Chapter 3

Simple Acyclic Graphoidal Covers in a Graph

In this chapter we introduce the concept of simple acyclic graphoidal cover and simple acyclic graphoidal covering number \( \eta_{as} \) of a graph and determine \( \eta_{as} \), for several families of graphs. We prove that \( \eta_{as}(G) \leq q - d + 1 \), where \( d \) is the diameter of \( G \) and obtain a characterization of graphs attaining this bound. We also prove that \( \eta_{as}(G) \geq \Delta - 1 \) and characterize graphs attaining this bound. Further we prove that \( \eta_{as}(G) \geq \binom{\omega}{2} \), where \( \omega \) is the clique number of \( G \). Also we extend the notion of isomorphism of graphoidal covers to simple acyclic graphoidal covers and characterize trees and unicyclic graphs having a unique minimum simple acyclic graphoidal cover.

Definition 3.1. A simple acyclic graphoidal cover of a graph \( G \) is an acyclic graphoidal cover \( \psi \) of \( G \) such that any two paths in \( \psi \) have at most one vertex in common.

Let \( \mathcal{G}_{as} \) denote the set of all simple acyclic graphoidal covers of a graph \( G \). Since \( E(G) \) is trivially a simple acyclic graphoidal cover of \( G \), we have \( \mathcal{G}_{as} \neq \phi \). Let \( \eta_{as}(G) = \min_{\psi \in \mathcal{G}_{as}} |\psi| \). Then \( \eta_{as}(G) \) is called the simple acyclic graphoidal covering number of \( G \). Any
simple acyclic graphoidal cover \( \psi \) of \( G \) for which \( |\psi| = \eta_{as}(G) \) is called a minimum simple acyclic graphoidal cover of \( G \).

**Example 3.2.** Consider the graph \( G \) given in Figure 3.1.

Let \( \psi_1 = \{(v_1, v_2, v_3, v_4), (v_5, v_6, v_7, v_8), (v_2, v_6), (v_3, v_7)\} \) and
\[
\psi_2 = \{(v_5, v_6, v_2, v_3, v_4), (v_3, v_7), (v_6, v_7), (v_1, v_2), (v_7, v_8)\}.
\]

Clearly \( \psi_1 \) and \( \psi_2 \) are simple acyclic graphoidal covers of \( G \) with \( |\psi_1| = 4 \) and \( |\psi_2| = 5 \). Hence \( \psi_2 \) is not a minimum simple acyclic graphoidal cover of \( G \). In fact \( \psi_1 \) is a minimum simple acyclic graphoidal cover of \( G \) and \( \eta_{as}(G) = 4 \).

We state without proof the following results, whose proofs are similar to the corresponding results for simple graphoidal covers.
Theorem 3.3. Every path in a simple acyclic graphoidal cover of a graph is an induced path.

Theorem 3.4. For any simple acyclic graphoidal cover \( \psi \) of a graph \( G \), let \( t_\psi \) denote the number of exterior vertices of \( \psi \) and let \( t = \min t_\psi \), where the minimum is taken over all simple acyclic graphoidal covers \( \psi \) of \( G \). Then \( \eta_{as}(G) = q - p + t \).

Corollary 3.5. For any graph \( G \), \( \eta_{as}(G) \geq q - p \). Moreover, the following are equivalent.

(i) \( \eta_{as}(G) = q - p \).

(ii) There exists a simple acyclic graphoidal cover \( \psi \) of \( G \) without exterior vertices.

(iii) There exists a set \( \mathcal{P} \) of internally disjoint and edge disjoint induced paths without exterior vertices such that any two paths in \( \mathcal{P} \) have at most one vertex in common. (From such a set \( \mathcal{P} \) of paths the required simple acyclic graphoidal cover of \( G \) can be obtained by adding the edges of \( G \) which are not covered by the paths in \( \mathcal{P} \).)

Corollary 3.6. If there exists a simple acyclic graphoidal cover \( \psi \) of a graph \( G \) such that every vertex \( v \) of \( G \) with \( \deg v > 1 \) is interior to \( \psi \), then \( \psi \) is a minimum simple acyclic graphoidal
cover of $G$ and $\eta_{as}(G) = q - p + n$, where $n$ is the number of pendant vertices of $G$.

**Corollary 3.7.** Let $G$ be a $(p, q)$-graph such that $\eta_{as}(G) = q - p$. Then $\delta \geq 2$ and $\Delta \geq 3$.

**Remark 3.8.** Since any graphoidal cover of a tree $T$ is obviously a simple acyclic graphoidal cover of $T$, it follows from Theorem 1.35 that $\eta_{as}(T) = \eta(T) = n - 1$, where $n$ is the number of pendant vertices of $T$.

**Theorem 3.9.** For any graph $G$, $\eta_{as}(G) \leq q$. Further, equality holds if and only if $G$ is complete.

**Proof.** The inequality is trivial.

Suppose $G$ is not complete. Let $u$ and $v$ be two non-adjacent vertices of $G$. Let $P$ be a shortest $u$-$v$ path in $G$ so that $P$ is an induced path. Then $\psi = \{P\} \cup \{E(G) - E(P)\}$ is a simple acyclic graphoidal cover of $G$ such that $|\psi| < q$ and hence $\eta_{as}(G) < q$.

Conversely, suppose $G$ is complete. Then any induced path in $G$ is of length 1 and hence $\eta_{as}(G) = q$. $\Box$

In the following theorems we determine the value of $\eta_{as}$ for unicyclic graphs, wheels and complete bipartite graphs.
Theorem 3.10. Let $G$ be a unicyclic graph with $n$ pendant vertices. Let $C$ be the unique cycle in $G$ and let $m$ denote the number of vertices of degree greater than 2 on $C$. Then

$$\eta_{as}(G) = \begin{cases} 
3 & \text{if } m = 0 \\
n + 2 & \text{if } m = 1 \\
n + 1 & \text{if } m = 2 \\
n & \text{if } m \geq 3 
\end{cases}$$

Proof. Let $C = (v_1, v_2, \ldots, v_k, v_1)$.

Case 1. $m = 0$

Then $G = C$ and $\eta_{as}(G) = 3$.

Case 2. $m = 1$.

Let $v_1$ be the unique vertex of degree greater than 2 on $C$. Let $T_i, 1 \leq i \leq (\deg v_1) - 2$, be the branches of $G$ at $v_1$. Let $\psi_i$, be a minimum simple acyclic graphoidal cover of the branch $T_i$. Let $P_1$ be the path in $\psi_1$ having $v_1$ as a terminal vertex. Let

$$Q_1 = P_1 \circ (v_1, v_2)$$
$$Q_2 = (v_2, v_3, \ldots, v_k)$$
$$Q_3 = (v_k, v_1).$$

Then

$$\psi = \left\{ \left( \bigcup_{i=1}^{(\deg v_1) - 2} \psi_i \right) \setminus \{P_1\} \right\} \cup \{Q_1, Q_2, Q_3\}$$
is a simple acyclic graphoidal cover of $G$ and the number of vertices exterior to $\psi$ is $n + 2$. Hence $\eta_{as}(G) \leq n + 2$. Further, for any simple acyclic graphoidal cover $\psi$ of $G$ the $n$ pendant vertices of $G$ and at least two vertices on $C$ are exterior to $\psi$ so that $t \geq n + 2$. Hence $\eta_{as}(G) \geq n + 2$.

