Chapter 8

Higher Order Asymptotics

8.1 Introduction

In the previous chapters several estimators and tests are proposed for normal mean with known $c.v.$ The small sample performance of the estimators and the tests are compared using simulations. In this chapter we derive Edgeworth expansions for the null distribution of the test statistics. The tests considered are the Wald, Locally Most Powerful, tests based on MDI and Bhat and Rao estimators. These higher order expansions would help us to determine the small sample type I error rates. They are also useful to find out the concentration of the estimator around the true value. The robustness of the tests for the violation of normality assumption can also be attempted by Edgeworth expansions for the distribution of the test statistic using a general distribution admitting eight central moments. Such an attempt has been used in the past and a good number of papers have appeared. Some of the salient works are the following:

The general theory of Edgeworth expansions and validity of this type of expansions have been described by many authors, Wallace (1958), Chambers (1967), Bhattacharya and Rao (1976), Bhattacharya and Ghosh (1978), Barndorff-Nielson and Cox (1979), Pederson (1979), Niki and Konishi (1986). A simple exposition of Edgeworth and saddle point approximation...
for some univariate, multivariate and conditional distributions with applications were illustrated by Barndorff-Nielsen and Cox (1979). Bhattacharya and Ghosh (1978) first derived the Edgeworth expansion for functions of sample mean. In their work, they provided a rigorous and very general development of the results of the earlier authors by assuming crucial Cramer c-condition on the joint distribution of all the components of the vector variable. This has motivated Bar and Rao (1991) to establish Edgeworth expansion for functions of sample means when only the partial Cramer c-condition is satisfied. Using the result of Bhattacharya and Ghosh (1978), Ghosh, Sinha and Subrahmaniam (1979), obtained the Edgeworth expansions of distributions of Fisher consistent estimators for curved exponential family or parent distribution.

Hall (1983), provided the method of inverting a general Edgeworth expansion, so as to correct a statistic for the effects of non-normality. Hall (1987), also considered the Edgeworth expansion of student's t-statistic under weaker moment condition. Rao and Bhatta (1989) derived the Edgeworth expansion for testing the hypothesis of c v. They derived the large sample tests based on normal approximation for sample c v. The Edgeworth expansion has been used to characterize the probability density function (pdf), the cosmic velocity-density function, cosmic microwave background fluctuations, gravitational evolution of the cosmic one-point pdf. For more details we refer (Gaztanaga et al (2000) the classical text book by Severmu (2000)).

The organization of the chapter is as follows. In section 8.2, we derive the product moments of sample mean and variance for any general distribution. In section 8.3, Edgeworth expansions for the null distributions of the above mentioned test statistics are derived using the normality assumption while the Edgeworth expansions for the null distributions of these tests based on a general distribution are derived in section 8.4 followed by the conclusion in 8.5.
8.2 Product Moments of Sample Mean and Sample Variance

We derive six cross product moments of sample mean $X$ and sample variance $S^2$ for any general distribution admitting first eight moments. The algebra is quite lengthy and the details are not reported here. In the appendix B 1, we just illustrate the derivation of the third cross moments of $X$ and $S^2$, namely, $\rho_{21} = E(\bar{X} - \mu)^2(S^2 - \sigma^2)$

Let $\rho_{ij} = E(\bar{X} - \mu)^i(S^2 - \sigma^2)^j$ for $i, j = 1, 2, \ldots, 6$ and $i + j \leq 6$. Then the product moments are,

