Chapter-2

Aggregate Directional Distance Formulation of DEA and with Integer Variables

2.1 Introduction:

Conventional DEA models make assumptions of non-negativity and real values in the inputs and outputs of the systems under study. However, there are many situations in performance measurements as some of the inputs and/or outputs ought to have negative integer values. For example, in the transport operations/courier services with inputs such as consignments for delivery to isolated places and outputs such as number of incorrect deliveries, growth in number of customers, in the efficiency measurement of hospitals outputs such as unsuccessful medical surgeries and in the measurement of banks efficiency variable like growth in number of customers are having integer and negative values as well. Furthermore, in a number of situations, units have negative and/or integer outputs and inputs. Sharp *et al.* (2007) pointed out a number of situations in which natural negative inputs and outputs were occurring. Plant of pollutant disposal, evaluating the efficiency of different advertising campaigns, logistic operation, evaluating different configurations of business system etc. could be best examples. Primarily, negativity is addressed in Chung *et al.* (1997), introducing the performance measure by reduction of the bad outputs like pollution and simultaneously increasing of good outputs. In this method directional distance function is used in the construction of Malmquist type productivity index. Scheel (2001), who resolves negativity problem by solving two BCC models. One, the objective function is maximized and another, where it is minimized for the case of undesirable outputs. Recently Watanabe and Tanaka (2007) developed balanced growth indicators using the directional output distance function. These papers did not consider aspects of projections or targets for the inefficient firms but, these approaches are found valid when all the units have non-positive values.

Over the last few years directional distance function in DEA has been applied to measure efficiency. This method was originated by Luenberger (1992) in a way to introduce distance function as benefit function followed by Chambers *et al.* (1996) who developed directional distance function. The standard directional distance function formulation of DEA to measure the technical efficiency is developed by Chambers *et al.* (1998) and by Fare and Grosskopf (2000). These models are restricted to non-negative data. Later on Portela *et al.* (2004) developed the range directional model to handle unrestricted data.

2.2 Range directional model (RDM) and non-radial DEA Model:

A usual choice for the directional vector is the observed input and output levels where inputs and outputs variables are positive in the model $\langle M\ 1.10\rangle$. When data are negative, the use of observed input and output levels would violate the constraints (1.10d) in the model $\langle M\ 1.10\rangle$. In order to overcome this problem Portela *et al.* (2004) modified the model to ensure that it yields improved solutions even when some data are negative. They defined the range of possible improvement of units as the direction vectors, defined as

$$R_{ro}^{+}=\underset{j}{Max}\left\{y_{rj}\right\}-y_{ro},\ r=1,2,\cdots,s \qquad (E\ 2.1)$$

$$R_{io}^{-}=x_{io}-\underset{j}{Min}\left\{x_{ij}\right\},\ i=1,2,\cdots,m;\ j=1,2,\cdots,n \qquad (E\ 2.2)$$

Based on these directional vectors, Portela *et al.* (2004) have developed range directional model given by

$$\left.\begin{array}{lll} & Max\ \beta_R & (2.1a)\\ subject\ to & \sum_{j=1}^{n}\lambda_j y_{rj}-\beta_R R_{ro}^{+}\geq y_{ro},\ r=1,2,...,s & (2.1b)\\ & \sum_{j=1}^{n}\lambda_j x_{ij}+\beta_R R_{io}^{-}\leq x_{io},\ i=1,2,...,m & (2.1c)\\ & \sum_{j=1}^{n}\lambda_j=1 & (2.1d)\\ & \lambda_j\geq 0,\ j=1,2,\cdots,n & (2.1e)\\ & \beta_R\ unrestricted & \end{array}\right\}\quad \langle M\ 2.1\rangle$$

The output (input) oriented model can be obtained by setting R_{io}^{-} (R_{ro}^{+}) equal to zero in model $\langle M\ 2.1\rangle$. The RDM model $\langle M\ 2.1\rangle$ is translation and unit invariant. At the optimal solution of the model at least one of the constraints of (2.1b) and (2.1c) holds equality, i.e., $\sum_{j=1}^{n}\lambda_j^{*}y_{rj}-\beta_R^{*}R_{ro}^{+}=y_{ro}$ for some r, then, $\beta_R^{*}=\frac{y_{ro}^{*}-y_{ro}}{R_{ro}^{+}}$ where $y_{ro}^{*}=\sum_{j=1}^{n}\lambda_j^{*}y_{rj}$. For output oriented model y_{ro}^{*} is the target value obtained at the optimal solution of the RDM. Efficient DMUs in the RDM necessarily have an optimal value $(1-\beta)$ equal to one, but it is not the sufficient condition for Pareto-Koopmans efficiency. If a DMU_o is Pareto-Koopmans efficiency, then it should satisfy i) $\beta_R=0$ and ii) all slacks are zero. In the RDM model efficiency score

$(1-\beta_R)$ is not able to incorporate all the sources of inefficiency. There may be slacks at nonzero level. Therefore RDM model does not assure projection on to the Pareto-Koopmans efficient target.

As suggested by Cooper *et al.* (1999), it is essential for a measure of efficiency to have properties that it is ought to be a single real number, incorporating all sources of inefficiency and is readily interpretable for use in a variety of managerial and scientific contexts. They also pointed out that the need for separately treating input oriented and output oriented approaches to efficiency measurement is eliminated. Hence it is necessary to simultaneously maximize outputs and minimize inputs. In DEA context Fare and Lovell (1978) introduced the input oriented non-radial measure of technical efficiency, which they called the Russell measure. The input (output) oriented non-radial measures ignore output (input) slacks present at the optimal solution. Pareto-Koopmans measure of efficiency only ensures that neither input nor output slacks will be present at the optimal solution of the relevant DEA model. The combination of the input and output Russell measures of technical efficiency is developed by Fare *et al.* (1985) and is given by

$$\left.\begin{array}{lllr}
 & Min\ R_o = \dfrac{1}{m+s}\left(\sum_{i=1}^{m}\eta_i + \sum_{r=1}^{s}\dfrac{1}{\varphi_r}\right) & & (2.2a) \\
subject\ to & \sum_{j=1}^{n}\lambda_j y_{rj} \geq \varphi_r y_{ro}, & r = 1,2,\cdots,s & (2.2b) \\
 & \sum_{j=1}^{n}\lambda_j x_{ij} \leq \eta_i x_{io}, & i = 1,2,\cdots,m & (2.2c) \\
 & \lambda_j \geq 0, & j = 1,2,\ldots,n & (2.2d) \\
 & \varphi_r \geq 1 \quad \forall r & & (2.2e) \\
 & 0 \leq \eta_i \leq 1 \quad \forall i & & (2.2f)
\end{array}\right\} \langle M\ 2.2\rangle$$

A DMU_o is Pareto-Koopmans efficient if and only if each of $\eta_i = 1$ *and* $\varphi_r = 1$. Here R_o is a weighted average of arithmetic and harmonic means. This model is simultaneously accounts for the inefficiency in both inputs and outputs. Even though this model is well defined, it does not satisfy the properties laid down by Cooper *et al.* (1999). Primarily, the solution for the model obtained through non-linear programming problem that cannot be easily found. Moreover, this model is not applicable when we deal with negative or undesirable variables. To address these shortcomings, we developed a model based on directional distance function in the next section.