Thus $\eta_{as}(G) = n + 2$.

**Case 3.** $m = 2$.

Let $v_1$ and $v_r$, where $1 < r \leq k$, be the vertices of degree greater than 2 on $C$. Let $S_1$ and $S_2$ denote respectively the $(v_1, v_r)$-section and $(v_r, v_1)$-section of the cycle $C$ and let $v_s$ be an internal vertex of $S_1$(say). Let $R_1$ and $R_2$ denote the $(v_1, v_s)$-section of $S_1$ and $(v_s, v_r)$-section of $S_1$ respectively. Let $\psi_i$ and $\psi_j'$, where $1 \leq i \leq (\text{deg } v_1) - 2$, $1 \leq j \leq (\text{deg } v_r) - 2$, be minimum simple acyclic graphoidal covers of the branches $T_i$ and $T_j'$ of $G$ at $v_1$ and $v_r$ respectively. Let $P_i$ and $P'_i$ denote respectively the paths in $\psi_i$ and $\psi'_i$ having the vertices $v_1$ and $v_r$ as terminal vertices. Let

\[Q_1 = P_i \circ R_1\]

\[Q_2 = P'_i \circ R_2^{-1}\]

\[Q_3 = S_2.\]
\[ \psi = \left\{ \left( \bigcup_{i=1}^{(\text{deg } v_i)-2} \psi_i \right) \bigcup \left( \bigcup_{j=1}^{(\text{deg } v_j)-2} \psi'_j \right) - \{P_1, P'_1\} \right\} \cup \{Q_1, Q_2, Q_3\} \]

is a simple acyclic graphoidal cover of \( G \) and the number of vertices exterior to \( \psi \) is \( n + 1 \). Hence \( \eta_{as}(G) \leq n + 1 \). Further, for any simple acyclic graphoidal cover \( \psi \) of \( G \) the \( n \) pendant vertices of \( G \) and at least one vertex on \( C \) are exterior to \( \psi \) so that \( t \geq n + 1 \). Hence \( \eta_{as}(G) \geq n + 1 \).

Thus \( \eta_{as}(G) = n + 1 \).

Case 4. \( m \geq 3 \).

Let \( v_{i_1}, v_{i_2}, \ldots, v_{i_r} \), where \( 1 \leq i_1 < i_2 < \ldots, < i_r \leq k \), be the vertices of degree greater than 2 on \( C \). Let \( \psi_{j_s}, 1 \leq j \leq r \) and \( 1 \leq s \leq (\text{deg } v_{i_j}) - 2 \), be minimum simple acyclic graphoidal covers of the branches \( T_{j_s} \) of \( G \) at \( v_{i_j} \). Let \( P_1, P_2 \) and \( P_3 \) respectively denote the paths in \( \psi_{1_1}, \psi_{2_1} \) and \( \psi_{3_1} \) having \( v_{i_1}, v_{i_2} \) and \( v_{i_3} \) as terminal vertices. Let

\[
Q_1 = P_1 \circ (v_{i_1}, v_{i_1+1}, \ldots, v_{i_2}) \\
Q_2 = P_2 \circ (v_{i_2}, v_{i_2+1}, \ldots, v_{i_3}) \text{ and} \\
Q_3 = P_3 \circ (v_{i_3}, v_{i_3+1}, \ldots, v_{i_1}).
\]

Then

\[
\psi = \left\{ \left( \bigcup_{j=1}^{r} \left( \bigcup_{s=1}^{(\text{deg } v_{i_j})-2} \psi_{j_s} \right) \right) - \{P_1, P_2, P_3\} \right\} \cup \{Q_1, Q_2, Q_3\}
\]

is a simple acyclic graphoidal cover of \( G \) such that every vertex
of degree greater than 1 is interior to $\psi$ and hence it follows from Corollary 3.6 that $\eta_{as}(G) = n$.

**Theorem 3.11.** For the wheel $W_n = K_1 + C_{n-1}$, we have

$$\eta_{as}(W_n) = \begin{cases} 6 & \text{if } n = 4 \\ n + 1 & \text{if } n \geq 5 \end{cases}$$

**Proof.** Let $V(W_n) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0v_i : 1 \leq i \leq n - 1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n - 2\} \cup \{v_1 v_{n-1}\}$.

If $n = 4$, then $W_n = K_4$ and hence it follows from Theorem 3.9 that $\eta_{as}(W_n) = 6$.

Now, suppose $n \geq 5$. Let $P_1 = (v_1, v_2, \ldots, v_{n-2})$ and $P_2 = (v_{n-3}, v_0, v_{n-1})$. Then $\psi = \{P_1, P_2\} \cup S$, where $S$ is the set of edges of $W_n$ not covered by $P_1$ and $P_2$ is a simple acyclic graphoidal cover of $W_n$ and $|\psi| = n + 1$. Hence $\eta_{as}(G) \leq n + 1$.

Now, let $\psi$ be any simple acyclic graphoidal cover of $W_n$. If $P$ is a path in $\psi$ containing edges of the cycle $C = (v_1, v_2, \ldots, v_{n-1}, v_1)$, then $V(P) \subseteq V(C)$. Hence at least three vertices on $C$ are exterior to $\psi$ so that $t \geq 3$. Hence $\eta_{as}(W_n) \geq q - p + 3 = n + 1$.

Thus $\eta_{as}(W_n) = n + 1$. 
Theorem 3.12.

(i) \( \eta_{\text{as}}(K_{1,n}) = n - 1 \), for all \( n \geq 2 \).

(ii)

\[
\eta_{\text{as}}(K_{2,n}) = \begin{cases} 
3 & \text{if } n = 2 \\
4 & \text{if } n = 3 \\
2n - 3 & \text{if } n \geq 4
\end{cases}
\]

(iii)

\[
\eta_{\text{as}}(K_{3,n}) = \begin{cases} 
5 & \text{if } n = 3 \\
3(n - 2) & \text{if } n \geq 4
\end{cases}
\]

(iv) Let \( m \) and \( n \) be integers with \( n \geq m \geq 4 \). Then

\[
\eta_{\text{as}}(K_{m,n}) = \begin{cases} 
mn - m - n & \text{if } n \leq \left(\binom{m}{2}\right) \\
mn - m - n + r & \text{if } n = \left(\binom{m}{2}\right) + r, r > 0
\end{cases}
\]

Proof. We observe that, for any simple acyclic graphoidal cover \( \psi \) of \( K_{m,n} \) any path in \( \psi \) is either a path of length 2 or an edge.

(i) Since \( K_{1,n} \) is a tree with \( n \) pendant vertices, it follows from Remark 3.8 that \( \eta_{\text{as}}(K_{1,n}) = n - 1 \).

(ii) Since \( K_{2,2} = C_4 \), we have \( \eta_{\text{as}}(K_{2,2}) = 3 \).

Now, let \( X = \{x_1, x_2\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \) be the bipartition of \( K_{2,n} \).
If \( n = 3 \), then \( \psi = \{(y_1, x_1, y_2), (y_1, x_2, y_3), (y_2, x_2), (y_3, x_1)\} \) is a simple acyclic graphoidal cover of \( K_{2,3} \). Hence \( \eta_{as}(K_{2,3}) \leq 4 \).

