CHAPTER 3

STUDIES ON UNIFORMLY CONVEX FUNCTIONS AND RELATED CLASSES

1. INTRODUCTION

Recently Goodman [15] introduced the geometrically defined classes of functions which are known as uniformly convex and uniformly starlike functions and are denoted by UCV and UST respectively.

Definition (3.1) : A function $f(z)$ is said to be uniformly convex (starlike) in $E$, denoted by UCV (UST), if $f(z)$ is convex (starlike) and has the property that each circular curve $\gamma$ contained in $E$ with center $\zeta$ also in $E$, the arc $f(\gamma)$ is convex (starlike w.r.t. $f(\zeta)$).

Goodman also analytically characterized these classes of functions as follows.

Lemma (3.1)[15] : Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E$. Then

(a) $f \in \text{UCV}$ if and only if for all $z, \zeta$ in $E$

$$\text{Re} \left[ 1 + \frac{(z-\zeta) f''(z)}{f'(z)} \right] \geq 0$$

and (b) $f \in \text{UST}$ if and only if for all $z, \zeta$ in $E$,

$$\text{Re} \left[ \frac{(z-\zeta) f'(z)}{f(z)-f(\zeta)} \right] \geq 0.$$
In analogy with the relationship between the convex and starlike functions one may expect that $f(z) \in UCV$ if and only if $zf'(z) \in UST$ holds. But Goodman [16] has shown that $zf'(z) \in UST$ does not imply that $f(z) \in UCV$. Still there may be a one way bridge in the sense that $f(z) \in UCV$ implies that $zf'(z) \in UST$. In exploring this one way bridge, F.Ronning [33] defined a subclass of starlike functions in the following way.

Definition (3.2): Let $S_p = \{ F \in S^* : F(z) = zf'(z); f \in UCV \}$.

Ronning [33] gave some simple examples supporting the conjecture $S_p \subset UST$. However quite recently F.Ronning [34] settled this conjecture negatively by showing that the function $f(z)$ defined by

$$\frac{zf'(z)}{f(z)} = \phi(z) = 1 + \frac{2}{\pi^2}\left\{\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right\}^2$$

is in the class $S_p$ but not in $UST$.

Using the two variable analytic characterization of the class $UCV$ given by Goodman [15], it is not easy to study many of the properties of the class $UCV$. Recently F.Ronning [33] and independently Ma and Minda [21] gave the one variable analytic characterization of the class $UCV$, which is very useful in obtaining the coefficient bounds, distortion and growth theorems.

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Theorem (3.1) [21,33]: Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic.

Then \( f \in \text{UCV} \) if and only if

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right|
\]

Remark (3.1): Since \( f \in \text{UCV} \) if and only if \( g(z) = zf' \in S_p \), we have \( g(z) \in S_p \) if and only if

\[
\text{Re} \left( \frac{zg'(z)}{g(z)} \right) > \left| \frac{zg'(z)}{g(z)} - 1 \right|
\]

Remark (3.2): It is known that the function

\[
\phi(z) = 1 + (2/\pi^2) \left\{ \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right\}^2
\]

(3.1)

maps the unit disc \( E \) onto the parabolic region \( \Omega \) defined by \( \Omega = \{ w : \text{Re}(w) > |w-1| \} \) (The branch of \( \sqrt{z} \) is chosen such that \( \text{Im} \sqrt{z} \geq 0 \)). Then from theorem (3.1) we have

\[
f(z) \in \text{UCV} \text{ if and only if } 1 + \frac{zf''(z)}{f'(z)} < \phi(z)
\]

and

\[
g(z) \in S_p \text{ if and only if } \frac{zf'(z)}{f(z)} < \phi(z).
\]

It follows from the shape of \( \Omega \) that each \( f(z) \in \text{UCV} \) is both convex of order \( 1/2 \) and strongly convex of order \( 1/2 \).
Remark (3.3): It turns that the function \( q(z) \) defined by

\[
1 + \frac{zq''(z)}{q'(z)} = \psi(z)
\]  

(3.2)

serves as an extremal function for certain extremal problems in the class UCV.

Some of the results of this chapter have been published in Journal of Ramanujan Mathematical Society [39].

2. ON THE CLASS UCV

Ruscheweyh obtained certain subordination characterization of the class of function \( f(z) \) defined by

\[
\frac{zf'(z)}{f(z)} < G(z) \text{ where } G(z) \text{ is convex univalent with } G(0) = 1.
\]

Indeed he proved the following theorem.