\[
\begin{align*}
\rho_{10} &= 0 \\
\rho_{01} &= 0 \\
\rho_{20} &= \frac{c^2 \mu^2}{n} \\
\rho_{11} &= \frac{\mu_3}{n} \\
\rho_{02} &= \frac{\mu_4 - c^4 \mu^4 + 2 c^4 \mu^4}{n^2} \\
\rho_{30} &= \frac{\mu_3}{n^2} \\
\rho_{21} &= \frac{\mu_4 - 3 c^4 \mu^4}{n^2} \\
\rho_{12} &= \frac{\mu_5 - 6 c^2 \mu^2 \mu_3}{n^2} + \frac{4 c^2 \mu^2 \mu_3}{n^3} + O(n^{-4}) \\
\rho_{03} &= \frac{\mu_6 - 3 c^2 \mu^2 \mu_4 - 6 \mu_3^2 + 2 c^3 \mu^6}{n^3} + \frac{4 c^2 \mu^2 (3 \mu_4 - 5 c^4 \mu^4)}{n^3} + O(n^{-4}) \\
\rho_{40} &= \frac{3 c^4 \mu^4}{n^2} + \frac{\mu_4 - 3 c^4 \mu^4}{n^3} + O(n^{-4}) \\
\rho_{31} &= \frac{3 c^2 \mu^2 \mu_3}{n^2} + \frac{\mu_5 - 10 c^2 \mu^2 \mu_3}{n^3} + O(n^{-4}) \\
\rho_{22} &= \frac{c^2 \mu^2 (\mu_4 - c^4 \mu^4) + 2 \mu_3^2 + \mu_5 - 11 c^2 \mu^2 \mu_4 - 6 \mu_3^2 + 20 c^6 \mu^6}{n^3} + O(n^{-4}) \\
\rho_{13} &= \frac{3 \mu_3 (\mu_4 - c^4 \mu^4)}{n^2} + \frac{\mu_7 - 9 c^2 \mu^2 \mu_5 - 15 \mu_3 (\mu_4 - 4 c^4 \mu^4)}{n^3} + O(n^{-4})
\end{align*}
\]
\[
\begin{align*}
\rho_{04} &= \frac{3 (\mu_4 - c^4 \mu^4)^2}{n^2} + \frac{\mu_8 - 4 c^2 \mu^2 \mu_6 - 24 \mu_3 \mu_5 + 3 \mu_4 (8 c^4 \mu^4 - \mu_4) + 96 c^2 \mu^2 \mu^2 - 18 c^3 \mu^3}{n^3} + O(n^{-4}) \\
\rho_{50} &= \frac{10 c^4 \mu^2 \mu_3}{n^3} + O(n^{-4}) \\
\rho_{41} &= \frac{6 c^2 \mu^2 (\mu_4 - 3 c^4 \mu^4) + 4 \mu_5^2}{n^3} + O(n^{-4}) \\
\rho_{32} &= \frac{7 \mu_3 (\mu_4 - 3 c^4 \mu^4) + 3 c^2 \mu^2 \mu_5}{n^3} + O(n^{-4}) \\
\rho_{23} &= \frac{c^2 \mu^2 (\mu_6 - 15 c^2 \mu^2 \mu_4 - 42 \mu_3^2 + 11 c^6 \mu^6) + 3 \mu_4^2 + 6 \mu_5 \mu_3}{n^3} + O(n^{-4}) \\
\rho_{14} &= \frac{4 \mu_3 (\mu_6 - 12 c^2 \mu^2 \mu_4 - 6 \mu_3^2 + 11 c^6 \mu^6) + 6 \mu_5 (\mu_4 - c^4 \mu^4)}{n^3} + O(n^{-4}) \\
\rho_{05} &= \frac{10 (\mu_6 - 3 c^2 \mu^2 \mu_4 - 6 \mu_3^2 + 2 c^6 \mu^6) (\mu_4 - c^4 \mu^4)}{n^3} + O(n^{-4}) \\
\rho_{60} &= \frac{15 c^6 \mu^6}{n^3} + O(n^{-4}) \\
\rho_{51} &= \frac{15 c^6 \mu^6 \mu_3}{n^3} + O(n^{-4}) \\
\rho_{42} &= \frac{3 c^2 \mu^2 (\mu_4 - c^4 \mu^4) + 12 c^2 \mu^2 \mu_3^2}{n^3} + O(n^{-4}) \\
\rho_{33} &= \frac{9 c^2 \mu^2 \mu_3 (\mu_4 - c^4 \mu^4) + 6 \mu_5^3}{n^3} + O(n^{-4}) \\
\rho_{24} &= \frac{3 c^2 \mu^2 (\mu_4 - c^4 \mu^4)^2 + 12 (\mu_4 - c^4 \mu^4) \mu_3^2}{n^3} + O(n^{-4}) \\
\rho_{15} &= \frac{15 \mu_3 (\mu_4 - c^4 \mu^4)^2}{n^3} + O(n^{-4}) \\
\rho_{06} &= \frac{15 (\mu_4 - c^4 \mu^4)^3}{n^3} + O(n^{-4}) \\
\end{align*}
\]

where \( \mu_i = E(X - \mu)^i \), for \( i = 1 \) to 8, the \( i \)th central moment of the population mean.