Now, let \( \psi \) be any simple acyclic graphoidal cover of \( K_{2,3} \). Then \( \psi \) contains at most two paths of length 2 and hence the number of vertices interior to \( \psi \) is at most 2 so that \( t \geq 3 \). Hence \( \eta_{as}(K_{2,3}) \geq q - p + 3 = 4 \). Thus \( \eta_{as}(K_{2,3}) = 4 \).

Now, suppose \( n \geq 4 \). Let \( P_1 = (x_1, y_1, x_2) \), \( P_2 = (y_2, x_1, y_3) \) and \( P_3 = (y_2, x_2, y_4) \). Then \( \psi = \{P_1, P_2, P_3\} \cup S \), where \( S \) is the set of edges of \( K_{2,n} \) not covered by \( P_1, P_2 \) and \( P_3 \) is a simple acyclic graphoidal cover of \( K_{2,n} \) and \( |\psi| = 2n - 3 \). Hence \( \eta_{as}(K_{2,n}) \leq 2n - 3 \). Further, for any simple acyclic graphoidal cover \( \psi \) of \( K_{2,n} \) at most one vertex in \( Y \) is interior to \( \psi \) so that \( t \geq n - 1 \). Hence \( \eta_{as}(K_{2,n}) \geq q - p + n - 1 = 2n - 3 \). Thus \( \eta_{as}(K_{2,n}) = 2n - 3 \).

(iii) Let \( X = \{x_1, x_2, x_3\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \) be the bipartition of \( K_{3,n} \).

If \( n = 3 \), then \( \psi = \{(x_1, y_1, x_3), (x_1, y_2, x_2), (y_1, x_2, y_3), (y_2, x_3, y_3), (x_1, y_3)\} \) is a simple acyclic graphoidal cover of \( K_{3,3} \). Hence \( \eta_{as}(K_{3,3}) \leq 5 \). Now, let \( \psi \) be any simple acyclic graphoidal cover of \( K_{3,3} \). Then \( \psi \) contains at most four paths of length 2 and hence the number of vertices interior to \( \psi \) is at most 4 so that \( t \geq 2 \). Hence \( \eta_{as}(K_{3,3}) \geq q - p + 2 = 5 \). Thus \( \eta_{as}(K_{3,3}) = 5 \).
Now, suppose \( n \geq 4 \). Let \( P_1 = (x_1, y_1, x_2), P_2 = (x_1, y_2, x_3), P_3 = (x_2, y_3, x_3), P_4 = (y_1, x_3, y_4), P_5 = (y_2, x_2, y_4) \) and \( P_6 = (y_3, x_1, y_4) \). Then \( \psi = \{P_1, P_2, P_3, P_4, P_5, P_6\} \cup S \), where \( S \) is the set of edges of \( K_{3,n} \) not covered by the paths \( P_i, 1 \leq i \leq 6 \), is a simple acyclic graphoidal cover of \( K_{3,n} \) and \( |\psi| = 3(n - 2) \). Hence \( \eta_{as}(K_{3,n}) \leq 3(n - 2) \). Further, for any simple acyclic graphoidal cover \( \psi \) of \( K_{3,n} \) at most three vertices in \( Y \) are interior to \( \psi \) so that \( t \geq n - 3 \). Hence \( \eta_{as}(K_{3,n}) \geq q - p + n - 3 = 3(n - 2) \). Thus \( \eta_{as}(K_{3,n}) = 3(n - 2) \).

(iv) Let \( m \) and \( n \) be integers with \( n \geq m \geq 4 \). Let \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \) be the bipartition of \( K_{m,n} \).

Suppose \( m = n = 4 \). Then \( \psi = \{(x_1, y_1, x_2), (x_1, y_2, x_3), (x_2, y_3, x_4), (x_3, y_4, x_4), (y_3, x_1, y_4), (y_2, x_2, x_4), (y_3, x_4, y_2)\} \) is a simple acyclic graphoidal cover of \( K_{4,4} \) without exterior vertices. Hence \( \eta_{as}(K_{4,4}) = q - p = 8 \).

Assume that \( m \geq 4 \) and \( n \geq 5 \).

Case 1. \( n \leq \binom{m}{2} \).

Let \( S = \{\{i, j\} : 1 \leq i < j \leq m\} \). Clearly \( |S| = \binom{m}{2} \). We define a relation \( " < " \) on \( S \) by \( (i, j) < (k, l) \) if either \( i < k \) or \( i = k \) and \( j < l \). We now, index the elements of \( Y \) by the set \( I \)
of the first \( n \) elements of \( S \). Thus \( Y = \{ y_{\{i,j\}} : \{i,j\} \in I \} \).

Let \( P_{i,j} = (x_i, y_{\{i,j\}}, x_j) \), for all \( \{i,j\} \in I \).

\[
Q_1 = (y_{\{2,3\}}, x_1, y_{\{2,4\}}) \\
Q_2 = (y_{\{1,3\}}, x_2, y_{\{1,4\}}) \\
Q_i = (y_{\{1,2\}}, x_i, y_{\{1,i+1\}}), \text{ for all } i, \text{ where } 3 \leq i \leq m - 1. \\
Q_m = (y_{\{1,2\}}, x_m, y_{\{2,3\}}). \text{ Then}
\]

\[
\psi = \{ P_{i,j} : 1 \leq i \leq m - 1, 1 \leq j \leq m \text{ and } i < j \} \cup \{Q_1, Q_2, \ldots, Q_m\}
\]

is a collection of internally disjoint and edge disjoint induced paths without exterior vertices such that any two paths in \( \psi \) have at most one vertex in common. Hence it follows from Corollary 3.5 that \( \eta_{as}(K_{m,n}) = q - p = mn - m - n \).

**Case 2.** \( n > \binom{m}{2} \).

Let \( n = \binom{m}{2} + r \), where \( r > 0 \).

Let \( Y = \{ y_{\{i,j\}} : \{i,j\} \in S \} \cup \{z_1, z_2, \ldots, z_r\} \).

Then the collection \( \psi \) given in (1) with \( I = S \) is a collection of internally disjoint and edge disjoint induced paths such that any two paths in this collection have at most one vertex in common in which the vertices \( z_1, z_2, \ldots, z_r \) are exterior to \( \psi \). Hence \( \eta_{as}(K_{m,n}) \leq q - p + r = mn - m - n + r \). Further, for any simple acyclic graphoidal cover \( \psi \) of \( K_{m,n} \) at least \( r \) vertices of \( Y \) are exterior to \( \psi \) so that \( \eta_{as}(K_{m,n}) \geq q - p + r = mn - m - n + r \).
Hence \( \eta_{as}(K_{m,n}) = mn - m - n + r \).

We now proceed to obtain some bounds for \( \eta_{as} \) and characterize graphs attaining the bounds.

**Theorem 3.13.** Let \( G \) be a graph with diameter \( d \). Then \( \eta_{as}(G) \leq q - d + 1 \). Further, equality holds if and only if for any diameter path \( P = (u = v_1, v_2, \ldots, v_{d+1} = v) \) the following are satisfied.

1. Any two neighbours of each of \( u \) and \( v \) not on \( P \) are adjacent.
2. For any vertex \( w \) not on \( P \),
   
   (i) \( d(w, P) = 1 \).
   