2.3 Measures of technical efficiency through aggregate directional distance model:

More recently, Fukuyama and Weber (2009) introduced directional slacks-based measure of technical inefficiency. They found that radial measure of efficiency overestimates technical efficiency when there are nonzero slacks in the constraints. Besides, radial measure as in the case of RDM, the maximum value of inefficiency is small when a particular outputs or inputs are used very efficiently by all the DMUs. It is required to develop a model with superior discrimination ability. In this chapter we introduce a non-oriented, non-proportional RDM model by aggregating inefficiencies of each inputs and outputs. Formally, we define the weighted average inefficiencies of individual input and output variables. The objective is to maximize the average inefficiencies of all the variables, which we call aggregate directional distance function model (ADDM). Assume that we have a set of '*n*' DMUs with '*m*' inputs and '*s*' outputs. In the usual directional distance function with real valued data, each observed DMU is classified by a pair of input and output vectors $(X_j, Y_j)=(x_{1j},x_{2j},\ldots,x_{mj}\ ,\ y_{1j},y_{2j},\ldots,y_{sj})$, $j=1,2,\ldots,n$. then, the production possibility set is given by a pair of input-output bundle (x_o,y_o) and the directional input and output bundle be $\left(R_{io}^{-}, R_{ro}^{+}\right)$ with following postulates.

$\langle R_1 \rangle$ $\left(x_j,y_j\right) \in T$ for $\forall j$

$\langle R_2 \rangle$ If $(x,y) \in T$ *and* $(u,v) \in \mathbb{R}^{+}$ $\Rightarrow$ $(x',-y') \geq (x,-y)$ where $x'=x+u$ and $y'=y-v$ then, $(x',y') \in T$

$\langle R_3 \rangle$ If (x,y) *and* $(x',y') \in T$, $(\tilde{x},\tilde{y}) = \lambda(x,y)+(1-\lambda)(x',y')$ such that $0 \leq \lambda \leq 1$ and $(\tilde{x},\tilde{y}) \in T$

$\langle R_4 \rangle$ If $(x,y) \in T$ $\Rightarrow$ $(\lambda x, \lambda y) \in T$ $\forall \lambda \in \mathbb{R}^{+}$

And the production possibility set is

$$T_{VRS}^{ADDM} = \left\{ (x,y) : x \geq \sum_{j=1}^{n} \lambda_j x_j \text{ and } y \leq \sum_{j=1}^{n} \lambda_j y_j ; \sum_{j=1}^{n} \lambda_j = 1;\ \lambda_j \geq 0; (j=1,2,\ldots,n) \right\} \quad (E\ 2.3)$$

Directional distance function for ADDM inefficiency measure can be defined as

$$\vec{D}\left(x,y;\ R_{io}^{-},R_{ro}^{+}\right)=Max\ \beta=\frac{1}{m+s}\left(\sum_{i=1}^{m}\beta_{io}^{-}+\sum_{r=1}^{s}\beta_{ro}^{+}\right):$$
$$(x_{io}-\beta_{io}^{-}R_{io}^{-},y_{ro}+\beta_{ro}^{+}R_{ro}^{+})\in T_{VRS}^{ADDM}$$
$$if\ (x_{io}-\beta_{io}^{-}R_{io}^{-},y_{ro}+\beta_{ro}^{+}R_{ro}^{+})\in T_{VRS}^{ADDM}\ \text{for some}\ \beta \qquad (E\ 2.4)$$

More details can be referred in Chambers *et al.* (1998).

The non-radial aggregate directional distance formulation of DEA can be developed through the following model

$$\left.\begin{aligned}
&I_e(x_o,y_o)_{ADDM}=Max\ \tfrac{1}{m+s}\left(\sum_{r=1}^{s}\beta_{ro}^{+}+\sum_{i=1}^{m}\beta_{io}^{-}\right) && (2.3a)\\
\textit{subject to}\quad &\sum_{j=1}^{n}\lambda_j y_{rj}-\beta_{ro}^{+}R_{ro}^{+}\geq y_{ro};\ r=1,2,\cdots,s && (2.3b)\\
&\sum_{j=1}^{n}\lambda_j x_{ij}+\beta_{io}^{-}R_{io}^{-}\leq x_{io};\ i=1,2,\cdots,m && (2.3c)\\
&\sum_{j=1}^{n}\lambda_j=1,\quad \lambda_j\geq 0,\quad j=1,2,\cdots,n, && (2.3d)\\
&\text{Where}\quad R_{io}^{-}=x_{io}-\min\{x_{ij}\},\quad R_{ro}^{+}=Max\{y_{rj}\}-y_{ro}
\end{aligned}\right\}\quad \langle M\ 2.3\rangle$$

This model measures Pareto-Koopmans technical efficiency of an evaluated DMU_o. A DMU_o is Pareto-Koopmans efficiency if and only if $\beta_{ro}^{+}=0$ for each output 'r' and $\beta_{io}^{-}=0$ for each output 'i'. In such case $I_e(x_o,y_o)=0$.

In the model $\langle M\ 2.3\rangle$, Pareto-Koopmans efficient input-output projection of an inefficient DMU_o is

$$x_{io}-\beta_{io}^{-*}R_{io}^{-}=\sum_{j=1}^{n}\lambda_j^{*}x_{ij}=x_{io}^{*}\quad \text{for all } i$$

$$y_{ro}+\beta_{ro}^{+*}R_{ro}^{+}=\sum_{j=1}^{n}\lambda_j^{*}y_{rj}=y_{ro}^{*}\ \text{for all } r$$

Where x_{io}^{*} and y_{ro}^{*} are target values for inputs and outputs respectively of an inefficient DMU to became efficient. Here, in this model, we maximize the inefficiency for each input and output. So there does not exist any input or output slacks. This model allows individual outputs to increase or inputs to decrease at different rates. Advantage of ADDM is that it yields the targets which attempts to reflect the priorities for improvement of inputs and outputs. A second advantage of efficiency measures is that the measure of efficiency is similar to non-radial DEA model. ADDM seeks the improvement for those variables on which the DMU has the largest difference from the best observed values.

Input (output) oriented model can be obtained by setting R^{+}_{ro} (R^{-}_{io}) equal to zero in model$\langle M\ 2.3\rangle$. In an input oriented model maximizing the inefficiency β^{-}_{io} for each input and keeping outputs same. Though there is no input slacks at the optimal solution, output slacks may remain in the optimal solution. The projection of inefficient DMUs on to the frontier may not be Pareto-efficient. Projection of the input of DMU$_0$ is $x^{*}_{io} = x_{io} - \beta^{-}_{io} R^{-}_{io}$. The inefficiency found by non-proportional oriented/non-oriented model of any DMU is more than or equal compared to its counterpart of the RDM. A program is developed using MATLAB 7.7 to solve this problem and it can be seen in Appendix, Program-1. This program is used to solve the numerical illustration given in Section-2.3.2.

2.3.1 Desirable properties of the ADDM:

Fare and Lovell (1978) have proposed a set of desirable properties that a measure of technical efficiency should posses. They restricted these properties to input based technical efficiency. Later, Borger *et al.* (1998) listed similar requirements for the efficiency model. Recently, Pastor *et al.* (1999) added few more properties in the context of enhanced Russell graph efficiency measure. Here we study these properties for the new efficiency of the proposed model.

