Chapter 4

Simple Path Covers in a Graph

The concept of path cover and path covering number \( \pi \) of a graph was introduced by Harary [22]. Preliminary results on \( \pi \) were obtained by Harary and Schwenk [23], Stanton et. al. [30, 31] and Peroche [28]. In this chapter, we obtain some bounds for \( \pi \). Also we introduce the concept of simple path cover and simple path covering number \( \pi_s \) of a graph and determine the value of \( \pi_s \) for several families of graphs. We prove that \( \pi_s(G) \geq \left\lfloor \frac{\Delta}{2} \right\rfloor \) and equality holds if and only if \( G \) is homeomorphic to a star. Further we discuss the relation between \( \pi_s \) and \( \eta_{as} \).

Arumugam and Suresh Suseela [16] observed that \( \pi \geq \left\lfloor \frac{\Delta}{2} \right\rfloor \) and characterized trees and unicyclic graphs attaining this bound. In the following theorem we characterize regular graphs attaining this bound.

**Theorem 4.1.** Let \( G \) be a regular graph. Then \( \pi(G) = \left\lceil \frac{\Delta}{2} \right\rceil \) if and only if \( G \) is a complete graph of even order.
Proof. Let $G$ be a $r$-regular graph on $p$ vertices with $\pi(G) = \left\lceil \frac{\Delta}{2} \right\rceil$.

We claim that $r$ is odd.

Suppose $r$ is even.

Then $q = \frac{pr}{2}$ and $\pi(G) = \frac{r}{2}$. Let $\psi$ be a path cover of $G$ with $|\psi| = \frac{r}{2}$. Now, the $\frac{r}{2}$ paths in $\psi$ can cover at most $\frac{r(p-1)}{2}$ edges of $G$ so that $q \leq \frac{r}{2}(p-1)$, which is a contradiction.

Hence $r$ is odd.

Now, it follows from Corollary 1.22 that $\pi(G) \geq \frac{p}{2}$ so that $\frac{r+1}{2} \geq \frac{p}{2}$ and hence $r \geq p - 1$.

Thus $r = p - 1$ so that $G = K_p$ and $p = r + 1$ is even.

The converse follows from Theorem 1.20.

**Theorem 4.2.** Let $G$ be a connected graph which is not a tree and let $\Delta = 3$. Then $\pi(G) = \left\lceil \frac{\Delta}{2} \right\rceil$ if and only if $G$ is homeomorphic to one of the graphs $G_i, 1 \leq i \leq 6$, given in Figure 4.1.

![Graphs G1, G2, G3](image-url)
Figure 4.1

Proof. Since $\Delta = 3$, it follows from Theorem 1.46 that $\pi = \eta_a$. Hence $\pi = \left\lceil \frac{\Delta}{2} \right\rceil$ if and only if $\eta_a = \Delta - 1$ so that the result follows from Theorem 1.45.

Remark 4.3. Let $\psi$ be a minimum path cover of a graph $G$. Since each path in $\psi$ covers at most $p - 1$ edges of $G$, it follows that $q \leq (p - 1)\pi$. Thus $\pi \geq \left\lfloor \frac{q}{p-1} \right\rfloor$. Further if $G$ is a $(p, q)$-graph such that $\frac{q}{p-1}$ is an integer, then $\pi(G) = \frac{q}{p-1}$ if and only if $G$ is decomposable into hamilton paths.

We now proceed to characterize trees, unicyclic graphs and cubic graphs for which $\pi = \left\lfloor \frac{q}{p-1} \right\rfloor$. We observe that for a tree $T$ $\frac{q}{p-1} = 1$ and hence $\pi(T) = \left\lfloor \frac{q}{p-1} \right\rfloor$ if and only if $T$ is a path.
Theorem 4.4. Let $G$ be a unicyclic graph. Then $\pi(G) = \left\lfloor \frac{q}{p-1} \right\rfloor$ if and only if $G$ is homeomorphic to one of the graphs $G_i$, $1 \leq i \leq 6$, given in Figure 4.2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure42.png}
\caption{Figure 4.2}
\end{figure}

Proof. Let $C$ be the unique cycle in $G$ and let $m$ denote the number of vertices of degree greater than 2 on $C$. Let $k$ be the number of vertices of odd degree in $G$.

Suppose $\pi(G) = \left\lfloor \frac{q}{p-1} \right\rfloor$. Since $q = p$, we have $\pi(G) = 2$.