   (ii) \( |N(w) \cap V(P)| \leq 3 \).
   
   (iii) If \( N(w) \cap V(P) = \{v_i, v_j, v_k\} \), where \( i < j < k \), then \( j = i + 1 \) and \( k = i + 2 \).
   
   (iv) If \( N(w) \cap V(P) = \{v_i, v_j\} \), where \( i < j \), then \( j = i + 1 \) or \( j = i + 2 \).
3. Every component of \( (V(G) - V(P)) \) is complete.
4. If \( x \) and \( y \) are two adjacent vertices not on \( P \), then \( N(x) \cap V(P) = N(y) \cap V(P) \).
(5) Suppose $x$ and $y$ are two non-adjacent vertices not on $P$. Then

(i) If $N(x) \cap V(P) = \{v_i, v_{i+2}\}$ or $\{v_i, v_{i+1}, v_{i+2}\}$ and $N(y) \cap V(P) = \{v_j\}$ with $i \leq j$, then $j \neq i + 1$.

(ii) If $N(x) \cap V(P) = \{v_i, v_{i+2}\}$ or $\{v_i, v_{i+1}, v_{i+2}\}$ and $N(y) \cap V(P) = \{v_i, v_{i+1}\}$ with $i \leq j$, then $j \geq i + 2$.

(iii) If $N(x) \cap V(P) = \{v_i, v_{i+2}\}$ or $\{v_i, v_{i+1}, v_{i+2}\}$ and $N(y) \cap V(P) = \{v_j, v_{j+2}\}$ with $i \leq j$, then $j \neq i + 1$.

(iv) If $N(x) \cap V(P) = \{v_i, v_{i+1}, v_{i+2}\}$ and $N(y) \cap V(P) = \{v_j, v_{j+1}, v_{j+2}\}$ with $i \leq j$, then $j \geq i + 2$.

Proof. Let $u$ and $v$ be two vertices in $G$ with $d(u, v) = d$ and let $P$ be a shortest $u$-$v$ path in $G$. Then $\psi = \{P\} \cup (E(G) - E(P))$ is a simple acyclic graphoidal cover of $G$ and $|\psi| = q - d + 1$. Hence $\eta_{as}(G) \leq q - d + 1$.

Suppose $\eta_{as}(G) = q - d + 1$. Let $P = (v_1, v_2, \ldots, v_{d+1})$ be a diameter path in $G$. Then conditions 2(i) to 2(iv) can be proved as in Theorem 2.19.

Claim 1 Any two neighbours of each of $u$ and $v$ not on $P$ are adjacent.
Suppose there exist two non-adjacent vertices $x$ and $y$ not on $P$ which are adjacent to $u$. Let $Q = (x, u, y)$. Then $\psi = \{P, Q\} \cup S$, where $S$ is the set of edges of $G$ not covered by $P$ and $Q$, is a simple acyclic graphoidal cover of $G$ and $|\psi| = q - d$, which is a contradiction. Hence any two neighbours of $u$ not on $P$ are adjacent. Similarly, any two neighbours of $v$ not on $P$ are adjacent. This proves condition (1) of the theorem.

Claim 2. Every component of $(V(G) - V(P))$ is complete.

Suppose there exists a component $H$ of $(V(G) - V(P))$ having two non-adjacent vertices, say $x$ and $y$. Let $Q$ be a shortest $x$-$y$ path in $H$. Then $\psi = \{P, Q\} \cup S$, where $S$ is the set of edges of $G$ not covered by $P$ and $Q$ is a simple acyclic graphoidal cover of $G$ and $|\psi| < q - d + 1$, which is a contradiction. Hence every component of $(V(G) - V(P))$ is complete. This proves condition (3) of the theorem.

Claim 3. If $x$ and $y$ are two adjacent vertices not on $P$, then $N(x) \cap V(P) = N(y) \cap V(P)$.

Suppose there exists a vertex $v_i$ on $P$ such that $v_i \in N(x) - N(y)$. Let $Q = (v_i, x, y)$. Then $\psi = \{P, Q\} \cup S$, where $S$ is the set of edges of $G$ not covered by $P$ and $Q$ is a simple acyclic graphoidal cover of $G$ and $|\psi| = q - d$, which is a contradiction.
Hence $N(x) \cap V(P) \subseteq N(y) \cap V(P)$. Similarly, $N(y) \cap V(P) \subseteq N(x) \cap V(P)$. Thus $N(x) \cap V(P) = N(y) \cap V(P)$. This proves condition (4) of the theorem.

Now, let $x$ and $y$ be two non-adjacent vertices not on $P$.

Suppose $N(x) \cap V(P) = \{v_i, v_{i+2}\}$ or $\{v_i, v_{i+1}, v_{i+2}\}$.

**Claim 4.** If $N(y) \cap V(P) = \{v_j\}$ with $i \leq j$, then $j \neq i + 1$.

Suppose $j = i + 1$. Let $P_1 = (u = v_1, v_2, \ldots, v_i, v_{i+1}, y)$, $P_2 = (v_{i+1}, v_{i+2}, \ldots, v_{d+1} = v)$ and $P_3 = (v_i, x, v_{i+2})$ (Refer Figure 3.2). Then $\psi = \{P_1, P_2, P_3\} \cup S$, where $S$ is the set of edges of $G$ not covered by $P_1, P_2$ and $P_3$ is a simple acyclic graphoidal cover of $G$ and $|\psi| = q - d$, which is a contradiction. Hence $j \neq i + 1$.

This proves condition 5(i) of the theorem.

![Figure 3.2](https://via.placeholder.com/150)
Claim 5. If \( N(y) \cap V(P) = \{v_j, v_{j+1}\} \) with \( i \leq j \), then \( j \geq i + 2 \).

Suppose \( j = i \) or \( j = i + 1 \). Let \( P_1 = (u = v_1, v_2, \ldots, v_j, y) \), \( P_2 = (y, v_{j+1}, v_{j+2}, \ldots, v_{d+1} = v) \) and \( P_3 = (v_i, x, v_{i+2}) \) (Refer Figure 3.3). Then \( \psi = \{P_1, P_2, P_3\} \cup S \), where \( S \) is the set of edges of \( G \) not covered by \( P_1, P_2 \) and \( P_3 \) is a simple acyclic graphoidal cover of \( G \) and \( |\psi| = q - d \), which is a contradiction. Hence \( j \geq i + 2 \). This proves condition 5(ii) of the theorem.

![Figure 3.3](image-url)
Claim 6. If \( N(y) \cap V(P) = \{v_j, v_{j+2}\} \) with \( i \leq j \), then \( j \neq i+1 \).

Suppose \( j = i + 1 \). Let \( P_1 = (u = v_1, v_2, \ldots, v_i, v_{i+1}, y) \), \( P_2 = (v_{i+1}, v_{i+2}, \ldots, v_{d+1} = y) \) and \( P_3 = (v_i, x, v_{i+2}) \) (Refer Figure 3.4). Then \( \psi = \{P_1, P_2, P_3\} \cup S \), where \( S \) is the set of edges of \( G \) not covered by \( P_1, P_2 \) and \( P_3 \) is a simple acyclic graphoidal cover of \( G \) and \( |\psi| = q - d \), which is a contradiction. Hence \( j \neq i + 1 \).

This proves condition 5(iii) of the theorem.