Theorem [3.2] [37]: Let \( G(z) \) be a convex conformal mapping on \( E, G(0) = 1 \) and let

\[
F(z) = z \exp \left\{ \int_0^z \frac{G(x) - 1}{x} \, dx \right\}.
\]

Let \( f(z) \) be analytic in \( E \), \( f(0) = f'(0) - 1 = 0 \). Then we have for \( z \in E \), for all \( s \) and \( t \), \( |s| \leq 1, |t| \leq 1 \),

\[
\frac{zf'(z)}{f(z)} < G(z) \text{ if and only if } \frac{tf(sz)}{sf(tz)} < \frac{tF(sz)}{sF(tz)} \text{ holds.}
\]

We have the following theorems for the class UCV.
Theorem (3.3): A function $f(z) \in \text{UCV}$ if and only if for each $s$ and $t$ with $|s| \leq 1$, $|t| \leq 1$,
$$\frac{f'(sz)}{f'(tz)} < \frac{q'(sz)}{q'(tz)}$$
where $q(z)$ is defined by (3.2).

Proof: Let $f(z) \in \text{UCV}$. Then $g(z) = zf'(z) \in S_p$.

Thus
$$1 + \frac{zf''(z)}{f'(z)} = \frac{zq''(z)}{q'(z)} < \phi(z)$$
where $\phi(z)$ is defined by (2.1).

Then by Theorem (3.2), $g(z) \in S_p$ if and only if
$$\frac{tg(sz)}{sg(tz)} < \frac{tF(sz)}{sF(tz)}$$
where $F(z) = z \exp \left( \int_0^z \frac{\phi(x)-1}{x} \, dx \right)$.

But
$$\frac{tg(sz)}{sg(tz)} = \frac{f'(sz)}{f'(tz)}.$$ (3.5)

Also for $q(z)$ as in (3.2), $zq'(z) = F(z)$ and
$$\frac{tF(sz)}{sF(tz)} = \frac{q'(sz)}{q'(tz)}.$$ (3.6)

Substituting (3.5) and (3.6) in (3.4) we get
$$f(z) \in \text{UCV} \quad \text{if and only if} \quad (3.3) \quad \text{is satisfied.}$$

Theorem (3.4): An analytic function $f(z)$ belonging to $A$ is in $\text{UCV}$ if and only if $f'(z) < q'(z)$ where $q(z)$ is as in (3.2).
Proof: Let \( f(z) \in \mathcal{U} \). Then Theorem (3.3) with \( s = 1 \) and \( t = 0 \) gives,
\[
f'(z) \prec q'(z).
\]
Conversely, let \( f(z) \in A \) satisfy the condition
\[
f'(z) \prec q'(z).
\]
Then by principle of subordination
\[
f'(E_r) \preceq q'(E_r) \text{ where } E_r = \{z : |z| < r \}
\]
for each \( r \) with \( 0 < r \leq 1 \).

1, e., \( f'(sz) \prec q'(sz) \) for each \( s \) with \( 0 < |s| \leq 1 \) and \( z \in E \). It follows that there exists an analytic function \( w_s(z) \) satisfying \( w_s(0) = 0 \) and \( |w_s(z)| \leq |z| < 1 \) such that
\[
f'(sz) = q'(sw_s(z)).
\]
Now we can find a sequence \( \{s_k\}_{k=1}^{\infty} \) such that \( s_k \to 1 \) as \( k \to \infty \) and \( w_{s_k} \to w \) locally uniformly in \( E \), where \( w(z) \) is analytic in \( E \) with \( |w(z)| \leq |z| \).

Hence for a fixed \( z \in E \),
\[
\frac{zf''(z)}{f'(z)} = \lim_{k \to \infty} \frac{f'(s_k z) - f'(z)}{(s_k - 1)} \quad \frac{f'(s_k z)}{f'(z)}
\]
\[
= \lim_{k \to \infty} \frac{w_{s_k}(z)}{q'(w_{s_k}(z))} \quad \frac{q'(s_k w_{s_k}(z)) - q'(w_{s_k}(z))}{(s_k - 1) w_{s_k}(z)}
\]
\[
= \frac{w(z)}{q'(w(z))} q''(w(z)) \quad \text{with} \quad |w(z)| < |z|
\]

Thus
\[
\text{Range of } \left[\frac{zf''(z)}{f'(z)}\right] \subseteq \text{Range of } \left[\frac{zq''(z)}{q'(z)}\right]
\]
which implies \( 1 + \frac{zf''(z)}{f'(z)} < \phi(z) \) or \( f(z) \in \text{UCV} \).