These expressions agree with the corresponding expressions for the normal distribution when the appropriate values for the moments of the normal distribution are substituted in the above expression.
8.3 Edgeworth Expansions for the Null Distribution of the Test Statistics using Normal Distribution

Let \( \hat{\theta} \) be a CAN estimator of \( \theta \). Then the expansion for the distribution of \( \sqrt{n}(\hat{\theta} - \theta)[V(\theta)]^{-1/2} \) is derived by Ghosh et al (1979) and is given below.

Theorem:

\[
P_{\theta}\left\{ \sqrt{n}(\hat{\theta} - \theta)[V(\theta)]^{-1/2} \leq x \right\} = \Phi(x) + \frac{\phi_{1x}(\theta)}{n^{1/2}} + \frac{\phi_{2x}(\theta)}{n} + o(n^{-1}), \tag{8.1}
\]

where

\[
\Phi(x) = \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt,
\]
\[
\phi(x) = \Phi'(x),
\]
\[
\phi_{1x}(\theta) = \int_{-\infty}^{x} \left\{ K_{11}(\theta)H_1(z) + \frac{K_{21}(\theta)}{2}H_2(z) + \frac{K_{31}(\theta)}{6}H_3(z) \right\} \phi(z) \, dz,
\]
\[
\phi_{2x}(\theta) = \int_{-\infty}^{x} \left\{ K_{12}(\theta)H_1(z) + \frac{K_{22}(\theta)}{2}H_2(z) + \frac{K_{32}(\theta)}{6}H_3(z) + \frac{K_{42}(\theta)}{24}H_4(z) \right\} \phi(z) \, dz,
\]
\[
H_p(x)\phi(x) = \left( -\frac{d}{dx} \right)^p \phi(x),
\]

\( H_p(x) \) denotes Hermite polynomial and the term \( o(n^{-1}) \) in (8.1) is uniform in \( \theta \).

\[
K_r(\theta) = \begin{cases} 
\frac{K_{r-1}(\theta)}{n^{1/2}} + \frac{K_{r2}(\theta)}{n} + o(n^{-1}) & \text{for } r = 1, 3, \\
1 + \frac{K_{r-1}(\theta)}{n^{1/2}} + \frac{K_{r2}(\theta)}{n} + o(n^{-1}) & \text{for } r = 2, \\
\frac{K_{r2}(\theta)}{n} + o(n^{-1}) & \text{for } r = 4, \\
o(n^{-1}) & \text{for } r \geq 5
\end{cases}
\]
We derive the Edgeworth expansions for the distribution of the tests statistics by computing expression for first four moments and substituting in (8.1)

8.3.1 Wald Test

The derivation of the Edgeworth expansion requires the computation of the first four cumulants of $\sqrt{n}(\hat{\mu}_{ML} - \mu_0) [I(\hat{\mu}_{ML})]^{-1/2}$ where $\hat{\mu}_{ML}$ and $I(\hat{\mu}_{ML})$ are given in (2.4) and (5.4) respectively. The cumulants are derived using Taylor series expansion of $(\hat{\mu}_{ML} - \mu_0)$ using delta method (Rao, 1965). Again the algebra is quite lengthy and the details are not shown here. In Appendix B 2, we have sketched the derivation of the moments for a general test statistic. The first four cumulants of $\sqrt{n}(\hat{\mu}_{ML} - \mu_0) [I(\hat{\mu}_{ML})]^{-1/2}$ to the order of $O(n^{-1})$ are given below