Figure 3.4

Claim 7. If \( x \) and \( y \) are two non-adjacent vertices not on \( P \) with \( N(x) \cap V(P) = \{v_i, v_{i+1}, v_{i+2}\} \) and \( N(y) \cap V(P) = \{v_j, v_{j+1}, v_{j+2}\} \), where \( i \leq j \), then \( j \geq i + 2 \).

If \( j = i \), let \( P_1 = (u = v_1, v_2, \ldots, v_j, x, v_{j+2}, \ldots, v_{d+1} = v) \),
$P_2 = (x, v_{j+1}, y)$ and $\psi = \{P_1, P_2\} \cup S$, where $S$ is the set of edges of $G$ not covered by $P_1$ and $P_2$ (Refer Figure 3.5).

If $j = i + 1$, let $P_1 = (u = v_1, v_2, \ldots, v_j, y), P_2 = (v_j, v_{j+1}, \ldots, v_{d+1} = v), P_3 = (v_i, x, v_{i+2})$ and $\psi = \{P_1, P_2, P_3\} \cup S$, where $S$ is the set of edges of $G$ not covered by $P_1, P_2$ and $P_3$ (Refer Figure 3.6).

Then $\psi$ is a simple acyclic graphoidal cover of $G$ and $|\psi| = q - d$, which is a contradiction. Hence $j \geq i + 2$. This proves condition 5(iv) of the theorem.
Conversely, suppose conditions (1)-(5) of the theorem are satisfied for any diameter path \( P = (u = v_1, \ldots, v_{d+1} = v) \). Let \( \psi \) be any minimum simple acyclic graphoidal cover of \( G \).

**Case 1.** \( P \) is a path in \( \psi \).

We claim that every vertex not on \( P \) is exterior to \( \psi \). Let \( w \) be a vertex not on \( P \). Let \( H \) be the component of \( G_1 = \langle V(G) - V(P) \rangle \) containing the vertex \( w \). If \( H = K_1 \), then \( N(w) \subseteq V(P) \) and hence \( w \) is exterior to \( \psi \). If \( |V(H)| \geq 2 \), then it follows from conditions (3) and (4) that any path having \( w \) as an internal vertex is not an induced path and hence \( w \) is exterior to \( \psi \). Thus every vertex not on \( P \) is exterior to \( \psi \).

Hence it follows from condition (1) that the number of vertices interior to \( \psi \) is exactly \( d-1 \) so that \( t = p - (d-1) = p - d + 1 \). Thus \( \eta_{as}(G) = q - d + 1 \).

**Case 2.** \( P \) is not a path in \( \psi \).

We claim that if there exists a vertex \( x \) not on \( P \) which is interior to \( \psi \), then there exists a vertex \( v_j \) on \( P \), where \( 2 \leq j \leq d \), which is exterior to \( \psi \).

Let \( Q \) be the path in \( \psi \) having \( x \) as an internal vertex. Then the two neighbours of \( x \) which are on \( Q \) are of the form \( \{v_i, v_{i+2}\} \), for some \( i \), where \( 1 \leq i \leq d - 1 \). We now claim that the vertex \( v_{i+1} \)
is exterior to \( \psi \). This is obvious if \( \text{deg } v_{i+1} = 2 \). Let \( \text{deg } v_{i+1} \geq 3 \).

We now consider the following cases.

**Subcase 2.1.** \(|N(x) \cap V(P)| = 2\).

Then \( N(x) \cap V(P) = \{v_i, v_{i+2}\} \). Let \( y \) be a vertex not on \( P \) which is adjacent to \( v_{i+1} \). Now by condition (4), the vertices \( x \) and \( y \) are not adjacent. Also it follows from conditions 5(i)-5(iii) that \( |N(y) \cap V(P)| = 3 \) and \( N(y) \cap V(P) = \{v_i, v_{i+1}, v_{i+2}\} \). Thus for any two neighbours \( y \) and \( z \) of \( v_{i+1} \) not on \( P \), we have \( N(y) \cap V(P) = N(z) \cap V(P) = \{v_i, v_{i+1}, v_{i+2}\} \) and hence it follows from condition 5(iv) that \( y \) and \( z \) are adjacent. Hence any path having \( v_{i+1} \) as an internal vertex is not an induced path. Thus \( v_{i+1} \) is exterior to \( \psi \).

**Subcase 2.2** \(|N(x) \cap V(P)| = 3\).

Then \( N(x) \cap V(P) = \{v_i, v_{i+1}, v_{i+2}\} \). If \( \text{deg } v_{i+1} = 3 \), then \( v_{i+1} \) is exterior to \( \psi \). Suppose \( \text{deg } v_{i+1} \geq 4 \). Let \( y \neq x \) be a vertex not on \( P \) which is adjacent to \( v_{i+1} \). It follows from conditions 5(i) to 5(iv) that the vertices \( x \) and \( y \) are adjacent and so by condition (4), we have \( N(y) \cap V(P) = \{v_i, v_{i+1}, v_{i+2}\} \). Hence any path having \( v_{i+1} \) as an internal vertex is not an induced path. Thus \( v_{i+1} \) is exterior to \( \psi \).
Thus for every vertex not on $P$ which is interior to $\psi$, there exists a vertex $v_j$, where $2 \leq j \leq d$, on $P$ which is exterior to $\psi$. Also it is clear that for any two distinct vertices not on $P$ which are interior to $\psi$, their corresponding vertices on $P$ which are exterior to $\psi$ are also distinct. Hence it follows from condition (1) that the number of vertices interior to $\psi$ is at most $d - 1$ so that $t \geq p - (d - 1) = p - d + 1$. Hence $\eta_{as}(G) \geq q - d + 1$.

Thus $\eta_{as}(G) = q - d + 1$. \qed

**Example 3.14.** Some graphs for which $\eta_{as} = q - d + 1$ are given in Figure 3.7.
Remark 3.15. Since any simple acyclic graphoidal cover of a graph $G$ is an acyclic graphoidal cover of $G$ and any acyclic graphoidal cover of $G$ is a graphoidal cover of $G$, we have $\eta \leq \eta_a \leq \eta_{as}$. These inequalities can be strict. For example, for the complete graph $K_3$, $\eta = 1$, $\eta_a = 2$ and $\eta_{as} = 3$.

Theorem 3.16. For any graph $G$, $\eta_{as}(G) \geq \Delta - 1$. Further, equality holds if and only if $G$ is homeomorphic to a star.

Proof. Since $\eta_{as}(G) \geq \eta_a(G)$, it follows from Theorem 1.44 that $\eta_{as}(G) \geq \Delta - 1$.

Suppose $\eta_{as}(G) = \Delta - 1$. Let $\psi = \{P_1, P_2, \ldots, P_{\Delta-1}\}$ be a minimum simple acyclic graphoidal cover of $G$. Let $v$ be a vertex of $G$ with $\text{deg} \ v = \Delta$. Then $v$ is interior to $\psi$ and $v$ lies on each $P_i$. Since $\psi$ is a simple acyclic graphoidal cover of $G$, we have...
\[ V(P_i) \cap V(P_j) = \{v\}, \text{ for all } i \neq j. \] Hence \( G \) is homeomorphic to a star.

The converse is obvious.