Remark (3.4) : The above theorem improves the result due to Ma and Minda [21, Theorem 3, Page 169] which states that if \( f(z) \in \text{UCV} \) then \( f'(z) \ll q'(z) \).

In [4] Burdick et.al. obtained the smallest values of \( \alpha \) for which \( \Re \left( \frac{f'(z)}{f'(-z)} \right)^\alpha > 0 \) holds for \( f(z) \) convex univalent. In the following theorem we obtain a similar result for the class \( \text{UCV} \).

Theorem (3.5) : If \( f \in \text{UCV} \) then \( \Re \frac{f'(z)}{f'(-z)} > 0 \).

Proof: \( f(z) \in \text{UCV} \).

If \( g(z) = \frac{f'(z)}{f'(-z)} \) then

\[
\log g(z) = \int_{-z}^{z} \frac{f''(t)}{f'(t)} \, dt.
\]

Let \(-z = re^{i\theta} \) then \( z = re^{i(\theta_0 + \pi)} \).

Put \( t = re^{i\theta}, \theta_0 < \theta < \theta_0 + \pi, \)

so that \( \arg g(z) = \Im(\log g(z)) \)

\[
= \Im \int_{\theta_0}^{\theta_0 + \pi} \left[ \frac{f''(re^{i\theta})}{f'(re^{i\theta})} re^{i\theta} \right] d\theta.
\]
\[
\int_{0}^{\theta_0 + \pi} \text{Re} \left[ \frac{f'(re^{i\theta})}{f'(re^{i0})} \right] d\theta.
\]

Since for \( f \in UCV \), \( \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 1/2 \), we get,

\[
\arg g(z) \geq \int_{0}^{\theta_0 + \pi} (-1 + (1/2))d\theta = -\pi/2. \quad (3.7)
\]

Since \( z \in E \) is arbitrary, replacing \( z \) by \(-z\) in (3.7), we get

\[
- \arg g(z) = \arg \left( \frac{1}{g(z)} \right) = \arg g(-z) > -\pi/2.
\]

i.e., \( \arg g(z) < \pi/2 \). \quad (3.8)

Thus from (3.7) and (3.8) we get \( |\arg g(z)| < \pi/2 \).

i.e., \( \text{Re} \ g(z) = \text{Re} \ \frac{f'(z)}{f'(-z)} > 0 \).

3. The Class UCC

In this section we define a new class of analytic functions and study its properties.

Definition (3.3): Let \( f \in A \). Then \( f \) is said to be uniformly close to convex in \( E \) if there exists a function \( g(z) \in C \) such that \( \frac{f'(z)}{g'(z)} < \phi(z) \) where \( \phi(z) \) is as in (3.1).

Let UCC denote the class of uniformly close to convex functions defined in \( E \).
Remark (3.5): We know that \( g(z) \in C \) if and only if 
\[
\psi(z) = zq'(z) \in S^*. 
\]
Thus \( f \in UCC \) if there exists a 
\( \psi(z) \in S^* \) such that
\[
\frac{zf'(z)}{\psi(z)} \prec \phi(z). 
\]
Remark (3.6): It is clear that \( UCV \subset S_p \subset UCC \subset K \).

Definition (3.4): Let \( UP' \) denote the subclass of \( A \) containing functions \( f(z) \) satisfying
\[
f'(z) \prec \phi(z). 
\]
Remark (3.7): The class \( UP' \) corresponds to the special choice \( g(z) = z \) in the definition (3.3).

Example (3.1): Let \( f(z) = z + Az^2, z \in E \).

Then \( f'(z) = 1 + 2Az \), so that \( |f'(z) - 1| \leq 2|A| \).
If \( |A| \leq 1/4 \) then \( |f'(z) - 1| \leq 1/2 \). Hence \( f'(z) \prec \phi(z) \).

Conversely if \( f(z) = z + Az^2 \in UP' \) then \( |A| \leq \frac{1}{4} \).

Thus \( f(z) = z + Az^2 \in UP' \) if and only if \( |A| \leq \frac{1}{4} \).

Example (3.2): Consider the function
\[
f(z) = \frac{1}{x} \log \frac{1}{1-xz}, z \in E, |x| \leq 1, \text{ for which}
\]
\[
f'(z) = \frac{1}{1-xz}
\]
If \( |x| \leq \frac{1}{\sqrt{2}} \) then \( f'(z) \prec \phi(z) \).
Thus \( |x| \leq \frac{1}{\sqrt{2}} \) then \( f(z) = \frac{1}{x} \log \left\{ \frac{1}{1-xz} \right\} \in UP' \).