If $m = 0$, then $G = C$ and $G$ is homeomorphic to $G_1$.

Suppose $m = 1$. Then by Theorem 1.23, we have $\pi(G) = \frac{k}{2} + 1 = 2$, so that $k = 2$. Thus $G$ has exactly two vertices of odd degree and hence $G$ is homeomorphic to $G_2$ or $G_3$. 

Suppose \( m > 1 \). Then by Theorem 1.23, \( \pi(G) = \frac{k}{2} = 2 \) so that \( k = 4 \). Hence \( m = 2 \). Let \( v \) and \( w \) be two vertices of degree greater than 2 on \( C \). If \( \text{deg } v \) and \( \text{deg } w \) are odd, then \( G \) is homeomorphic to \( G_4 \). If \( \text{deg } v \) and \( \text{deg } w \) are even, then \( G \) is homeomorphic to \( G_5 \). If \( \text{deg } v \) is odd and \( \text{deg } w \) is even, then \( G \) is homeomorphic to \( G_6 \).

The converse is obvious.

**Theorem 4.5.** Let \( G \) be a cubic graph. Then \( \pi(G) = \left\lfloor \frac{q}{p-1} \right\rfloor \) if and only \( G = K_4 \).

*Proof.* Suppose \( \pi(G) = \left\lfloor \frac{q}{p-1} \right\rfloor \). Since \( q = \frac{3p}{2} \), we have \( \pi(G) = 2 \). Also by Theorem 1.19, \( \pi(G) = \frac{p}{2} \). Hence \( p = 4 \) and consequently \( G = K_4 \). The converse is obvious.

**Remark 4.6.** Let \( G \) be a \((p,q)\)-graph with \( q \) even. Then it follows from Theorem 1.14 that \( G \) is decomposable into subgraphs each isomorphic to a path of length two and hence \( \pi(G) \leq \frac{q}{2} \).

In the following theorems we characterize trees and unicyclic graphs for which \( \pi = \frac{q}{2} \) when \( q \) is even.

**Theorem 4.7.** Let \( T \) be a tree with even size \( q \). Then \( \pi(T) = \frac{q}{2} \) if and only if \( T \) has exactly one vertex of even degree.
Proof. By Theorem 1.18, \( \pi(T) = \frac{k}{2} \), where \( k \) is the number of vertices of odd degree in \( T \). Hence \( \pi(T) = \frac{q}{2} \) if and only if \( k = q = p - 1 \), so that \( T \) has exactly one vertex of even degree.

**Theorem 4.8.** Let \( G \) be a unicyclic graph of even size \( q \) with cycle \( C \). Let \( m \) denote the number of vertices of degree greater than 2 on \( C \). Then \( \pi(G) = \frac{q}{2} \) if and only if the following hold.

(i) If \( m = 0 \), then \( G = C_4 \).

(ii) If \( m = 1 \), then \( G \) has exactly two vertices of even degree.

(iii) If \( m > 1 \), then \( G \) has no vertex of even degree.

Proof. Suppose \( m = 0 \). Then \( G = C \) and \( \pi(G) = 2 \). Hence \( \pi(G) = \frac{q}{2} \) if and only if \( q = 4 \), so that \( G = C_4 \).

Suppose \( m = 1 \). Then \( \pi(G) = \frac{k}{2} + 1 \), where \( k \) is the number of vertices of odd degree in \( G \). Hence \( \pi(G) = \frac{q}{2} \) if and only if \( k = p - 2 \) so that \( G \) has exactly two vertices of even degree.

Suppose \( m > 1 \). Then \( \pi(G) = \frac{k}{2} \). Hence \( \pi(G) = \frac{q}{2} \) if and only if \( k = p \) so that \( G \) has no vertex of even degree.

We now introduce the concept of simple path cover and simple path covering number of a graph.
Definition 4.9. A *simple path cover* of a graph $G$ is a path cover $\psi$ of $G$ such that any two paths in $\psi$ have at most one vertex in common.

Since for any graph $G$, the edge set $E(G)$ is a simple path cover, the collection $\mathcal{P}_s$ of all simple path covers of $G$ is non-empty. Let $\pi_s(G) = \min_{\psi \in \mathcal{P}_s} |\psi|$. Then $\pi_s(G)$ is called the *simple path covering number* of $G$ and any simple path cover $\psi$ of $G$ for which $|\psi| = \pi_s(G)$ is called a *minimum simple path cover* of $G$.