$$
K_1 = \frac{c^2 (c + 1)}{\sqrt{n} (1 + 2c^2)^{3/2}} + O(n^{-3/2}),
$$

$$
K_2 = 1 + \frac{2 (7c^4 + 12c^2 + 4)c^2}{n (1 + 2c^2)^3} + O(n^{-2}),
$$

$$
K_3 = \frac{2 (5c^3 + 3)c}{\sqrt{n} (1 + 2c^2)^{3/2}} + O(n^{-3/2}),
$$

and

$$
K_4 = \frac{12 (7c^4 + 6) c^2}{n (1 + 2c^2)^2} + O(n^{-2})
$$

8.3.2 Locally Most Powerful Test

To test $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$, Cox and Hinkley (1974) suggested two one sided LMP test when the test statistic is not traceable. This test is same as that of score test statistic and is given by $\sqrt{I(\mu_0) \frac{\partial \log L}{\partial \mu_0}}$. The simplified expression for $\frac{\partial \log L}{\partial \mu_0}$ is given in (5.3). Thus the
The first four cumulants of this statistic are given by,
\[ K_1 = \frac{c}{\sqrt{n}(1 + 2c^2)^{3/2}} + O(n^{-3/2}), \]
\[ K_2 = 1 + \frac{2c^2}{n(1 + 2c^2)} + O(n^{-2}), \]
\[ K_3 = \frac{2c(4c^2 + 3)}{\sqrt{n}(1 + 2c^2)^{3/2}} + O(n^{-3/2}), \]
and
\[ K_4 = \frac{12c^3(3c^2 + 4)}{n(1 + 2c^2)^2} + O(n^{-2}) \] (8.3)

### 8.3.3 Test Based on MDI Estimator

The test statistic based on MDI estimator is given by \( \sqrt{n}(\hat{\mu}_{MDI} - \mu_0) [V(\hat{\mu}_{MDI})]^{-1/2} \), where \( \hat{\mu}_{MDI} \) and \( V(\hat{\mu}_{MDI}) \) are given in (6.1) and (6.2) respectively. The first four cumulants of this test statistic are given by,
\[ K_1 = \frac{\sqrt{n}(3c^2 + 1)}{(1 + 2c^2)^{3/2}} + O(n^{-3/2}), \]
\[ K_2 = 1 + \frac{2(15c^4 + 14c^2 + 8)c^2}{n(1 + 2c^2)^3} + O(n^{-2}), \]
\[ K_3 = \frac{-2(5c^2 + 3)c}{\sqrt{n}(1 + 2c^2)^{3/2}} + O(n^{-3/2}), \]
\[ K_4 = \frac{12(7c^2 + 6)c^2}{n(1 + 2c^2)^2} + O(n^{-2}) \] (8.4)

### 8.3.4 Test Based on Bhat and Rao Estimator

The test statistic based on Bhat and Rao estimator is given by \( \sqrt{n}(\hat{\mu}_{RB} - \mu_0) [V(\hat{\mu}_{RB})]^{-1/2} \), where \( \hat{\mu}_{RB} \) and \( V(\hat{\mu}_{RB}) \) are given in (6.7) and (6.8) respectively. The first four cumulants of
Using the expressions for the first four cumulants derived above, one can identify $K_{r,1}$, for $r=1$ to $4$, for each of the test statistics, which can be used to estimate the P-values using (8.1)

\[
K_1 = \frac{3 \left(c^2 + 1\right)c}{\sqrt{n} \sqrt{1 + 2c^2}} + O(n^{-3/2}),
\]

\[
K_2 = 1 + \frac{2 \left(18c^6 + 45c^4 + 36c^2 + 10\right)c^2}{n \left(1 + 2c^2\right)^2} + O(n^{-2}),
\]

\[
K_3 = \frac{2 \left(5c^2 + 3\right)c}{\sqrt{n} \left(1 + 2c^2\right)^{3/2}} + O(n^{-3/2}),
\]

and

\[
K_4 = \frac{12 \left(7c^2 + 6\right)c^2}{n \left(1 + 2c^2\right)^2} + O(n^{-2}) \quad (8.5)
\]

8.4 **Edgeworth Expansions for the Null Distribution of the Test Statistics using General Distribution**

We have also derived the Edgeworth expansion for the null distribution of above mentioned test statistics using the first eight moments of a general distribution