We now give a necessary analytic condition for an analytic function \( f(z) \) to be in the class UCC which is similar to Kaplan's [18] condition for the class K of close to convex functions. However we are unable to give a necessary and sufficient condition analogous to Kaplan's condition for K.

Theorem (3.6): If \( f \in UCC \) then for \( 0 < r < 1 \),

\[
\begin{align*}
\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ 1 + \frac{\text{re}^i \theta f''(\text{re}^i \theta)}{f'(\text{re}^i \theta)} \right\} d\theta \geq -(\pi/2) \quad (3.9)
\end{align*}
\]

for all \( z = \text{re}^i \theta \), \( \theta_1 \) and \( \theta_2 \) with \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \).

Proof: Let \( f(z) \in UCC \). Then there exists a convex univalent function \( g(z) \) such that

\[
\frac{f'(z)}{g'(z)} < \phi(z) \quad (3.10)
\]

where

\[
\phi(z) = 1 + \frac{2}{r^2} \left[ \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right]^2.
\]

As we have seen \( \phi(z) \) maps the unit disc \( E \) onto the parabolic region \( \Omega = \{ w : \text{Rew} > |w-1| \} \) which lies inside the sector \( |\arg w| \leq \pi/4 \).

Let \( q(z) = \arg f'(z) \) and \( \psi(z) = \arg g'(z) \). Since \( f(z) \) and \( g(z) \) are univalent, \( f'(z) \) and \( g'(z) \) do not vanish in the unit disc \( E \). Hence \( q(z) \) and \( \psi(z) \) are well defined.
Then from (3.10) we have
\[ |\arg f'(z) - \arg g'(z)| = |q(z) - \psi(z)| < \pi/4. \quad (3.11) \]
Here suitable branches of \( q(z) \) and \( \psi(z) \) are chosen. (This is possible since \( q(z) \) and \( \psi(z) \) are continuous and periodic with period \( 2\pi \)).

Let \( Q(r, \theta) = q(re^{i\theta}) + \theta = \arg zf'(z) \)
and \( H(r, \theta) = \psi(re^{i\theta}) + \theta = \arg zg'(z) \).

Since \( g(z) \) is convex, \( \arg zg'(z) \) is an increasing function of \( \theta \). Then for \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \),
\[ H(r, \theta_1) < H(r, \theta_2). \quad (3.12) \]
Now for \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \), \( 0 < r < 1 \), we have
\[
Q(r, \theta_2) - Q(r, \theta_1) = g(re^{i\theta_2}) + \theta_2 - q(re^{i\theta_1}) - \theta_1 \\
= [q(re^{i\theta_2}) - \psi(re^{i\theta_2})] + [\psi(re^{i\theta_2}) + \theta_2 - (\psi(re^{i\theta_1}) + \theta_1)] + \\
+ [q(re^{i\theta_1}) - \psi(re^{i\theta_1})] \\
\geq -\frac{\pi}{4} + 0 - (\pi/4) = -\pi/2.
\]
But \( \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} = \frac{d}{d\vartheta} \left( \arg re^{i\vartheta}f'(re^{i\vartheta}) \right) \)
implies that
\[
\int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{re^{i\vartheta}f'(re^{i\vartheta})}{f''(re^{i\vartheta})} \right\} d\vartheta = \arg zf'(z) \bigg|_{\theta_1}^{\theta_2}
\]
The proof is complete.

The following result on the ratio of the derivatives of functions in UCC is analogous to the corresponding result of the class $K$ due to Burdick et al. [4].

Theorem (3.7). If $f(z) \in UCC$ then \[ \Re \left( \frac{f'(z)}{f'(-z)} \right)^{1/3} > 0. \]

Proof: Let $f(z) \in UCC$. Then by theorem (3.6), we have

\[
\theta_2 \int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{\Re e^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} \right\} d\theta > -\pi/2
\]

for $0 < \theta_1 < \theta_2 \leq 2\pi$, $0 < r < 1$.