1. Inefficiency measure of the DMUs satisfy $0 \le I_e(x_o, y_o)_{ADDM} \le 1$.

Proof: At the optimal solution to the ADDM, the constraints hold equality. Then,

$$y^{*}_{ro} = y_{ro} + \beta^{+}_{ro} R^{+}_{ro},\ \beta^{*}_{ro} = \frac{y^{*}_{ro} - y_{ro}}{R^{+}_{ro}} = \frac{y^{*}_{ro} - y_{ro}}{Max\{y_{rj}\} - y_{ro}} \Rightarrow$$

$$\beta^{*}_{io} = \frac{x_{io} - x^{*}_{io}}{R^{-}_{io}} = \frac{x_{io} - x^{*}_{io}}{x_{io} - Min\{x_{ij}\}}$$ i.e β^{-*}_{io} and β^{+*}_{ro} equals to the ratio of optimal slack to the maximum slack. Optimum value of the variable lies between observed value and the maximum value. If the optimal value is the $Max\{y_{rj}\}$, for all outputs and optimum value for inputs is the $Min\{x_{ij}\}$ then I^{*}_{e} equals to 1, implies fully inefficient.

2. If $I_e(x_o, y_o)_{ADDM}$ =0 $\Leftrightarrow$ DMU$_o$ being evaluated is Pareto-Koopmans efficient.

Proof: $I_e(x_o, y_o)_{ADDM}$=0 if and only if each $\beta^+_{ro} = 0$ and $\beta^-_{io} = 0$ $\Rightarrow$ at the optimal solution all the constraints are holding equality.

3. $I_e(x_o, y_o)_{ADDM}$ is translation invariant.

Proof: Let an amount u_r is added to each output and v_i to each input, then the constraints in the model becomes

$$\sum_{j=1}^{n} \lambda_j (y_{rj} + u_r) - \beta^+_{ro} R^+_{ro} \geq (y_{ro} + u_r) \ and$$

$$\sum_{j=1}^{n} \lambda_j (x_{ij} + v_i) + \beta^-_{io} R^-_{io} \leq (x_{io} + v_i)$$

The range of the variable does not change

$$\sum_{j=1}^{n} \lambda_j y_{rj} + u_r \geq y_{ro} + u_r + \beta^+_{ro} R^+_{ro} \ and$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} + v_i \leq x_{io} + v_i - \beta^-_{io} R^-_{io} \quad \because \sum_{j=1}^{n} \lambda_j = 1$$

Hence the model is translation invariant. This holds in case of VRS model.

4. $I_e(x_o, y_o)_{ADDM}$ is unit invariance.

Let an amount μ_r is multiplied to each output and γ_i to each input, then the constraints in the model becomes

$$\sum_{j=1}^{n} \lambda_j \mu_r y_{rj} \geq \mu_r y_{ro} + \beta^+_{ro} \mu_r R_y \ and$$

$$\sum_{j=1}^{n} \lambda_j \gamma_i x_{ij} \leq \gamma_i x_{io} - \beta^-_{io} \gamma_i R_x$$

These constraints reduce to ADDM of DEA formulation, whose solution is therefore does not change when the unit of measurement changes.

5. $I_e((x^+_o;\text{-}x^-_o), (y^+_o;\text{-}y^-_o))$ satisfies the following relationships as interpretation of the notions of inputs and outputs $\text{-}x^-_o$ is the inputs which are undesirable input. Such inputs are reflected by assigning them negative values. In usual DEA efficiency measurements, lesser the DMU uses inputs and/or higher the DMU produces outputs better for the efficiency. Such inputs and outputs are called desirable inputs and outputs. Contrary to these situations as in the case of pollutant disposal plants, sewage sludge as inputs, more the DMU uses this kind

of inputs better for the efficiency are called undesirable inputs. Likewise, $-y_o^-$ are the undesirable outputs. In a production system undesirable outputs may occur but it is to be minimized. Lesser the DMU produces undesirable outputs better for the efficiency.

i) $I_e\left((\gamma x_o^+;\frac{-x_o^-}{\gamma}),(y_o^+;-y_o^-)\right) \geq \frac{1}{\gamma} I_e\left((x_o^+;-x_o^-),(y_o^+;-y_o^-)\right)$ if $\gamma > 1$

ii) $I_e\left((\gamma x_o^+;\frac{-x_o^-}{\gamma}),(y_o^+;-y_o^-)\right) \leq \frac{1}{\gamma} I_e\left((x_o^+;-x_o^-),(y_o^+;-y_o^-)\right)$ if $\gamma < 1$

iii) $I_e\left((x_o^+;-x_o^-),(\mu y_o^+;\frac{-y_o^-}{\mu})\right) \geq \mu I_e\left((x_o^+;-x_o^-),(y_o^+;-y_o^-)\right)$ if $\mu < 1$

iv) $I_e\left((x_o^+;-x_o^-),(\mu y_o^+;\frac{-y_o^-}{\mu})\right) \leq \mu I_e\left((x_o^+;-x_o^-),(y_o^+;-y_o^-)\right)$ if $\mu > 1$

v) $I_e\left((\kappa x_o^+;\frac{-x_o^-}{\kappa}),(\frac{y_o^+}{\kappa};-\kappa y_o^-)\right) \geq \frac{1}{\kappa^2} I_e\left((x_o^+;-x_o^-),(y_o^+;-y_o^-)\right)$ if $\kappa > 1$

vi) $I_e\left((\kappa x_o^+;\frac{-x_o^-}{\kappa}),(\frac{y_o^+}{\kappa};-\kappa y_o^-)\right) \leq \frac{1}{k^2} I_e\left((x_o^+;-x_o^-),(y_o^+;-y_o^-)\right)$ if $\kappa < 1$

Proof:

i) $\gamma > 1$ and $(\beta_{io}^{-*},\beta_{ro}^{+*},\lambda^*)$ is the optimal solution to the model $\langle M\ 2.3\rangle$, when DMU$_o$ being evaluated. Then $(\gamma\beta_{io}^{-*},\beta_{ro}^{+*},\lambda^*)$ is a feasible solution to the model $\langle M\ 2.3\rangle$, when $\left((\gamma x_o^+;\frac{-x_o^-}{\gamma}),(y_o^+;-y_o^-)\right)$ is under evaluation, because the constraints for inputs and outputs are clearly satisfied and $\beta_{io}^{-*} \leq \gamma\beta_{io}^{-*} \leq 1$.

Therefore $I_e\left((\gamma x_o^+;\frac{-x_o^-}{\gamma}),(y_o^+;-y_o^-)\right) \geq I_e\left((x_o^+;-x_o^-),(y_o^+;-y_o^-)\right)$

$\Rightarrow I_e\left((\gamma x_o^+;\frac{-x_o^-}{\gamma}),(y_o^+;-y_o^-)\right) \geq \frac{1}{\gamma} I_e\left((x_o^+;-x_o^-),(y_o^+;-y_o^-)\right)$ when $\gamma > 1$

Proof is similar for ii to vi.

2.3.2 Numerical example:

In order to illustrate the performance of our new model we have taken data pertaining to the pollutant processing system analyzed in Sharp *et al.* (2007) shown in Table-2.1.

