 - \sigma^2) D_{01}\right] + \frac{1}{2} \left[(\bar{X} - \mu)^2 D_{20} + 2(\bar{X} - \mu) (S^2 - \sigma^2) D_{11} + (S^2 - \sigma^2)^2 D_{01}\right] + \frac{1}{6} \left[(\bar{X} - \mu)^3 D_{30} + 3(\bar{X} - \mu)^2 (S^2 - \sigma^2) D_{21} + 3(\bar{X} - \mu) (S^2 - \sigma^2)^2 D_{12} + (S^2 - \sigma^2)^3 D_{03}\right] + \frac{1}{24} \left[(\bar{X} - \mu)^4 D_{40} + 4(\bar{X} - \mu)^3 (S^2 - \sigma^2) D_{31} + 6(\bar{X} - \mu)^2 (S^2 - \sigma^2)^2 D_{22} + 4(\bar{X} - \mu) (S^2 - \sigma^2)^3 D_{13} + (S^2 - \sigma^2)^4 D_{04}\right]
$$

(2.34)
The asymptotic bias of the estimator \( \hat{\mu} \) is given by,

\[
B(\hat{\mu}) = E(\hat{\mu} - \mu) = (\hat{\mu} |_{x=\mu, \sigma^2=\sigma^2} - \mu) + \frac{1}{2} [\rho_{20} D_{20} + 2 \rho_{11} D_{11} + \rho_{02} D_{02}] + O(n^{-2})
\]

(2.35)

If the estimator \( \hat{\mu} \) is unbiased to the order of \( O(n^{-1}) \), the first term in (2.35) \( e \) \( (\hat{\mu} |_{x=\mu, \sigma^2=\sigma^2} - \mu) \) is equal to zero.

The asymptotic MSE of the estimator \( \hat{\mu} \) is given by,

\[
MSE(\hat{\mu}) = E(\hat{\mu} - \mu)^2 = (\hat{\mu} |_{x=\mu, \sigma^2=\sigma^2} - \mu)^2 + \left[ \rho_{20} D_{10}^2 + 2 \rho_{11} D_{10} D_{01} + \rho_{02} D_{01}^2 \right] + \rho_{30} D_{10} D_{20}
\]

\[
+ \rho_{21} (2 D_{10} D_{11} + D_{20} D_{01}) + \rho_{12} (D_{10} D_{02} + 2 D_{01} D_{11}) + \rho_{03} D_{01} D_{02}
\]

\[
+ \rho_{40} \left( \frac{D_{30}^2}{4} + \frac{D_{10} D_{30}}{3} \right) + \rho_{31} \left( D_{20} D_{11} + D_{10} D_{21} + \frac{D_{01} D_{30}}{3} \right)
\]

\[
+ \rho_{22} \left( D_{11}^2 + D_{10} D_{12} + D_{01} D_{21} + \frac{D_{20} D_{02}}{2} \right)
\]

\[
+ \rho_{13} \left( D_{11} D_{02} + \frac{D_{10} D_{03}}{3} + D_{01} D_{12} \right) + \rho_{04} \left( \frac{D_{01} D_{03}}{3} + \frac{D_{02}^2}{4} \right)
\]

\[
+ 2 (\hat{\mu} |_{x=\mu, \sigma^2=\sigma^2} - \mu) \{ \frac{1}{2} [\rho_{20} D_{20} + 2 \rho_{11} D_{11} + \rho_{02} D_{02}] + \rho_{30} D_{30} + 3 \rho_{21} D_{21} + 3 \rho_{12} D_{12} + \rho_{03} D_{03} \}
\]

\[
+ \frac{1}{6} [\rho_{30} D_{30} + 3 \rho_{21} D_{21} + 3 \rho_{12} D_{12} + \rho_{03} D_{03}] + \frac{1}{24} [\rho_{40} D_{40} + 4 \rho_{31} D_{31} + 6 \rho_{22} D_{22} + 4 \rho_{13} D_{13} + \rho_{04} D_{04}] + O(n^{-3})
\]

(2.36)

Using the (2.35) and (2.36), we can find the asymptotic bias and MSE for an estimator.