Let $g(z) = \frac{f'(z)}{f'(-z)}$. Then \[ \log g(z) = \int_{\theta_0}^{\theta_0 + \pi} \frac{f''(t)}{f'(t)} dt. \]

Let $-z = re^{i\theta_0}$, then $z = re^{i\theta_0 + \pi}$ and put $t = re^{i\theta}$ for $\theta_0 \leq \theta \leq \theta_0 + \pi$, so that

\[
\arg g(z) = \text{Im}(\log g(z)) = \int_{\theta_0}^{\theta_0 + \pi} \Re \left\{ \Re e^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right\} d\theta
\]

\[ > -\frac{\pi}{2} - \pi = -\frac{3\pi}{2}. \quad (3.13) \]

Since $z \in E$ is arbitrary, replacing $z$ by $-z$ in (3.13), we get

\[-\arg g(z) = \arg \left( -\frac{1}{g(z)} \right) = \arg g(-z) > -3\pi/2.\]

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which implies
\[ \arg g(z) \leq 3\pi/2. \tag{3.14} \]
Then from (3.13) and (3.14) we get
\[ |\arg g(z)| < 3\pi/2, \]
which is equivalent to
\[ \Re (g(z))^{1/3} > 0. \]

The following theorem shows that $\text{UP}'$ is closed under convolution.

**Theorem (3.8)**: If $f, g \in \text{UP}'$ then so is $f * g$.

For the proof of the above theorem we need the following lemma and its corollary.

**Lemma (3.2)**: If $f \in \text{UCC}$ then there exists a $g \in C$ such that
\[ \frac{f(z)}{g(z)} < \phi(z). \]

**Proof** : If $f \in \text{UCC}$ then there exists of $g \in C$ such that
\[ \frac{f'(z)}{g'(z)} < \phi(z). \]

Let \[ \frac{f(z)}{g(z)} = p(z), \] so that $p(z)$ is analytic in $E$ with $p(0) = 1$.

But \[ \left\{ \frac{g(z)}{zg'(z)} \right\} zp'(z) + p(z) = \frac{f'(z)}{g'(z)} < \phi(z). \]

Since $g \in C \subset S^*$, \[ \Re \frac{g(z)}{zg'(z)} > 0. \]
Hence by lemma (1.4), \[ p(z) = \frac{f(z)}{g(z)} \prec \phi(z). \]

Corollary (3.1): If \( f \in \operatorname{UP}' \) then \[ \frac{f(z)}{z} \prec \phi(z) \]
and hence \[ \operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}. \]

Proof: If \( f \in \operatorname{UP}' \) i.e., \( f'(z) \prec \phi(z) \), then by Lemma (3.2) with \( g(z) = z \), we get \[ \frac{f(z)}{z} \prec \phi(z). \]

Since \( \phi(z) \) maps the unit disc \( E \) onto the parabolic region \( \Omega = \{ w \mid \text{Rew} > |w-1| \} \) which is contained in the half plane \( \{ w \mid \text{Rew} > \frac{1}{2} \} \), we have,
\[ \operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}. \]

Proof of Theorem (3.8): Let \( \psi = f \ast g \).

Then \( \psi'(z) = f'(z) \ast \frac{g(z)}{z} \).

Since \( \operatorname{Re} \frac{g(z)}{z} > \frac{1}{2} \), by Lemma (1.1), \( \psi'(z) \) takes values in the convex hull of the image of \( E \) under \( f'(z) \).

But \( f'(z) \prec \phi(z) \) implies that \( f'(E) \) lies in the parabolic region \( \Omega = \phi(E) \).

Thus \( \psi'(z) \in \phi(E) \) for each \( z \in E \). Since \( \phi \) is convex univalent we have,
\[ \psi'(z) \ll \phi(z). \]

i.e.,
\[ \psi(z) \in \operatorname{UP}' . \]

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The Class UM(α)

Here we consider a class which is a subclass of α-convex functions defined by Mocanu [13].

Definition (3.5): Let α be complex. We say that an analytic function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) in \( A \) belongs to the class UM(α) if

\[
\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1-\alpha) \frac{zf'(z)}{f(z)} \leq \phi(z)
\]

where \( \phi(z) \) is as in (3.1).

Remark (3.8): The class UM(α) is a subset of M(α), the class of α-convex functions defined by Mocanu [13]. Though the definition is meaningful for all complex α, in the following we assume that \( |\arg \alpha| < \pi/4 \).

Theorem (3.9): For \( |\arg \alpha| < \pi/4 \), \( UM(\alpha) \subseteq S_p \).

Proof: Let \( f(z) \in UM(\alpha) \) and \( p(z) = \frac{zf'(z)}{f(z)} \).