These data contains 13 DMUs has one positive input (cost), one negative input (effluent), one positive output (saleable output) and two negative outputs (methane and CO_2). We have applied RDM (Non-Oriented) and ADDM (Non-Oriented), to data set given in Table-2.1. It reveals that five efficient DMUs (3, 7, 8, 11, 13) is efficient in both the models as shown in Table-2.2. Figure-2.1 shows that inefficiency

DMU	Cost (I1)	Effluent (I2)	Saleable Output (O1)	CO2 (O2)	Methane (O3)
1	1.03	-0.05	0.56	-0.09	-0.44
2	1.75	-0.17	0.74	-0.24	-0.31
3	1.44	-0.56	1.37	-0.35	-0.21
4	10.8	-0.22	5.61	-0.98	-3.79
5	1.3	-0.07	0.49	-1.08	-0.34
6	1.98	-0.1	1.61	-0.44	-0.34
7	0.97	-0.17	0.82	-0.08	-0.43
8	9.82	-2.32	5.61	-1.42	-1.94
9	1.59	0	0.52	0	-0.37
10	5.96	-0.15	2.14	-0.52	-0.18
11	1.29	-0.11	0.57	0	-0.24
12	2.38	-0.25	0.57	-0.67	-0.43
13	10.3	-0.16	9.56	-0.58	0

Table-2.1 Data on pollutant processing system.

of RDM is less than or equal to inefficiency of ADDM. Thus RDM overstates efficiency measure by including the slacks at the optimal solution.

Models	DMUs												
	1	2	3	4	5	6	7	8	9	10	11	12	13
RDM(NO)	0.0351	0.0819	0	0.2648	0.0757	0.0292	0	0	0.0055	0.1404	0	0.1552	0
ADDM(NO)	0.2431	0.3031	0	0.4789	0.3114	0.2138	0	0	0.1769	0.2547	0	0.4137	0

Table-2.2: Comparison of inefficiency score by different models.

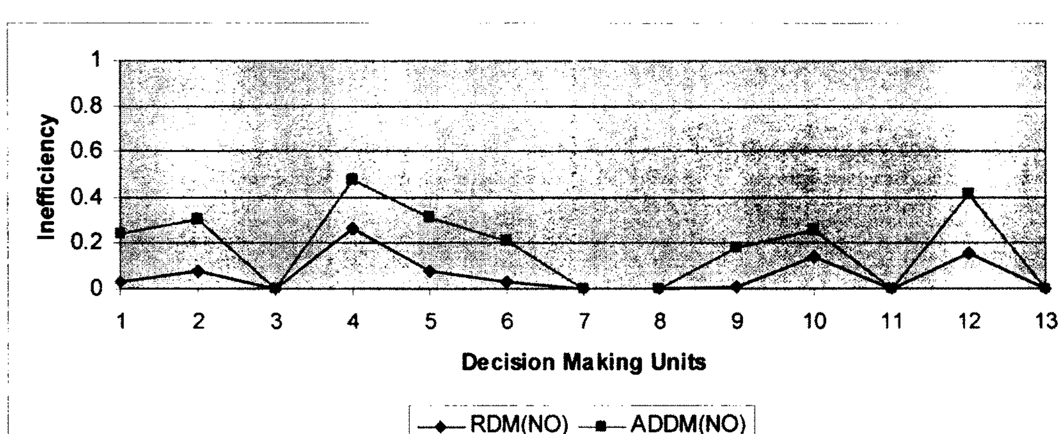

Figure-2.1: Comparison of inefficiency score by RDM and ADDM.

To compare with the Portela *et al.* (2004) non-oriented RDM model with the new model $\langle M\ 2.3\rangle$, we have taken results of the non oriented RDM and is shown in Table-2.3. It gives the λ^* reference set i.e weights corresponding to reference DMUs, inefficiency β^* and its efficiency $(1-\beta^*)$.

DMUs	1	2	3	4	5	6	7	8	9	10	11	12	13
1-β	0.9649	0.9181	1	0.7352	0.9243	0.9708	1	1	0.9945	0.8596	1	0.8488	1
β	0.0351	0.0819	0	0.2648	0.0757	0.0292	0	0	0.0055	0.1404	0	0.1552	0
λ3	0.0153	0.464	1	0.0812	0.649	0.9424				0.7368		0.9175	
λ7	0.9793	0.3528		0.1344	0.351		1						
λ8		0.0017		0.2695				1				0.0205	
λ11		0.1344							1		1		
λ13	0.0054	0.0472		0.5149		0.0576				0.2632		0.0612	1

Table-2.3: Results of non-oriented range directional distance model of VRS-DEA formulation.

Note: Values of λ1, λ2, λ4, λ5, λ6, λ9, λ10, λ13 are equal to zero for all DMUs. Hence those rows are omitted.

Results of the non-oriented ADDM by applying the model $\langle M\ 2.3\rangle$ are shown in Table-2.4. It can be seen from Table-2.4 that model$\langle M\ 2.3\rangle$ finds only those efficient units which are also efficient generated by the RDM non-oriented model (DMUs 3, 7, 8, 11, 13). Efficient scores of inefficient DMUs generated from the model $\langle M\ 2.3\rangle$ are less than or equal to RDM non-oriented model as shown in Table-2.3. This model simultaneously maximizes outputs and minimizes the inputs. Since the model $\langle M\ 2.3\rangle$, takes into account of individual variations in inputs and outputs, where in Portela *et al* (2004) RDM model attempting uniform variations in all inputs and outputs. Hence, in the RDM model the slacks remain in the optimal solutions that leads to inefficiency under RDM is less than or equal to the inefficiency under model$\langle M\ 2.3\rangle$. Portela *et al.*(2004) model overestimates the efficiency measure by including the slacks at the optimal solution.

DMUs	1	2	3	4	5	6	7	8	9	10	11	12	13
1-β	0.7569	0.6969	1	0.5211	0.6886	0.7862	1	1	0.8231	0.7453	1	0.5863	1
β	0.2431	0.3031	0	0.4789	0.3114	0.2138	0	0	0.1769	0.2547	0	0.4137	0
β1	0.0289	0	0	0.9722	0.0233	0	0	0	0.0046	0.0886	0	0.0277	0
β2	0.1111	0.7895	0	0.3844	0.961	0.1892	0	0	0.0001	0.6881	0	0.8375	0
β3	0.0227	0.2441	0	0.9858	0	0.4005	0	0	0.3506	0	0	0.4636	0
β4	1	0.4821	0	0.0522	0.5407	0.2776	0	0	0.483	0.4966	0	0.74	0
β5	0.0529	0	0	0	0.0318	0.2019	0	0	0.0462	0	0	0	0
λ3		0.1325	1			0.9707				0.062		0.3111	
λ7	1				0.5263		1						
λ8				0.0278				1					
λ11		0.8603			0.4737				1	0.6958	1	0.6889	
λ13		0.0071		0.9722		0.0293				0.2423			1

Table-2.4: Results of ADDM.

Note: Values of λ1, λ2, λ4, λ5, λ6, λ9, λ10, λ13 are equal to zero for all DMUs. Hence those rows are omitted.

2.4 Integer directional distance model:

Another aspect discussed in this chapter is integer restriction for the variables under performance evaluation. There are numerous instances of categorical data which are integrated in DEA analysis. Initially, Banker and Morey (1986b) incorporated the categorical variable into DEA formulation, to which Kamakura (1988) contributed a variation to its mixed integer linear programming formulation (MILP), and subsequently, Roussseau and Semple (1993) improving over the shortcomings in Kamakura's method. In recent times, Lozano and Villa (2006) pioneered work on integer inputs and outputs in DEA by incorporating integrality constraints to DEA framework and developed MILP procedure for solving DEA with integer variables. However, their model violated the DEA properties of convexity, free disposability and returns to scale. As a consequence, this model was not in agreement with Banker *et al.* (1984) of minimum extrapolation principle. Moreover, it overestimates the efficiency of evaluated DMU. In order to overcome these limitations, Kuosmanen and Kazemi Matin (2009) developed axiomatic approach to DEA formulation in constant returns to scale. Further, Kazemi Matin and Kuosmanen (2009) extended to variable returns to scale. No work is carried out to measure the efficiency and fixing the targets for inefficient DMUs where negative and integer variables simultaneously occur. Here, we integrate axiomatic approach to newly developed aggregate directional distance formulation of DEA and thereby responding to the shortcomings in the system involving both negative and integer data set.