Example 4.10. Consider the graph $G$ given in Figure 4.3

![Figure 4.3](image)

Let $\psi_1 = \{(v_1, v_4, v_7, v_8), (v_2, v_4, v_3), (v_4, v_5), (v_5, v_6), (v_5, v_7)\}$ and $\psi_2 = \{(v_1, v_4, v_7, v_8), (v_3, v_4, v_5, v_6), (v_2, v_4), (v_7, v_5)\}$.

Then $\psi_1$ and $\psi_2$ are simple path covers of $G$. In fact $\psi_2$ is a minimum simple path cover of $G$ and $\pi_s(G) = 4$. 
Theorem 4.11. Every path in a simple path cover of a graph $G$ is an induced path.

Proof. Similar to that of Theorem 2.3.

Theorem 4.12. For any simple path cover $\psi$ of a graph $G$, let $t_\psi = \sum_{P \in \psi} t(P)$, where $t(P)$ denotes the number of internal vertices of $P$ and let $t = \max t_\psi$, where the maximum is taken over all simple path covers $\psi$ of $G$. Then $\pi_s(G) = q - t$.

Proof. Let $\psi$ be any simple path cover of $G$.

Then $q = \sum_{P \in \psi} |E(P)|$

$= \sum_{P \in \psi} (t(P) + 1)$

$= |\psi| + \sum_{P \in \psi} t(P)$

$= |\psi| + t_\psi$

Hence $|\psi| = q - t_\psi$ so that $\pi_s(G) = q - t$.

Corollary 4.13. For any graph $G$ with $k$ vertices of odd degree

$$\pi_s(G) = \frac{k}{2} + \sum_{v \in V(G)} \left\lfloor \frac{\deg v}{2} \right\rfloor - t.$$

Proof. Since $q = \frac{k}{2} + \sum_{v \in V(G)} \left\lfloor \frac{\deg v}{2} \right\rfloor$, the result follows.

Corollary 4.14. For any graph $G$, $\pi_s(G) \geq \frac{k}{2}$ where $k$ is the number of vertices of odd degree in $G$. Further, the following are equivalent.
(i) \( \pi_s(G) = \frac{k}{2} \).

(ii) There exists a simple path cover \( \psi \) of \( G \) such that every vertex \( v \) in \( G \) is an internal vertex of \( \left\lfloor \frac{\deg(v)}{2} \right\rfloor \) paths in \( \psi \).

(iii) There exists a simple path cover \( \psi \) of \( G \) such that every vertex of odd degree is an external vertex in exactly one path in \( \psi \) and no vertex of even degree is an external vertex in any path in \( \psi \).

Remark 4.15. For any \((p,q)\)-graph \( G \), \( \pi_s(G) \leq q \). Further, equality holds if and only if \( G \) is complete.

Remark 4.16. Since any path cover of a tree \( T \) is a simple path cover of \( T \), it follows from Theorem 1.18 that \( \pi_s(T) = \pi(T) = \frac{k}{2} \), where \( k \) is the number of vertices of odd degree in \( T \).

We now proceed to determine the value of \( \pi_s \) for unicyclic graphs and wheels.

Theorem 4.17. Let \( G \) be a unicyclic graph with cycle \( C \). Let \( m \) denote the number of vertices of degree greater than 2 on \( C \). Let \( k \) be the number of vertices of odd degree. Then
\[
\pi_s(G) = \begin{cases} 
3 & \text{if } m = 0 \\
\frac{k}{2} + 2 & \text{if } m = 1 \\
\frac{k}{2} + 1 & \text{if } m = 2 \\
\frac{k}{2} & \text{if } m \geq 3 
\end{cases}
\]

Proof. Let \( C = (v_1, v_2, \ldots, v_r, v_1) \).

Case 1. \( m = 0 \).

Then \( G = C \) so that \( \pi_s(G) = 3 \).

Case 2. \( m = 1 \).

Let \( v_1 \) be the unique vertex of degree greater than 2 on \( C \). Let \( G_1 \) be the tree rooted at \( v_1 \). Then \( G_1 \) has \( k \) vertices of odd degree and hence \( \pi_s(G_1) = \frac{k}{2} \). Let \( \psi_1 \) be a minimum simple path cover of \( G_1 \).