**Theorem 3.17.** For any graph \( G \), \( \eta_{as}(G) \geq \binom{\omega}{2} \), where \( \omega \) is the clique number of \( G \). Further, if \( \eta_{as}(G) = \binom{\omega}{2} \), then the following are satisfied.

(i) There exists a unique maximum clique \( H \) in \( G \).

(ii) If \( v \in V(H) \), then \( \deg v = \omega \) or \( \omega - 1 \).

(iii) If \( v \in V(G) - V(H) \), then \( \deg v \leq \left\lfloor \frac{\omega}{2} \right\rfloor + 1 \).

**Proof.** Let \( H \) be a maximum clique in \( G \) so that \( |E(H)| = \binom{\omega}{2} \).

Let \( \psi \) be a simple acyclic graphoidal cover of \( G \). Since any path in \( \psi \) covers at most one edge of \( H \), it follows that \( \eta_{as}(G) \geq \binom{\omega}{2} \).

Now, let \( G \) be a graph with \( \eta_{as}(G) = \binom{\omega}{2} \). Let \( \psi \) be a minimum simple acyclic graphoidal cover of \( G \).

Suppose there exists a vertex \( v \in V(H) \) with \( \deg v > \omega \). Let \( x \) and \( y \) be two vertices not on \( H \) which are adjacent to \( v \). Let \( P \) and \( Q \) be paths in \( \psi \) covering the edges \( xv \) and \( yv \) respectively. Since \( \eta_{as}(G) = \binom{\omega}{2} \), each of the paths \( P \) and \( Q \) covers exactly one edge of \( H \) and both of them are induced paths. Hence it
follows that \( P \neq Q \) and \( v \) is interior to both \( P \) and \( Q \), which is a contradiction. Hence \( \deg v = \omega \) or \( \deg v = \omega - 1 \). This proves condition (ii) of the theorem.

Now, let \( v \in V(G) - V(H) \). Since any path in \( \psi \) which contains \( v \) covers exactly two vertices of \( H \) and \( v \) is an internal vertex of at most one path in \( \psi \), it follows that \( \deg v \leq \left\lfloor \frac{\omega}{2} \right\rfloor + 1 \). This proves condition (iii) of the theorem.

Now, it follows from (iii) that \( H \) is the unique maximum clique in \( G \).

**Remark 3.18.** Since any simple acyclic graphoidal cover of a graph \( G \) is a simple graphoidal cover of \( G \) and any simple graphoidal cover of \( G \) is a graphoidal cover of \( G \), we have \( \eta \leq \eta_s \leq \eta_{as} \). These inequalities can be strict. For example, for the complete graph \( K_4 \), \( \eta = 2, \eta_s = 4 \) and \( \eta_{as} = 6 \).

The following example shows that there is no relation between \( \eta_{as} \) and \( \eta_g \).
Example 3.19. Consider the graphs $G_1, G_2$ and $G_3$ given in Figure 3.8.

Now, $\eta_{as}(G_1) = 5$, $\eta_{as}(G_2) = 5$ and $\eta_{as}(G_3) = 4$. Also it follows from Theorem 1.53 and Theorem 1.54 that $\eta_g(G_1) = 4$, $\eta_g(G_2) = 6$ and $\eta_g(G_3) = 4$.

Thus $\eta_{as}(G_1) > \eta_g(G_1)$, $\eta_{as}(G_2) < \eta_g(G_2)$ and $\eta_{as}(G_3) = \eta_g(G_3)$. Hence there is no relation between $\eta_{as}$ and $\eta_g$.

Obviously for any tree $T$, we have $\eta = \eta_a = \eta_g = \eta_s = \eta_{as}$. Also there exist graphs which are not trees for which the above equations are valid as shown in the following lemma.

**Lemma 3.20.** Let $G$ be a $(p, q)$-graph. Then $\eta(G') = \eta_a(G') = \eta_g(G') = \eta_s(G') = \eta_{as}(G') = p + q$, where $G'$ is the graph obtained from $G$ by attaching two pendant edges to every vertex of $G$. 
Proof. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$.

Let $u_i$ and $w_i$, $1 \leq i \leq p$, be the pendant vertices of $G'$ adjacent to $v_i$. Then $\psi = \{(u_i, v_i, w_i) : 1 \leq i \leq p\} \cup E(G)$ is a (acyclic / geodesic / simple /simple acyclic) graphoidal cover of $G'$ such that every vertex of degree greater than 1 is interior to $\psi$. Hence $\eta(G') = \eta_a(G') = \eta_g(G') = \eta_s(G') = \eta_{as}(G') = |\psi| = p + q$.

The above lemma leads to the following problem.

Problem 3.21. Characterize the class of graphs for which $\eta = \eta_a = \eta_g = \eta_s = \eta_{as}$.

We now proceed to investigate the structure of graphs which admit a (minimum) simple acyclic graphoidal cover satisfying the Helly property.

Theorem 3.22. A graph $G$ has a simple acyclic graphoidal cover satisfying the Helly property if and only if $G$ is triangle-free.

Proof. Suppose $G$ is triangle-free. Then $\psi = E(G)$ is a simple acyclic graphoidal cover of $G$ satisfying the Helly property.

Conversely, suppose $G$ has a triangle, say $C = (u, v, w, u)$. Let $\psi$ be any simple acyclic graphoidal cover of $G$. Then the edges $uv, vw$ and $uw$ lie on three different paths in $\psi$, say $P_1, P_2$
and \( P_3 \) respectively. Clearly \( \{P_1, P_2, P_3\} \) is a pairwise intersecting family of paths in \( \psi \). If there exists a vertex \( x \) which is common to the paths \( P_1, P_2 \) and \( P_3 \), then the vertices \( x \) and \( v \) are common to both \( P_1 \) and \( P_2 \), which is a contradiction. Hence \( V(P_1) \cap V(P_2) \cap V(P_3) = \emptyset \). Thus \( \psi \) does not satisfy the Helly property.

**Theorem 3.23.** Every simple acyclic graphoidal cover of a graph \( G \) satisfies the Helly property if and only if \( G \) is a tree.

**Proof.** Suppose \( G \) is a graph in which every simple acyclic graphoidal cover satisfies the Helly property. Suppose \( G \) contains a cycle, say \( C = (v_1, v_2, \ldots, v_k, v_1) \), where \( k \geq 3 \). Let \( P_1 = (v_1, v_2) \), \( P_2 = (v_2, v_3, \ldots, v_k) \) and \( P_3 = (v_k, v_1) \). Then \( \psi = \{P_1, P_2, P_3\} \cup (E(G) - E(C)) \) is a simple acyclic graphoidal cover of \( G \). Clearly \( \{P_1, P_2, P_3\} \) is pairwise intersecting family of paths in \( \psi \), whereas there exists no vertex in \( G \) common to the paths \( P_1, P_2 \) and \( P_3 \). Hence \( \psi \) does not satisfy the Helly property, which is a contradiction. Hence \( G \) is a tree.

The converse follows from Theorem 1.16.

We now construct some classes of graphs with a minimum simple acyclic graphoidal cover satisfying the Helly property.
Theorem 3.24. Let $C$ be a cycle of length greater than 3. Then the graph $G$ obtained from $C$ by attaching a pendant edge to every vertex of $C$ has a minimum simple acyclic graphoidal cover satisfying the Helly property.