8.4.1 Wald Test

\[
K_1 = \frac{-7\mu_0^4c^6 + \mu_0^4c^2 + 3c^2\mu_4 + \mu_4 + 8\mu_3\mu_0c^2 + 3\mu_3\mu_0}{\sqrt{n} \left(1 + 2c^2\right)^{5/2} \mu_0^4c} + O(n^{-3/2}),
\]
\[ K_2 = \frac{2\mu_3 \mu_0 + \mu_0^4 c^2 - \mu_0^4 c^4 + \mu_4}{\mu_0^4 c^2 (1 + 2 c^2)} - \frac{1}{n(n+2)^2 c^2 \mu_0^8} \left[ 2 \mu_0^2 \mu_6 (3 c^2 + 1) (1 + 2 c^2)^2 + 2 \mu_0^2 \mu_5 (11 c^2 + 4) (1 + 2 c^2)^2 - 2 \mu_4^2 (39 c^4 + 24 c^2 + 4) - 2 \mu_4 \mu_3 \right. \\
\left. (100 c^8 - 172 c^6 - 96 c^4 + 13 c^2 + 8) - \mu_0^2 \mu_2 (548 c^4 + 384 c^3 + 69) + 2 \mu_5 \mu_0^5 (162 c^6 - 267 c^5 - 249 c^4 - 46 c^3 + 1) \\
- \mu_0^8 c^4 (46 c^6 - 252 c^5 + 40 c^4 + 141 c^3 + 34)] + O(n^{-2}), \right. \\
K_3 = \frac{1}{\sqrt{n \mu_3 \mu_0^3} (1 + 2 c^2)^{3/2} \left[ \mu_0^2 \mu_6 (1 + 2 c^2)^3 + 3 \mu_0^3 \mu_5 (1 + 2 c^2)^2 - 6 \mu_0^4 (3 c^2 + 1) \\
- 2 \mu_0 \mu_4 \mu_3 (15 + 42 c^2) + 3 \mu_0^4 \mu_4 (8 c^6 - 12 c^4 - 3 c^2 + 1) - 12 \mu_0^2 \mu_3^2 (8 c^2 + 3) \\
+ \mu_0^5 \mu_3 (60 c^6 - 98 c^4 - 44 c^2 + 1) - \mu_0^8 c^4 (10 c^6 - 38 c^4 + 22 c^2 + 15)] + O(n^{-3/2}), \right. \\
K_4 = \frac{1}{n \mu_0^4 (1 + 2 c^2)^{6/2} c^4} \left[ \mu_0^4 \mu_6 (1 + 2 c^2)^4 + 4 \mu_0^5 \mu_5 (1 + 2 c^2)^3 - 24 \mu_0^2 \mu_6 \mu_4 (1 + 2 c^2)^2 \\
(3 c^3 + 1) - 12 \mu_0^3 \mu_5 \mu_3 (1 + 2 c^2)^2 (14 c^2 + 5) + 2 \mu_0^5 \mu_0 (1 + 2 c^2)^2 \\
(28 c^6 - 8 c^4 - 32 c^2 + 8) - 30 \mu_0^3 \mu_5 \mu_4 (1 + 2 c^2)^2 (20 c^2 + 7) - 12 \mu_0^4 \mu_5 \mu_3 (1 + 2 c^2)^2 \\
(46 c^3 + 17) + 4 \mu_0^7 \mu_5 (1 + 2 c^2)^2 (48 c^6 - 65 c^4 - 29 c^2 + 1) + 24 \mu_4^3 (28 c^4 + 18 c^2 + 3) \\
+ 12 \mu_0^3 \mu_4 \mu_3^2 (996 c^4 + 700 c^2 + 125) + 12 \mu_0 \mu_4 \mu_3 \mu_1 (139 c^2 + 24 + 206 c^4) \mu_4 - \\
(-618 c^2 + 512 c^8 - 808 c^6 - 33 c^4) \mu_0^4] - 3 \mu_0 \mu_0^4 (288 c^4 + 76 c^2 + 400 c^8 \\
+ 33 - 560 c^6) + \mu_0^8 (2096 c^6 - 3216 c^{10} - 84 c^4 + 1 + 2224 c^8 + 768 c^{12} - 142 c^2) \mu_4 \\
+ 24 \mu_0^3 \mu_3^3 (288 c^2 + 53 + 396 c^6) - 12 \mu_0^6 \mu_3^2 (8 + 644 c^8 - 1023 c^4 - 1096 c^6 - 177 c^2) \\
+ 4 \mu_0^6 c^2 \mu_3 (45 c^4 (-80 c^6 + 7 + 24 c^2 + 16 c^8 - 17 c^4) \mu_0^2 - (141 c^2 - 74 c^6 + 454 c^4 \\
+ 16 + 567 c^6 - 564 c^{10}) - 3 \mu_0^4 (1 - 136 c^2 - 530 c^4 - 496 c^{10} - 20 c^6 + 928 c^8 \\
+ 64 c^{12}) \mu_0^4)] + O(n^{-2}) \right. (8.6)
8.4.2 LMP Test