Then logarithmic differentiation gives

\[
\frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}.
\]

Thus

\[
1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} + p(z) \quad \text{so that}
\]
\[ \alpha \left[ 1 + \frac{zf''(z)}{f'(z)} \right] + (1-\alpha) \frac{zf'(z)}{f(z)} = \alpha \frac{zp'(z)}{p(z)} + p(z) < \phi(z). \]

Further if \(|\arg \alpha| < \pi/4\), then

\[ \text{Re}(\alpha \phi(z)) > 0. \]

From Lemma (1.5), we have

\[ p(z) = \frac{zf'(z)}{f(z)} < \phi(z). \]

Thus

\[ f(z) \in S_p. \]

Remark (3.9): It is shown that for \(0 \leq \beta \leq \alpha\),

\[ \text{UM}(\alpha) \subset \text{UM}(\beta). \]

Remark (3.10): We have \(\text{UM}(0) = S_p, \text{UM}(1) = \text{UCV}\), so that by Remark (3.9) we have \(\text{UM}(\alpha) \subset \text{UCV}\) for \(\alpha \geq 1\) and \(\text{UM}(\alpha) \subset S_p\) for \(\alpha > 0\). Thus for \(0 \leq \alpha \leq 1\), \(\{ \text{UM}(\alpha) \}\) gives a continuous passage from the class \(\text{UCV}\) to \(S_p\).

For \(f \in \text{UM}(\alpha)\), we have the integral representation given by,

**Theorem (3.10):** An analytic function \(f \in A\) is in \(\text{UM}(\alpha)\) if and only if there exists a function \(F(z) \in S_p\) such that

\[ f(z) = \left[ \frac{1}{\alpha} \int_0^z \frac{F(t)^{1/\alpha}}{t} \, dt \right]^\alpha \quad (3.15) \]

(Here suitable branch of \(f(z)\) is chosen to satisfy conditions \(f(0) = 0\) and \(f'(0) = 1\).)
Proof: Let \( f(z) \in UM(\alpha) \).

Put

\[
F(z) = f(z) \left( \frac{zf'(z)}{f(z)} \right)^\alpha. \tag{3.16}
\]

Then \( F(z) \) is analytic in \( E \) with \( F(0) = 0 \). Also

\[
\frac{zF'(z)}{F(z)} = \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1-\alpha) \frac{zf'(z)}{f(z)} < \phi(z).
\]

i.e., \( F(z) \in S_p \) and from (3.16) we get (3.15).

Conversely if \( f(z) \) has the integral representation (3.15) with \( F \in S_p \), then equation (3.16) again holds, which in turn implies that \( f \in UM(\alpha) \).

We can use Ruscheweyh's result namely Theorem (3.2) to obtain the following subordination theorem for functions in \( UM(\alpha) \).

**Theorem (3.11):** An analytic function \( f(z) \) in \( A \) is in \( UM(\alpha) \) if and only if for all \( s \) and \( t \) with \( |s| \leq 1, |t| \leq 1 \),

\[
\left( \frac{f'(sz)}{f'(tz)} \right)^\alpha \left( \frac{tf(sz)}{sf(tz)} \right)^{1-\alpha} < \frac{tF(sz)}{sF(tz)}
\]

where

\[
F(z) = z \exp \left[ \int_0^z \left( \frac{\phi(x)-1}{x} \right) \, dx \right]. \tag{3.17}
\]

Proof: Let \( G(z) = f(z) \left( \frac{zf'(z)}{f(z)} \right)^\alpha \).
Then \[
\frac{zG'(z)}{G(z)} = \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1-\alpha) \frac{zf'(z)}{f(z)} . \tag{3.18}
\]

It follows from (3.18) that
\[f(z) \in \text{UM}(\alpha) \text{ if and only if } G(z) \in S_p.\]

But by Theorem (3.2), \(G(z) \in S_p\) if and only if for each \(s\) and \(t\) with \(|s| \leq 1, |t| \leq 1,\)
\[
\frac{tG(sz)}{sG(tz)} \prec \frac{tF(sz)}{sF(tz)}
\]
where \(F(z)\) is defined by (3.17).

But from the definition of \(G(z)\) we have,
\[
\frac{tG(sz)}{sG(tz)} = \left( \frac{f'(sz)}{f'(tz)} \right)^\alpha \left( \frac{tf(sz)}{sf(tz)} \right)^{1-\alpha}.
\]

Thus \(f(z) \in \text{UM}(\alpha)\) if and only if
\[
\left( \frac{f'(sz)}{f'(tz)} \right)^\alpha \left( \frac{tf(sz)}{sf(tz)} \right)^{1-\alpha} \prec \frac{tF(sz)}{sF(tz)} .
\]