Proof. Let $C = (v_1, v_2, \ldots, v_n, v_1)$, where $n \geq 4$. Let $u_1, u_2, \ldots, u_n$ be the pendant vertices of $G$ which are adjacent to $v_1, v_2, \ldots, v_n$ respectively. Then $\psi = \{(u_i, v_i, v_{i+1}) : 1 \leq i \leq n-1\} \cup \{(u_n, v_n, v_1)\}$ is a minimum simple acyclic graphoidal cover of $G$. Clearly any pairwise intersecting family of paths in $\psi$ contains at most two paths and hence $\psi$ satisfies the Helly property.

Theorem 3.25. Let $G$ be a graph. Then the graph $G'$ obtained from $G$ by attaching two pendant edges to every vertex of $G$ has a minimum simple acyclic graphoidal cover satisfying the Helly property.

Proof. Let $\psi$ be the minimum simple acyclic graphoidal cover of $G'$ given in Lemma 3.20. Then any pairwise intersecting family of paths in $\psi$ has at most two paths and hence $\psi$ satisfies the Helly property.

The above results lead to the following problems.

Problem 3.26. Characterize graphs which admit a minimum simple acyclic graphoidal cover satisfying the Helly property.
Problem 3.27. Characterize graphs in which every minimum simple acyclic graphoidal cover satisfies the Helly property.

We now extend the notion of isomorphism of graphoidal covers to simple acyclic graphoidal covers.

Definition 3.28. Two simple acyclic graphoidal covers $\psi_1$ and $\psi_2$ of a graph $G$ are said to be isomorphic if there exists an isomorphism $\alpha$ of $G$ such that $\psi_2 = \{\alpha(P) : P \in \psi_1\}$. A graph $G$ is said to have a unique minimum simple acyclic graphoidal cover if any two minimum simple acyclic graphoidal covers of $G$ are isomorphic.

Remark 3.29. Clearly any two isomorphic simple acyclic graphoidal covers of a $(p, q)$ - graph $G$ give rise to the same partition of the integer $q$. However the converse is not true, for example consider the graph $G$ given in Figure 3.9.

Figure 3.9
Let \( \psi_1 = \{(v_5, v_1, v_2, v_6), (v_1, v_4, v_8), (v_2, v_3, v_7), (v_4, v_3)\} \) and \\
\( \psi_2 = \{(v_5, v_1, v_2, v_3)(v_1, v_4, v_8), (v_4, v_3, v_7), (v_6, v_2)\} \).

Then \( \psi_1 \) and \( \psi_2 \) are two minimum simple acyclic graphoidal covers of \( G \) which are not isomorphic. However \( \psi_1 \) and \( \psi_2 \) determine the same partition of \( q \).

**Remark 3.30.** Since every graphoidal cover of a tree is a simple acyclic graphoidal cover it follows from Theorem 1.48 that a tree \( T \) has a unique minimum simple acyclic graphoidal cover if and only if there exists at most one vertex \( v \) in \( T \) with \( \deg v > 2 \) and the distance from \( v \) to all the pendant vertices of \( T \) are equal.

We now proceed to characterize unicyclic graphs with a unique minimum simple acyclic graphoidal cover.

**Theorem 3.31.** A unicyclic graph \( G \) has a unique minimum simple acyclic graphoidal cover if and only if \( G \) is either \( C_3 \) or \( C_4 \) or a graph obtained by attaching a path to a vertex of a triangle.

**Proof.** Let \( C = (v_1, v_2, \ldots, v_k, v_1) \) be the cycle in \( G \). Let \( m \) denote the number of vertices of degree greater than 2 on \( C \). Let \( n \) be the number of pendant vertices in \( G \). Suppose \( G \) has a unique minimum simple acyclic graphoidal cover.

We claim that \( m \leq 1 \).
Suppose \( m \geq 3 \). Let \( v_i, v_j \), where \( 1 < i < j \leq k \), be vertices of degree greater than 2 on \( C \). Let \( P = (v_1, w_1, w_2, \ldots, w_r) \) be the longest path in \( G \) such that \( V(P) \cap V(C) = \{v_1\} \). Let \( P' = (v_i, u_1, \ldots, u_s) \) be the longest path in \( G \) such that \( V(P') \cap V(C) = \{v_i\} \). Let \( P'' = (v_j, z_1, z_2, \ldots, z_t) \) be the longest path in \( G \) such that \( V(P'') \cap V(C) = \{v_j\} \).

Suppose \( m = 2 \). Let \( v_{i_1} \) and \( v_{i_2} \), where \( 1 \leq i_1 < i_2 \leq k \), be the vertices of degree greater than 2 on \( C \). Let \( S_1 \) and \( S_2 \) denote the \((v_{i_1}, v_{i_2})\)-section and \((v_{i_2}, v_{i_1})\)-section of the cycle \( C \) respectively and let \( v_r \) be an internal vertex of \( S_1 \) (say). Let \( R_1 \) and \( R_2 \) denote respectively the \((v_{i_1}, v_r)\) and \((v_r, v_{i_2})\)-sections of \( S_1 \). Let \( Q \) be the longest path in \( G \) with origin \( v_{i_1} \) such that \( V(Q) \cap V(C) = \{v_{i_1}\} \) and let \( Q' \) be the longest path in \( G \) with origin \( v_{i_2} \) such that \( V(Q') \cap V(C) = \{v_{i_2}\} \). Now, let \( P_1 = Q^{-1} \circ S_2 \circ Q' \), \( P_2 = R_1 \), \( P_3 = R_2 \), \( Q_1 = Q^{-1} \circ S_2 \), \( Q_2 = R_1 \) and \( Q_3 = R_2 \circ Q' \).

Let \( \mathcal{P}_1 = \{P_1, P_2, P_3\} \) and \( \mathcal{P}_2 = \{Q_1, Q_2, Q_3\} \). In both cases
the two collections of paths \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) in \( G \) satisfy the following conditions:

(i) Both \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) cover the same set of edges and these edges cannot be covered by a fewer number of paths.

(ii) Both \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) have the same set of vertices as internal vertices.

(iii) The paths in \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) cannot be extended to cover more edges of \( G \).

(iv) Any two paths in each of \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) have at most one vertex in common.

(v) Both \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) cover all the edges of the cycle \( C \).

Hence we can find two minimum simple acyclic graphoidal covers \( \psi_1 \) and \( \psi_2 \) of \( G \) such that \( \mathcal{P}_1 \subseteq \psi_1 \) and \( \mathcal{P}_2 \subseteq \psi_2 \). It follows from condition (v) that any automorphism of \( G \) maps \( \mathcal{P}_1 \) to \( \mathcal{P}_2 \). However, in \( \mathcal{P}_1 \), both end vertices of \( p_1 \) are pendant vertices, whereas no path in \( \mathcal{P}_2 \) has this property. Hence there is no automorphism of \( G \) which maps \( \psi_1 \) to \( \psi_2 \) so that \( \psi_1 \) and \( \psi_2 \) are not isomorphic, which is a contradiction.

Hence \( m \leq 1 \).
Case 1. \( m = 0 \).