\[ K_1 = \frac{c}{\sqrt{n}\sqrt{1 + 2c^2}} + O(n^{-3/2}), \]

\[ K_2 = \frac{2\mu_3\mu_0 + c^2\mu_0^4 - c^4\mu_0^4 + \mu_4}{c^2(1 + 2c^2)\mu_0^4} + \frac{2(\mu_3\mu_0 + \mu_4 - c^4\mu_0^4)}{n c^2(1 + 2c^2)\mu_0^4} + O(n^{-2}), \]

\[ K_3 = \frac{\mu_6 + 3\mu_0\mu_5 + 3\mu_0^2\mu_4 + (1 - c^2) + c^4\mu_0^4(1 - 6c^2) + c^4\mu_0^4(2c^2 - 3) + O(n^{-3/2})}{\sqrt{n}c^3(1 + 2c^2)^{3/2}\mu_0} + O(n^{-3/2}), \]

\[ K_4 = \frac{1}{n c^4(1 + 2c^2)^2\mu_0^8} \left\{ \mu_8 + 4\mu_0\mu_7 + 2\mu_0^2\mu_6(3 - 2c^2) + 4\mu_0^3\mu_5(1 - 3c^2) - 3\mu_4 + \mu_4 \mu_0 \left[ (12c^4 - 18c^2 + 1)\mu_0^3 - 12\mu_3 - 12\mu_0^3\mu_3^2 \right] + 4c^2\mu_0^5\mu_3 \left[ 45c^4\mu_0^2 + 9(c^2 - 4) \right] - 3c^4\mu_0^8 \left( 2c^4 - 4c^2 + 1 \right) \right\} + O(n^{-2}) \]  

(8.7)

8.4.3 Test Based on MDI Estimator

\[ K_1 = \frac{c^2\mu_0^4(3c^4 + c^2 - 1) - \mu_4(3c^2 + 2) - \mu_3\mu_0(4c^2 + 3)}{\mu_0^4c(1 + 2c^2)^{5/2}\sqrt{n}} + O(n^{-3/2}), \]

\[ K_2 = \frac{2\mu_3\mu_0 + \mu_0^4c^2 - \mu_0^4c^4 + \mu_4}{\mu_0^4c^2(1 + 2c^2)} + \frac{1}{n c^2(1 + 2c^2)^5\mu_0^8} \left\{ -2\mu_0^2\mu_0(3c^2 + 2) \right\} (1 + 2c^2)^3 - 2\mu_0^3\mu_0(7c^2 + 5)(1 + 2c^2)^3 + 2\mu_4(39c^4 + 54c^2 + 19) + 2\mu_0\mu_4\mu_5(102c^4 + 151c^2 + 57) - \mu_4\mu_0^4(84c^6 + 76c^4 + 15c^2 + 12c^2 - 8) + \mu_0^2\mu_3^2(144c^6 + 356c^4 + 324c^2 + 105) + 2\mu_0^5\mu_5(66c^6 + 159c^4 + 150c^4 + 52c^2 - 1) + c^4\mu_0^8(62c^8 + 152c^6 + 162c^4 + 103c^2 + 34) \right\} + O(n^{-2}) \]

\[ K_3 = \frac{1}{\sqrt{n}(1 + 2c^2)^{7/2}\mu_0^8c^3} \left\{ \mu_0^2\mu_0(1 + 2c^2)^2 + 3\mu_0^3\mu_5(1 + 2c^2)^2 - 6\mu_0^4(3c^2 + 2) \right\} - 6\mu_0^4\mu_4(10c^2 + 7) + 3\mu_0^4\mu_4(8c^6 - 3c^2 + 1) - 6\mu_0^2\mu_3^2(4c^4 + 12c^2 + 7) - \mu_0^5\mu_3(12c^8 + 62c^4 + 44c^2 - 1) - c^4\mu_0^8(10c^6 + 16c^4 + 22c^2 + 15) \right\} + O(n^{-3/2}), \]