If \( \deg v_1 \) is odd, then \( \deg_{G_1} v_1 \) is odd. Let \( P \) be the path in \( \psi_1 \) having \( v_1 \) as a terminal vertex. Now, let

\[
P_1 = P \circ (v_1, v_2)
\]

\[
P_2 = (v_2, v_3, \ldots, v_r) \text{ and}
\]

\[
P_3 = (v_r, v_1).
\]

If \( \deg v_1 \) is even, then \( \deg_{G_1} v_1 \) is even. Let \( P = (x_1, x_2, \ldots, x_r, v_1, x_{r+1}, \ldots, x_s) \) be a path in \( \psi_1 \) having \( v_1 \) as an internal vertex.

Now, let
\[ P_1 = (x_1, x_2, \ldots, x_r, v_1, v_2) \]

\[ P_2 = (x_s, x_{s-1}, \ldots, x_{r+1}, v_1, v_r) \]

\[ P_3 = (v_2, v_3, \ldots, v_r). \]

Then \[ \psi = \{ \psi_1 \} \cup \{ P_1, P_2, P_3 \} \] is a simple path cover of \( G \) and hence \( \pi_s(G) \leq |\psi_1| + 2 = \frac{k}{2} + 2 \). Further, for any simple path cover \( \psi \) of \( G \), all the \( k \) vertices of odd degree and at least two vertices on \( C \) are terminal vertices of paths in \( \psi \). Hence \( t \leq q - \frac{k}{2} - 2 \), so that \( \pi_s(G) = q - t \geq \frac{k}{2} + 2 \).

Thus \( \pi_s(G) = \frac{k}{2} + 2 \).

Case 3. \( m = 2 \).

Let \( v_1 \) and \( v_i \), where \( 2 \leq i \leq r \), be the vertices of degree greater than 2 on \( C \). Let \( P \) and \( Q \) denote respectively the \( (v_1, v_i) \)-section and \( (v_i, v_1) \)-section of \( C \). Let \( v_j \) be an internal vertex of \( P \) (say). Let \( R_1 \) and \( R_2 \) be the \( (v_1, v_j) \)-section of \( P \) and \( (v_j, v_1) \)-section \( P \) respectively. Let \( G_1 \) be the graph obtained by deleting all the internal vertices of \( P \).

Subcase 3.1. Both \( \deg v_1 \) and \( \deg v_i \) are odd.

Then both \( \deg_{G_1} v_1 \) and \( \deg_{G_1} v_i \) are even. Hence \( G_1 \) is a tree with \( k - 2 \) odd vertices so that \( \pi_s(G_1) = \frac{k}{2} - 1 \). Let \( \psi_1 \) be a minimum simple path cover of \( G_1 \). Then \( \psi = \psi_1 \cup \{ R_1, R_2 \} \) is a simple path cover of \( G \) and \( |\psi| = \frac{k}{2} + 1 \). Hence \( \pi_s(G) \leq \frac{k}{2} + 1 \).
Subcase 3.2. Both $\text{deg } v_1$ and $\text{deg } v_i$ are even.

Then $\text{deg}_{G_1} v_1$ and $\text{deg}_{G_1} v_i$ are odd. Hence $G_1$ is a tree with $k + 2$ vertices of odd degree so that $\pi_s(G_1) = \frac{k}{2} + 1$. Let $\psi_1$ be a minimum simple path cover of $G_1$.

Suppose $v_1$ and $v_i$ are terminal vertices of two different paths in $\psi_1$, say $P_1$ and $P_2$ respectively. Now, let

$$Q_1 = P_1 \circ R_1$$
$$Q_2 = P_2 \circ R_2^{-1}$$
$$\psi = \{\psi_1 - \{P_1, P_2\}\} \cup \{Q_1, Q_2\}.$$

Suppose there exists a path $P_1$ in $\psi_1$ having both $v_1$ and $v_i$ as its end vertices. Then let $P_1 = Q$. Let $P_2$ be an $u_1-w_1$ path in $\psi_1$ having $v_1$ as an internal vertex and $P_3$ be an $u_2-w_2$ path in $\psi_1$ having $v_i$ as an internal vertex. Let $S_1$ and $S_2$ be the $(u_1, v_1)$-section of $P_2$ and $(w_1, v_1)$-section of $P_2$ respectively. Let $S_3$ and $S_4$ be the $(u_2, v_i)$-section of $P_3$ and $(w_2, v_i)$-section of $P_3$ respectively. Now, let

$$Q_1 = S_1 \circ P_1 \circ S_3^{-1}$$
$$Q_2 = S_2 \circ R_1$$
$$Q_3 = S_4 \circ R_2^{-1}$$
$$\psi = \{\psi_1 - \{P_1, P_2, P_3\}\} \cup \{Q_1, Q_2, Q_3\}.$$