Then \( G = C \) and \( \eta_{as}(G) = 3 \). Suppose \( k \geq 5 \). Let \( P_1 = (v_1, v_k), P_2 = (v_k, v_{k-1}, \ldots, v_2), P_3 = (v_1, v_2), Q_1 = (v_1, v_k, v_{k-1}) \) and \( Q_2 = (v_{k-1}, v_{k-2}, \ldots, v_2) \). Then \( \psi_1 = \{P_1, P_2, P_3\} \) and \( \psi_2 = \{Q_1, Q_2, P_3\} \) are two minimum simple acyclic graphoidal covers of \( G \) which determine two different partitions of \( G \), which is a contradiction. Thus \( k \leq 4 \).

Hence \( G \) is either \( C_3 \) or \( C_4 \).

Case 2 \( m = 1 \).

Then \( \eta_{as}(G) = n + 2 \). Let \( v_1 \) be the vertex of degree greater than 2 on \( C \).

Claim 1. Every vertex not on \( C \) has degree either 1 or 2.

Suppose there exists a vertex \( w \) not on \( C \) such that \( \text{deg } w \geq 3 \). Let \( P = (w, w_1, w_2, \ldots, w_l, v_1) \). Let \( Q_1 = (u_1, u_2, \ldots, u_r, w, u_{r+1}, \ldots, u_s) \) be the longest path such that \( V(Q_1) \cap V(P) = \{w\} \).

Let \( v_i \) and \( v_j \), where \( 1 < i < j \leq k \) be two vertices on \( C \).

\[ Q = (v_i, v_{i+1}, \ldots, v_j) \]
\[ R = (v_j, v_{j+1}, \ldots, v_k, v_1) \]
\[ P_1 = (u_s, u_{s-1}, \ldots, u_{r+1}, w, w_1, w_2, \ldots, w_l, v_1, v_2, \ldots, v_i) \]
\[ P_2 = (u_1, u_2, \ldots, u_r, w) \]
\[ Q_2 = (w, w_1, \ldots, w_l, v_1, v_2, \ldots, v_i) \].
Then \( S_1 = \{P_1, P_2, Q, R\} \) and \( S_2 = \{Q_1, Q_2, Q, R\} \) are two collections of paths satisfying the conditions (i) - (v) as stated earlier. Hence we can find paths \( P_3, P_6, \ldots, P_{n+2} \) such that \( \psi_1 = \{Q, R, P_1, P_2, P_3, \ldots, P_{n+2}\} \) and \( \psi_2 = \{Q, R, Q_1, Q_2, P_3, P_6, \ldots, P_{n+2}\} \) are two minimum simple acyclic graphoidal covers of \( G \).

Since, \( P_1 \) and \( Q_2 \) are respectively the paths in \( \psi_1 \) and \( \psi_2 \) having \( v_1 \) as an internal vertex, any automorphism \( \alpha \) of \( G \) maps \( P_1 \) to \( Q_2 \). However the path \( P_1 \) has a pendant vertex, whereas the path \( Q_2 \) has no pendant vertex. Hence there is no automorphism of \( G \) which maps \( \psi_1 \) to \( \psi_2 \) so that \( \psi_1 \) and \( \psi_2 \) are not isomorphic, which is a contradiction.

Thus every vertex not on \( C \) has degree either 1 or 2.

**Claim 2.** \( \deg v_1 = 3 \).

Suppose \( \deg v_1 = r \geq 4 \). Let \( u_1, u_2, \ldots, u_{r-2} \) be the pendant vertices of \( G \). Let \( P_i \), where \( 1 \leq i \leq r-2 \), be the \( u_i-v_1 \) path in \( G \). Let \( v_{i_1} \) and \( v_{i_2} \), where \( 1 < i_1 < i_2 \leq k \), be two vertices on \( C \). Let

\[
Q_1 = (v_1, v_2, \ldots, v_{i_1}) \\
Q_2 = (v_{i_1}, v_{i_1+1}, \ldots, v_{i_2}) \\
Q_3 = (v_{i_2}, v_{i_2+1}, \ldots, v_k, v_1).
\]
Then $\psi_1 = \{P_1 \circ P_2^{-1}, P_3, \ldots, P_{r-2}, Q_1, Q_2, Q_3\}$ and

$$\psi_2 = \{P_1 \circ Q_1, P_2, P_3, \ldots, P_{r-2}, Q_2, Q_3\}$$

are two minimum simple acyclic graphoidal covers of $G$. Now, the path in $\psi_1$ having $v_1$ as an internal vertex has two pendant vertices, whereas the path in $\psi_2$ having $v_1$ as an internal vertex has exactly one pendant vertex. Hence $\psi_1$ and $\psi_2$ are not isomorphic, which is a contradiction. Thus $\deg v_1 = 3$.

**Claim 3.** $k = 3$.

By Claim 2, $G$ contains exactly one pendant vertex, say $u_1$. Let $P$ be the $u_1-v_1$ path of length $l > 0$.

Suppose $k \geq 5$. Then $\psi_1 = \{P \circ (v_1, v_2), (v_2, v_3), (v_3, v_4, \ldots, v_l, v_1)\}$ and $\psi_2 = \{P \circ (v_1, v_2), (v_2, v_3, v_4), (v_4, v_5, \ldots, v_k, v_1)\}$ are two minimum simple acyclic graphoidal covers of $G$ giving rise to the following partitions of $q$ respectively.

(i) $l + 1, 1, k - 2$

(ii) $l + 1, 2, k - 3$

Since $l > 0$ and $k \geq 5$, $\psi_1$ and $\psi_2$ are non-isomorphic, which is a contradiction. Hence $k \leq 4$.

Suppose $k = 4$. Then $\psi_1 = \{P \circ (v_1, v_2), (v_2, v_3), (v_3, v_4, v_1)\}$ and $\psi_2 = \{P \circ (v_1, v_2, v_3), (v_3, v_4), (v_4, v_1)\}$ are two minimum simple
acyclic graphoidal covers of $G$ which determine two different partitions of $q$ and hence $\psi_1$ and $\psi_2$ are non-isomorphic, which is a contradiction. Thus $k = 3$.

Thus by Claim 1, Claim 2 and Claim 3, $G$ is a graph obtained by attaching a path to a vertex of a triangle.

The converse is obvious.

**Conclusion and Scope**

In this chapter we have introduced the concept of simple acyclic graphoidal cover and simple acyclic graphoidal covering number $\eta_{as}$ of a graph $G$ and several results concerning this parameter have been presented. We have constructed graphs for which the graphoidal covering number $\eta$, the acyclic graphoidal covering number $\eta_a$, the geodesic graphoidal covering number, $\eta_g$, simple graphoidal covering number $\eta_s$ and the simple acyclic graphoidal covering number $\eta_{as}$ are equal. We also obtained the necessary conditions for $\eta_{as} = \binom{\omega}{2}$, where $\omega$ is the clique number of $G$. The following are some interesting problems for further investigation.

(i) Characterize graphs for which $\eta_{as} = q - p$.

(ii) Characterize graphs for which $\eta = \eta_a = \eta_g = \eta_s = \eta_{as}$.

(iii) Characterize graphs for which $\eta_{as} = \binom{\omega}{2}$. 
(iv) Characterize graphs which admit a minimum simple acyclic graphoidal cover satisfying the Helly property.

(v) Characterize graphs in which every minimum simple acyclic graphoidal cover satisfies the Helly property.

(vi) Characterize graphs having a unique minimum simple acyclic graphoidal cover.