146
\[ K_4 = \frac{1}{n\mu_0^{12}c^4} \left\{ \mu_0^4 \mu_5(2c^2 + 1)^4 + 4 \mu_0^5 \mu_7(2c^2 + 1)^4 - 24 \mu_0^2 \mu_5 \mu_4(3c^2 + 2) \\
(2c^2 + 1)^2 - 12 \mu_0^3 \mu_5 \mu_4(16c^2 + 11)(2c^2 + 1)^2 - 12 \mu_0^4 \mu_5 \mu_3(4c^2 + 3)(2c^2 + 7) \\
(2c^2 + 1)^2 + 4 \mu_0^7 \mu_6(12c^6 - 29c^4 - 29c^2 + 1)(2c^2 + 1)^2 + 24 \mu_0^4(28c^4 + 38c^2 + 13) \\
+24 \mu_0^2 \mu_5 \mu_3(65 + 130c^4 + 183c^2) - 3 \mu_0^4 \mu_4^2(400c^8 + 304c^6 - 112c^4 - 36c^2 + 41) \\
+12 \mu_0^2 \mu_4 \mu_3^2(144c^8 + 628c^4 + 704c^2 + 237) - 12 \mu_0^5 \mu_4 \mu_3(96c^8 - 128c^6 - 338c^4 \\
-111c^2 + 25) + \mu_0^8 \mu_4(768c^12 + 624c^10 + 160c^8 + 392c^6 - 142c^2 + 1) \\
+48 \mu_0^3 \mu_3^3(60c^6 + 148c^4 + 129c^2 + 39) - 12 \mu_0^4 \mu_3^2(16c^10 - 300c^8 - 796c^6 - 727c^4 \\
-223c^2 + 8) + 4c^2 \mu_0 \mu_3(252c^10 - 94c^8 - 673c^6 - 500c^4 - 139c^2 - 16) + 315 \mu_0^2 c^4 \\
(1 + c^2)^2 - 3 \mu_0^{12} c^4(64c^{12} - 112c^{10} - 560c^8 - 716c^6 - 458c^4 + 136c^2 + 1) + \mathcal{O}(n^{-2}) \right\} \\
(8.8) \\

8.4.4 Test Based on Bhat and Rao Estimator \\

\[ K_1 = \frac{6 \mu_0^4 c^8 + 3 \mu_0 c^4 + \mu_4 - 6 \mu_0^3 \mu_2 c^2 - \mu_3 c^2 \mu_0}{\sqrt{n_c^2 \mu_0^4 (1 + 2c^2)^{3/2}}} + \mathcal{O}(n^{-3/2}), \]

\[ K_2 = \frac{2 \mu_0 \mu_3 + c^2 \mu_0^4 - c^4 \mu_0^4 + \mu_4}{c^2 (1 + 2c^2) \mu_0^4} + \frac{1}{n_c^2 (1 + 2c^2)^{3/2} \mu_0^6} \left\{ -2 c^2 \mu_0 \mu_6 (1 + 2c^2) (3c^2 + 1) \\
+2 c^2 \mu_0 \mu_5 (2c^2 + 1) (6c^6 - 2c^4 - 1) + 8 (3c^2 + 1)^2 \mu_4^2 - 2 \mu_0 \mu_4 \mu_3 (3c^2 + 1) \\
(36c^6 - 4c^2 - 3) + c^4 \mu_4 \mu_4 \mu_0^4 (60c^6 - 132c^4 - 81c^2 - 12) + c^4 \mu_4 \mu_0^4 \\
(60c^6 - 132c^4 - 81c^2 - 12) + c^2 \mu_2 \mu_3^2 (204c^6 + 12 + 41c^2 + 72c^4) - 2 c^4 \mu_0^5 \mu_3 \\
(84c^6 - 72c^4 - 15c^2 - 7c^2 - 2) + c^8 \mu_0^8 (16 + 81c^2 + 36c^6 + 72c^8 + 170c^{10}) \right\} + \mathcal{O}(n^{-2}), \]
\[ K_3 = \frac{1}{\sqrt{\pi c_2^2 c_0^2 (1 + 2 c_2)^{3/2}}} \left\{ -c_2^2 \mu_0^2 \mu_6 (1 + 2 c_2) - 3 c_2^2 \mu_0^3 \mu_5 (1 + 2 c_2) + 6 \mu_4^2 (1 + 3 c_2) \\
-6 \mu_0 \mu_4 \mu_3 (6 c_4 - 5 c_2 - 2) - 3 c_2^2 \mu_0 \mu_4 (22 c_4 + 7 c_2 + 1) + 6 \mu_2^2 \mu_4^2 \\
(6 c_6 - c_4 + 4 c_2 + 1) + \mu_0^3 \mu_3 (108 c_6 + 10 c_2 + 6 c^4 - 1) + \mu_6^8 (86 c_4 + 46 c_2 + 15) \right\} + O(n^{-3/2}), \]