There are many applications in which one or more inputs/or outputs are necessarily having negative integer quantities. When the values of the integer variable are large, then it is reasonable to consider them as continuous since overall effect of rounding off the solution to the nearest integer is small. However, this is not the case with small integer values, which are usually associated to high cost. For example, measurement of hospital or clinical efficiency output variable like unsuccessful medical surgeries and also in the case of categorical variables. In such cases a difference of one unit makes a significant difference and rounding off the solution is not adequate. In order to overcome this limitation we propose a mixed integer directional distance function formulation.

2.4.1 Mixed integer ADDM:

Let the set of input variables be $IP = IP^{I} \cup IP^{NI}$ and the set of output variables $OP = OP^{I} \cup OP^{NI}$. The subsets IP^{I} *and* OP^{I} are integer valued and the subsets IP^{NI} *and* OP^{NI} are real valued. The subsets IP^{I} *and* IP^{NI} as well as OP^{I} *and* OP^{NI} are be mutually exclusive. There are 'm' inputs variable out of which $p(\le m)$ are integers and 's' output variables of which $q(\le s)$ are integers. That is, $\left(IP^{I}, OP^{I}\right) \in Z^{p+q}$ and $\left(IP^{NI}, OP^{NI}\right) \in R^{(m-p)+(s-q)}$. Therefore, we can define input and output vectors of observed DMUs which are characterized by a pair of $\left(x_j, y_j\right)$ and is decomposed as $\left(\begin{pmatrix} x_j^{I} \\ x_j^{NI} \end{pmatrix}, \begin{pmatrix} y_j^{I} \\ y_j^{NI} \end{pmatrix}\right) \in \left(\begin{pmatrix} Z^{p} \\ R^{m-p} \end{pmatrix}, \begin{pmatrix} Z^{q} \\ R^{s-q} \end{pmatrix}\right)$.

Following the axiomatic approach of Kuosmanen and Kazemi Matin (2009), the production possibility set T is defined such that it satisfies the following assumptions.

$\langle I_1 \rangle$ The production possiblity set T is closed. $\left(\begin{pmatrix} x_j^{I} \\ x_j^{NI} \end{pmatrix}, \begin{pmatrix} y_j^{I} \\ y_j^{NI} \end{pmatrix}\right) \in T$ for $\forall j$

$\langle I_2 \rangle$ Disposability: $\left(\begin{pmatrix} x_j^{I} \\ x_j^{NI} \end{pmatrix}, \begin{pmatrix} y_j^{I} \\ y_j^{NI} \end{pmatrix}\right) \in T$ *and* $\left(\begin{pmatrix} u^{I} \\ u^{NI} \end{pmatrix}, \begin{pmatrix} v^{I} \\ v^{NI} \end{pmatrix}\right) \in \begin{pmatrix} Z \\ R \end{pmatrix} \Rightarrow \left(\begin{pmatrix} \bar{x}^{I} \\ \bar{x}^{NI} \end{pmatrix}, \begin{pmatrix} -\bar{y}^{I} \\ -\bar{y}^{NI} \end{pmatrix}\right) \ge \left(\begin{pmatrix} x^{I} \\ x^{NI} \end{pmatrix}, \begin{pmatrix} -y^{I} \\ -y^{NI} \end{pmatrix}\right)$

where $\begin{pmatrix} \bar{x}^{I} \\ \bar{x}^{NI} \end{pmatrix} = \begin{pmatrix} x^{I}+u^{I} \\ x^{NI}+u^{NI} \end{pmatrix}$ and $\begin{pmatrix} \bar{y}^{I} \\ \bar{y}^{NI} \end{pmatrix} = \begin{pmatrix} y^{I} \\ y^{NI} \end{pmatrix} - \begin{pmatrix} v^{I} \\ v^{NI} \end{pmatrix} \Rightarrow \left(\begin{pmatrix} \bar{x}^{I} \\ \bar{x}^{NI} \end{pmatrix}, \begin{pmatrix} \bar{y}^{I} \\ \bar{y}^{NI} \end{pmatrix}\right) \in T$

$\langle I_3 \rangle$ Convexity: $\left(\begin{pmatrix} x^I \\ x^{NI} \end{pmatrix}, \begin{pmatrix} y^I \\ Y^{NI} \end{pmatrix}\right)$ *and* $\left(\begin{pmatrix} \bar{x}^I \\ \bar{x}^{NI} \end{pmatrix}, \begin{pmatrix} \bar{y}^I \\ \bar{y}^{NI} \end{pmatrix}\right) \in T,$

$$\left(\begin{pmatrix} \tilde{x}^I \\ \tilde{x}^{NI} \end{pmatrix}, \begin{pmatrix} \tilde{y}^I \\ \tilde{y}^{NI} \end{pmatrix}\right) = \lambda\left(\begin{pmatrix} x^I \\ x^{NI} \end{pmatrix}, \begin{pmatrix} y^I \\ Y^{NI} \end{pmatrix}\right) + (1-\lambda)\left(\begin{pmatrix} \bar{x}^I \\ \bar{x}^{NI} \end{pmatrix}, \begin{pmatrix} \bar{y}^I \\ \bar{y}^{NI} \end{pmatrix}\right)$$

such that $0 \le \lambda \le 1$ and $\left(\begin{pmatrix} \tilde{x}^I \\ \tilde{x}^{NI} \end{pmatrix}, \begin{pmatrix} \tilde{y}^I \\ \tilde{y}^{NI} \end{pmatrix}\right) \in \begin{pmatrix} Z \\ R \end{pmatrix}^{m+s} \Rightarrow \left(\begin{pmatrix} \tilde{x}^I \\ \tilde{x}^{NI} \end{pmatrix}, \begin{pmatrix} \tilde{y}^I \\ \tilde{y}^{NI} \end{pmatrix}\right) \in T$

$\langle I_4 \rangle$ Divisibility: If $\left(\begin{pmatrix} x_j^I \\ x_j^{NI} \end{pmatrix}, \begin{pmatrix} y_j^I \\ y_j^{NI} \end{pmatrix}\right) \in T$ *and* $\exists\, \lambda, 0 \le \lambda \le 1,$ $\left(\lambda\begin{pmatrix} x^I \\ x^{NI} \end{pmatrix}, \lambda\begin{pmatrix} y^I \\ y^{NI} \end{pmatrix}\right) \in \begin{pmatrix} Z \\ R \end{pmatrix}^{m+s}$

$$\Rightarrow \left(\lambda\begin{pmatrix} x^I \\ x^{NI} \end{pmatrix}, \lambda\begin{pmatrix} y^I \\ y^{NI} \end{pmatrix}\right) \in T$$

Then we define production possibility set with respect to the VRS reference technology for mixed integer aggregate directional distance model satisfying $\langle I_1 \rangle$ to $\langle I_4 \rangle$ and is given by

$$T_{VRS}^{MIADDM} = \left\{ \begin{array}{l} \left(\begin{pmatrix} x^I \\ x^{NI} \end{pmatrix}, \begin{pmatrix} y^I \\ y^{NI} \end{pmatrix}\right) \in \left(\begin{pmatrix} Z^p \\ R^{m-p} \end{pmatrix}, \begin{pmatrix} Z^q \\ R^{s-q} \end{pmatrix}\right) : \begin{pmatrix} x^I \\ x^{NI} \end{pmatrix} \ge \sum_{j=1}^{n} \lambda_j \begin{pmatrix} x_j^I \\ x_j^{NI} \end{pmatrix} \\ \text{and } \begin{pmatrix} y^I \\ y^{NI} \end{pmatrix} \le \sum_{j=1}^{n} \lambda_j \begin{pmatrix} y_j^I \\ y_j^{NI} \end{pmatrix},\ \sum_{j=1}^{n} \lambda_j = 1;\ \lambda_j \ge 0;\ (j = 1, 2, \ldots, n). \end{array} \right\} \qquad (E\ 2.5)$$

The aim of this section is to develop mixed integer aggregate directional distance model. As pointed out earlier, integer valued negative data may arise in efficiency measurement in many situations and occur in dealing with undesirable inputs and outputs. The standard directional distance measure of efficiency is given in (E 2.4) may yield the targets for inefficient DMUs on non-integer values. So, projected operating point is not feasible. Based on the axioms defined in the sub Section-2.4.1, we adapt the directional distance function for the non-radial measure for mixed variables as

$$\left.\begin{array}{l} \vec{D}\left(\begin{pmatrix} X_o^I \\ X_o^{NI} \end{pmatrix}, \begin{pmatrix} Y_o^I \\ Y_o^{NI} \end{pmatrix}; \begin{pmatrix} R_{io}^{-I} \\ R_{io}^{-NI} \end{pmatrix}, \begin{pmatrix} R_{ro}^{+I} \\ R_{ro}^{NI} \end{pmatrix}\right)^{\text{int}} = \text{Max } \beta_{ADDM}^{\text{int}} = \dfrac{1}{m+s}\left(\sum_{i=1}^{p} \beta_{io}^{-} + \sum_{i=p+1}^{m} \beta_{io}^{-} + \sum_{r=1}^{q} \beta_{ro}^{+} + \sum_{r=q+1}^{s} \beta_{ro}^{+}\right) \\ : \left(\begin{pmatrix} x_o^I - \beta_{io}^{-I} R_{io}^{-I} \\ x_o^{NI} - \beta_{io}^{-NI} R_{io}^{-NI} \end{pmatrix}, \begin{pmatrix} y_o^I + \beta_{ro}^{+I} R_{ro}^{+I} \\ y_o^{NI} + \beta_{ro}^{+NI} R_{ro}^{+NI} \end{pmatrix}\right) \in T \text{ and} \\ \left(\begin{pmatrix} \tilde{x}^I \\ \tilde{x}^{NI} \end{pmatrix}, \begin{pmatrix} \tilde{y}^I \\ \tilde{y}^{NI} \end{pmatrix}\right) \in T\text{: } (\tilde{x}^I, \tilde{y}^I) \in Z^{p \times q}, \left(\tilde{x}^{NI}, \tilde{y}^{NI}\right) \in R^{(m-p)\times(s-q)} \\ \text{such that } \left(\begin{pmatrix} x_o^I - \beta_{io}^{-I} R_{io}^{-I} \ge \tilde{x}^I \\ x_o^{NI} - \beta_{io}^{-NI} R_{io}^{-NI} \ge \tilde{x}^{NI} \end{pmatrix}, \begin{pmatrix} y_o^I + \beta_{ro}^{+I} R_{ro}^{+I} \le \tilde{y}^I \\ y_o^{NI} + \beta_{ro}^{+NI} R_{ro}^{+NI} \le \tilde{y}^{NI} \end{pmatrix}\right) \end{array}\right\} (E\ 2.6)$$

Here $\left(\begin{pmatrix} R_{io}^{-I} \\ R_{io}^{-NI} \end{pmatrix}, \begin{pmatrix} R_{ro}^{+I} \\ R_{ro}^{+NI} \end{pmatrix}\right)$ is a non-negative directional vector. This function is simultaneously reducing the inputs and expanding outputs in the direction of $\left(\begin{pmatrix} R_{io}^{-I} \\ R_{io}^{-NI} \end{pmatrix}, \begin{pmatrix} R_{ro}^{+I} \\ R_{ro}^{+NI} \end{pmatrix}\right)$. The mixed integer directional distance formulation of DEA model can be formulated based on direction distance function defined in $(E\ 2.6)$ as

$$\left.\begin{array}{ll} I_e(x_o, y_o)_{ADDM}^{\text{int}} = Max\ \beta_{ADDM}^{\text{int}} = \dfrac{1}{m+s}\left(\sum_{i=1}^{p}\beta_{io}^{-} + \sum_{i=p+1}^{m}\beta_{io}^{-} + \sum_{r=1}^{q}\beta_{ro}^{+} + \sum_{r=q+1}^{s}\beta_{ro}^{+}\right) & \\ \text{subject to} \quad \sum_{j=1}^{n}\lambda_j y_{rj} - \beta_{ro}^{+}R_{ro}^{+} \geq y_{ro} \quad r \in OP^{NI} & \\ \sum_{j=1}^{n}\lambda_j y_{rj} - \tilde{y}_{ro} \geq 0 \quad r \in OP^{I} & \\ \beta_{ro}^{+}R_{ro}^{+} - \tilde{y}_{ro} \leq -y_{ro} \quad r \in OP^{I} & \\ \sum_{j=1}^{n}\lambda_j x_{ij} + \beta_{io}^{-}R_{io}^{-} \leq x_{io} \quad i \in IP^{NI} & \\ \sum_{j=1}^{n}\lambda_j x_{ij} - \tilde{x}_{io} \leq 0 \quad i \in IP^{I} & \\ \beta_{io}^{-}R_{io}^{-} + \tilde{x}_{io} \leq x_{io} \quad i \in IP^{I} & \\ \sum_{j=1}^{n}\lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, 2, \cdots, n. & \\ 0 \leq \beta_{io}^{-}, \beta_{ro}^{+} \leq 1 & \\ \tilde{x}_{io} \in Z^{p}, \tilde{y}_{ro} \in Z^{q} \text{ are unrestricted} & \end{array}\right\} \quad \langle M\ 2.4\rangle$$

$\beta_{ADDM}^{\text{int}}$ is the measure of technical inefficiency of the evaluated DMU and its efficiency is $\left(1-\beta_{ADDM}^{\text{int}}\right)$. It non-radially reduces the inputs and expands outputs. $\sum_{j=1}^{n}\lambda_j y_{rj}$ and $\sum_{j=1}^{n}\lambda_j x_{ij}$ are the convex combination of outputs and inputs respectively. The input and output reference points are

$$\left(\begin{pmatrix} \tilde{x}_{io} \\ x_{io}^{NI} - \beta_{io}^{-} R_{io}^{-NI} \end{pmatrix}, \begin{pmatrix} \tilde{y}_{ro} \\ y_{ro}^{NI} + \beta_{ro}^{+} R_{ro}^{+NI} \end{pmatrix}\right) \in \left(\begin{pmatrix} Z^{p} \\ R^{m-p} \end{pmatrix}, \begin{pmatrix} Z^{q} \\ R^{s-q} \end{pmatrix}\right).$$