\[ K_4 = \frac{1}{\pi c_2^2 c_0^2 (1 + 2 c_2)^3} \left\{ c_4^4 \mu_0^4 \mu_8 (1 + 2 c_2)^2 + 4 c_4^4 \mu_0^5 \mu_7 (1 + 2 c_2)^2 - 24 c_2^3 \mu_0^2 \mu_6 \mu_4 \\
(1 + 2 c_2)^2 (3 c^2 + 1) + 12 c_2^2 \mu_0^3 \mu_5 \mu_3 (1 + 2 c_2) (6 c_4 - 5 c_2 - 2) + 2 c^6 \mu_0^6 \mu_6 (1 + 2 c_2) \\
(68 c_4 + 22 c_2^2 + 3) + 12 c^2 \mu_0^3 \mu_4 \mu_5 \mu_4 (1 + 2 c_2) (6 c_4 - 11 c^2 - 4) - 12 c_2^3 \mu_0^4 \mu_5 \mu_3 \\
(1 + 2 c_2)^2 (12 c_6 - 8 c_4 + 13 c^2 + 4) - 4 c_4^4 \mu_0^7 \mu_5 (1 + 2 c_2) (54 c_6 - 33 c^4 - 5 c^2 - 1) \\
+72 \mu_4^3 (3 c^2 + 1)^2 - 24 \mu_0 \mu_4^2 \mu_3 (3 c^2 + 1) (32 c_4 - 16 c^2 - 7) + 3 \mu_2^2 \mu_4^2 \\
(144 c_6 - 844 c_4 - 528 c^4 - 105 c_2 - 8) + 12 \mu_0^2 \mu_4 \mu_3^2 (300 c_4 - 72 c^4 + 49 c_2 + 64 c_2^2 + 10) \\
-12 c^2 \mu_0^5 \mu_4 \mu_3 (144 c_4 - 588 c^6 - 36 c_6 + 57 c^4 + 13 c_2^2 + 2) - c_4^4 \mu_0^6 \mu_4 (2592 c_4 - 1560 c_4 \\
-792 c_6 - 160 c_4 + 2 c^2 - 1) - 24 \mu_0^3 \mu_4 \mu_3 (76 c_4 - 56 c_6 - 13 c_2 - 17 c^2 - 31 c^2 - 1) \\
+12 c^2 \mu_0^6 \mu_3^2 (144 c_4 - 468 c^6 + 128 c_6 + 49 c_4 + 65 c_2^2 + 16) + 4 c^8 \mu_0^9 \mu_3 [c^4 \mu_0^2 (1620 c^8 \\
-180 c^6 + 585 c^4 + 180 c^2 + 45) - (324 c^4 + 972 c^8 + 405 c_6 + 118 c^4 + 69 c^2 + 14)] \\
+3 \mu_0^{12} c^8 (1296 c_4^{10} + 992 c^8 + 836 c^6 + 310 c^4 + 68 c^2 - 1) \right\} + O(n^{-2}) \quad (8.9) \]

### 8.5 Conclusion

We have derived Edgeworth expansion to the order of $O(n^{-1})$ for the null distribution of the test statistics for Wald, LMP, tests based on MDI and Bhat and Rao estimators using normal and general distribution. When the underlying distribution is $N(\mu, c_2^2)$, the order of the approximation is order of $O(n^{-1})$, uniformly for all $\mu_0$. Such approximation is close
to the true distribution Thus the result can be used to estimate the P-values.

If the QQ plot indicates that, the distribution is not normal, one can estimate the first four moments of the distribution using the sample moments and use the expression derived under general distribution to estimate the P-values. Such P-values are robust for violation of normality assumptions.

We would not derive the Edgeworth expansion of the LR test statistic. The reason is that the algebra is very tedious to obtain the third and higher order derivatives of this statistic. This is an open problem for the future research.