$\left(\tilde{x}_{io}, \tilde{y}_{ro}\right) \in Z^{p} \times Z^{q}$ are the feasible integer valued inputs and outputs and $\left(x_{io} - \beta_{io}^{-} R_{io}^{-}, y_{ro} + \beta_{ro}^{+} R_{ro}^{+}\right) \in \left(R^{m-p}, R^{s-q}\right)$ are the non-integer inputs and outputs. The modified inefficiency measure of non-radial distance from the reference point i.e

$\left(x_{io}-\beta_{io}^{-}R_{io}^{-},\ y_{ro}+\beta_{ro}^{+}R_{ro}^{+}\right)$ with integer valued coordinates and the optimal solution of this model is $\left(x_{io}-\beta_{io}^{-*}R_{io}^{-}\right)=\tilde{x}_{io}=\sum_{j=1}^{n}\lambda_j^{*}x_{ij}$ and $\left(y_{ro}+\beta_{ro}^{+*}R_{ro}^{+}\right)=\tilde{y}_{ro}^{*}=\sum_{j=1}^{n}\lambda_j^{*}y_{rj}$.

2.4.2 Mixed integer radial directional distance model:

We can modify the directional distance function for the radial measure for mixed variables as

$$\left.\begin{array}{l}
\vec{D}\begin{pmatrix} X_o^{I},Y_o^{I};R_{io}^{-^{I}},R_{ro}^{+^{I}} \\ X_o^{NI},Y_o^{NI};R_{io}^{-^{NI}},R_{ro}^{+^{NI}}\end{pmatrix}^{\text{int}} = Max\ \beta_{RDM}^{\text{int}} \\
\text{such that}\begin{pmatrix} x_o^{I}-\beta R_{io}^{-^{I}}, & y_o^{I}+\beta R_{ro}^{+^{I}} \\ x_o^{NI}-\beta R_{io}^{-^{NI}}, & y_o^{NI}+\beta R_{ro}^{+^{NI}}\end{pmatrix}\in T \\
\text{and}\begin{pmatrix}\tilde{x}^{I},\tilde{y}^{I} \\ \tilde{x}^{NI},\tilde{y}^{NI}\end{pmatrix}\in T:\ (\tilde{x}^{I},\tilde{y}^{I})\in Z,\left(\tilde{x}^{NI},\tilde{y}^{NI}\right)\in R \\
\begin{pmatrix} x_o^{I}-\beta R_{io}^{-^{I}}\geq\tilde{x}^{I}; & y_o^{I}+\beta R_{ro}^{+^{I}}\leq\tilde{y}^{I} \\ x_o^{NI}-\beta R_{io}^{-^{NI}}\geq\tilde{x}^{NI}; & y_o^{NI}+\beta R_{ro}^{+^{NI}}\leq\tilde{y}^{NI}\end{pmatrix}
\end{array}\right\} \qquad (E\ 2.7)$$

The modified inefficiency, measures the radial distance from the reference point to the evaluated DMU. Based on directional distance function defined in the $(E\ 2.7)$, we can modify the directional distance formulation of DEA as shown in $\langle M\ 2.6\rangle$ and called as mixed integer radial directional distance model.

$$\left.\begin{array}{lll}
I_e(x_o,y_o)_{RDM}^{\text{int}}=\beta_{RDM}^{\text{int}}=Max\ \ \beta+\varepsilon\left(\sum_{i=1}^{m}s_i^{-}+\sum_{r=1}^{s}s_r^{+}+\sum_{i=1}^{p}s_i^{-^{I}}+\sum_{r=1}^{q}s_r^{+^{I}}\right) & & \\
\textit{subject to}\ \ \sum_{j=1}^{n}\lambda_j y_{rj}-\beta R_{ro}^{+}-s_{ro}^{+}=y_{ro} & r\in OP^{NI} & \\
\qquad\sum_{j=1}^{n}\lambda_j y_{rj}-\tilde{y}_{ro}-s_{ro}^{+}=0 & r\in OP^{I} & \\
\qquad\beta R_{ro}^{+}-\tilde{y}_{ro}+s_{ro}^{+^{I}}=-y_{ro} & r\in OP^{I} & \\
\qquad\sum_{j=1}^{n}\lambda_j x_{ij}+\beta R_{io}^{-}+s_{io}^{-}=x_{io} & i\in IP^{NI} & \\
\qquad\sum_{j=1}^{n}\lambda_j x_{ij}-\tilde{x}_{io}+s_{io}^{-}=0 & i\in IP^{I} & \\
\qquad\beta R_{io}^{-}+\tilde{x}_{io}+s_{io}^{-^{I}}=x_{io} & i\in IP^{I} & \\
\qquad\sum_{j=1}^{n}\lambda_j=1,\ \ \lambda_j\geq 0,\ \ j=1,2,\cdots,n. & & \\
\qquad s_{ro}^{+},s_{io}^{-},s_{ro}^{+^{I}},s_{io}^{-^{I}}\geq 0 & & \\
\qquad\tilde{x}_{io}\in Z^{p},\tilde{y}_{ro}\in Z^{q}\ \text{are unrestricted} & &
\end{array}\right\} \qquad \langle M\ 2.5\rangle$$

β_{RDM}^{int} is the measure of technical inefficiency of the evaluated DMU and its efficiency is $\left(1-\beta_{RDM}^{\text{int}}\right)$. It radially reduces the inputs and expands outputs. Symbol ε represents the non-Archimedean infinitesimal, variables $s_r^+, s_i^-, s_r^{+I}, s_i^{-I}$ denotes the non-radial slacks. $\sum_{j=1}^{n}\lambda_j y_{rj}$ and $\sum_{j=1}^{n}\lambda_j x_{ij}$ are the convex combination of outputs and inputs respectively. The input and output reference points are $\left(\begin{pmatrix}\tilde{x}_o^I \\ x_o^{NI}-\beta R_{io}^{-NI}\end{pmatrix}, \begin{pmatrix}\tilde{y}_o^I \\ y_o^{NI}+\beta R_{ro}^{+NI}\end{pmatrix}\right) \in \left(\begin{pmatrix}Z^p \\ R^{m-p}\end{pmatrix}, \begin{pmatrix}Z^q \\ R^{s-q}\end{pmatrix}\right)$ and $\left(\tilde{x}_o^I, \tilde{y}_o^I\right) \in Z^p \times Z^q$ are the integer valued inputs and outputs and $\left(x_o^{NI}-\beta R_{io}^{-NI}, y_o^{NI}+\beta R_{ro}^{+NI}\right) \in \left(R^{m-p}, R^{s-q}\right)$ are the non-integer inputs and outputs. Here we define two types of slacks, s_i^-, s_r^+ are of one type. These represent the difference between the convex combination $\left(\sum_{j=1}^{n}\lambda_j x_{ij}, \sum_{j=1}^{n}\lambda_j y_{rj}\right)$ and the reference points $\left(\begin{pmatrix}\tilde{x}_{io} \\ x_{io}-\beta R_{io}^-\end{pmatrix}, \begin{pmatrix}\tilde{y}_{ro} \\ y_{ro}+\beta R_{ro}^+\end{pmatrix}\right)$. The other type of slacks s_i^{-I}, s_r^{+I} are the difference between the reference points $\left(\tilde{x}_{io}, \tilde{y}_{ro}\right)$ and the projection points $\left(x_{io}-\beta R_{io}^-, y_{ro}+\beta R_{ro}^+\right)$ in the integer subset. Therefore $\left(x_{io}-\beta R_{io}^-\right) \geq \tilde{x}_{io} \geq \sum_{j=1}^{n}\lambda_j x_{ij}$ and $\left(y_{ro}+\beta R_{ro}^+\right) \leq \tilde{y}_{ro} \leq \sum_{j=1}^{n}\lambda_j y_{rj}$. Here, in this model there may be slacks in the optimal solution. The reference and the projection points may not be the Pareto efficient.

2.4.3 Results:

Result 2.1: The mixed integer directional distance inefficiency computed by the model $\langle M\ 2.4\rangle$ is never greater than the inefficiency through the ADDM $\langle M\ 2.3\rangle$.

Proof: The production possibility set T_{VRS}^{MIADDM} of (E 2.4) is contained in T_{VRS}^{ADDM} of (E2.3) i.e. $T_{VRS}^{MIADDM} \subset T_{VRS}^{ADDM}$, then inefficiencies computed through the model $\langle M\ 2.3\rangle \geq \langle M\ 2.4\rangle$

Therefore $I_e(x_o, y_o)_{ADDM}^{\text{int}} \leq I_e(x_o, y_o)_{ADDM}$.

Result 2.2: When variables are integer, if an existing DMU is ADDM efficient, then it is ADDM-integer efficient.

Proof can be seen easily.

2.4.4 Illustrative example:

Consider a case for performance measure of two outputs and single input as shown in Table-2.5. It lists 10 clinical units. Input 'x_1' is the number of physicians normalized to one. Output-1 'y_1' is the growth in the number of patients; output-2 'y_2' is the income. We plot each clinic taking output$_1$/input$_1$ and output$_2$/input$_1$ in the x and y axis respectively. Applying the output oriented RDM model by taking $R_{io}^{-}=0$ in the model $\langle M\ 2.1\rangle$. Also the model $\langle M\ 2.3\rangle$ to the data set given in Table-2.5, we found that there are three efficient clinics A, B and C with inputs and outputs as $(x_1^A, y_1^A, y_2^A) = (1, -2, 8.8)$, $(x_1^B, y_1^B, y_2^B) = (1, 4, 5.8)$ and $(x_1^C, y_1^C, y_2^C) = (1, 5, 3.4)$ in both the models. We obtain the efficiency score for an inefficient DMU_D is 0.3889, 0.5915 respectively for ADDM and RDM output oriented models and this will project the inefficient DMU_D on the frontier $(y_1^*, y_2^*) = (1.5, 7.05)$ at D_3, $(y_1^*, y_2^*) = (1.1279, 7.1239)$ at D_1. The thick continuous black piecewise linear line represents the efficient frontier. Projection of inefficient DMU_D on to the efficient frontier in the direction of ideal DMU is indicated by the continuous black line in the case of radial and thin dotted line in the case of non-radial. Both RDM and ADDM models will yield the targets for inefficient DMU to a non-integer value. A program is developed using MATLAB 7.7 to solve this problem and it can be seen in Appendix, Program-2.

DMU	A	B	C	D	E	F	G	H	I	J
No. of clinicians	1	1	1	1	1	1	1	1	1	1
Growth in number of patients	-2	4	5	-4	-1	-3	-1	2	-2	-4
Income (in Rs. 10,000)	8.8	5.8	3.4	4.3	6.5	5.9	2.3	3.9	7.1	1.9

Table-2.5: Two outputs and single input.

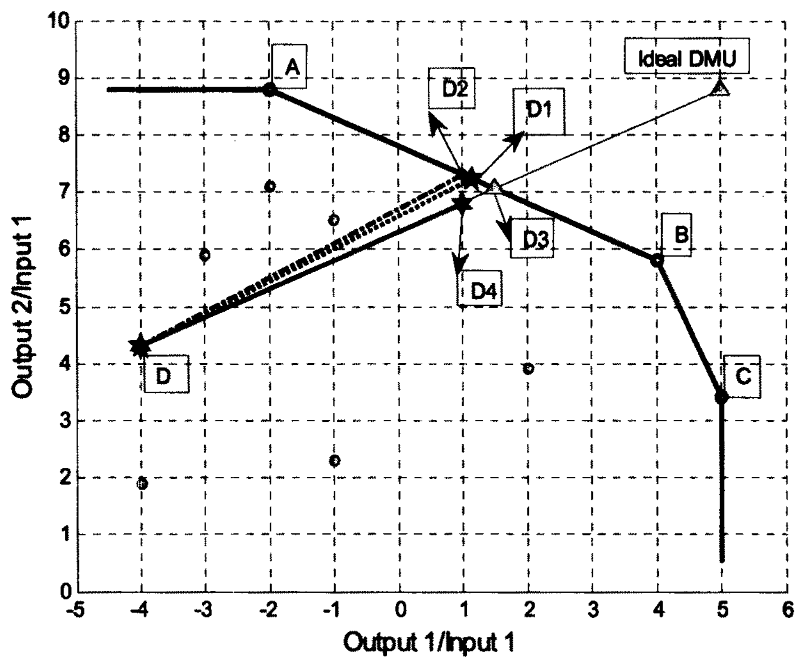

Figure-2.2: Illustration of integer targets for inefficient DMUs.

The modified integer programming models $\langle M\ 2.4\rangle$ (output oriented) and $\langle M\ 2.5\rangle$ finds the integer targets as $\left(y_1^*, y_2^*\right)$ = (1, 6.8) with efficiency score 0.4444 at the point D_4 and $\left(y_1^*, y_2^*\right)$ = (1,7.2271) at the point D_2 with efficiency score 0.5926 respectively. But the model $\langle M\ 2.5\rangle$ yields the integer target on to the frontier. In the case of radial integer directional distance model finds the integer target just below the frontier.

2.5 Conclusions:

This chapter is concerned with the non-radial measurement of efficiency from directional distance DEA formulation perspective. Negative integer valued inputs and outputs occur in a variety of situations. The radial measure of inefficiency proposed by Portela *et al.* (2004) to the negative inputs and outputs may have slacks in the optimal solution. The ADDM model presented in this chapter overcomes this problem. Proposed ADDM model takes into account of individual input and output variables for inefficiency measure. In RDM, proportionate increase of outputs or decrease of inputs is considered. Hence, it leads to the lower value of the inefficiency. We defined a non-proportional, non-oriented efficiency measure, which leads to the Pareto-Koopmans measure of technical efficiency. Inefficiency measure is bounded by 0 and 1, attaining minimum value of zero if and only if the DMU being rated Pareto-Koopmans efficiency. It monotonically decreases for any increase in outputs and/or reduction in inputs or increases monotonically for decrease in outputs and or increase in inputs. This model is translation and unit invariant. The new model allows us to alter the variable weights.

Further, integer directional distance function is also discussed in this chapter. Usual directional distance formulation of DEA and ADDM projection of efficient targets for inefficient DMUs generally has non-integer values. In this chapter we modified directional distance function and mixed integer directional distance formulation of DEA has been proposed. This model guarantees integer targets for inefficient DMUs and this can be applied to measure efficiency where input and output variables having negative integer values.