Then $\psi$ is a simple path cover of $G$ and $|\psi| = |\psi_1| = \frac{k}{2} + 1$ and hence $\pi_s(G) \leq \frac{k}{2} + 1$. 
Subcase 3.3  \( \deg v_1 \) is odd and \( \deg v_i \) is even.

Then \( \deg_{G_1} v_i \) is even and \( \deg_{G_1} v_i \) is odd. Hence \( G_1 \) is a tree with \( k \) vertices of odd degree so that \( \pi_\ast(G_1) = \frac{k}{2} \). Let \( \psi_1 \) be a minimum simple path cover of \( G_1 \). Let \( P_1 \) be the path in \( \psi_1 \) having \( v_i \) as a terminal vertex.

If \( E(P_1) \cap E(Q) = \emptyset \), let

\[
Q_1 = P_1 \circ R_2^{-1} \\
Q_2 = R_1 \text{ and} \\
\psi = \{\psi_1 \setminus \{P_1\}\} \cup \{Q_1, Q_2\}.
\]

Suppose \( E(P_1) \cap E(Q) \neq \emptyset \). Since \( \deg_{G_1} v_i \geq 3 \), there exists an \( u_1-w_1 \) path in \( \psi_1 \), say \( P_2 \), having \( v_i \) as an internal vertex. Let \( S_1 \) and \( S_2 \) be the \((w_1, v_i)\)-section of \( P_2 \) and \((u_1, v_i)\)-section of \( P_2 \) respectively. Now, let

\[
Q_1 = P_1 \circ S_1^{-1} \\
Q_2 = S_2 \circ R_2^{-1} \\
Q_3 = R_1 \text{ and} \\
\psi = \{\psi_1 \setminus \{P_1, P_2\}\} \cup \{Q_1, Q_2, Q_3\}.
\]

Then \( \psi \) is a simple path cover of \( G \) and \(|\psi| = |\psi_1| + 1 = \frac{k}{2} + 1\).

Hence \( \pi_\ast(G) \leq \frac{k}{2} + 1 \).

Thus in either of the above subcases, we have \( \pi_\ast(G) \leq \frac{k}{2} + 1 \).

Also, for any simple path cover \( \psi \) of \( G \) all the \( k \) vertices of odd degree and at least one vertex on \( C \) are terminal vertices of paths...
in $\psi$. Hence $t \leq q - \frac{1}{2} - 1$, so that $\pi_s(G) = q - t \geq \frac{b}{2} + 1$.

Hence $\pi_s(G) = \frac{b}{2} + 1$.

**Case 4.** $m \geq 3$.

Let $v_{i_1}, v_{i_2}, \ldots, v_{i_s}$, where $1 \leq i_1 < i_2 < \cdots < i_s \leq r$ and

$s \geq 3$, be the vertices of degree greater than 2 on $C$. Let $\psi_{i_j},$

$1 \leq j \leq s$, be a minimum simple path cover of the tree rooted

at $v_{i_j}$. Consider the vertices $v_{i_1}, v_{i_2}$ and $v_{i_3}$. For each $j$, where

$1 \leq j \leq 3$, let $P_j$ be the path in $\psi_{i_j}$ in which $v_{i_j}$ is a terminal vertex

if $\text{deg } v_{i_j}$ is odd, otherwise let $P_j$ be an $u_j$-$w_j$ path in $\psi_{i_j}$ in which

$v_{i_j}$ is an internal vertex and $R_j$ and $S_j$ be the $(u_j, v_{i_j})$ and $(w_j, v_{i_j})$-

sections of $P_j$ respectively. Further, let $P = (v_{i_1}, v_{i_1+1}, \ldots, v_{i_s}),$

$Q = (v_{i_2}, v_{i_2+1}, \ldots, v_{i_3})$ and $R = (v_{i_3}, v_{i_3+1}, \ldots, v_{i_1})$.

If $\text{deg } v_{i_1}, \text{deg } v_{i_2}$ and $\text{deg } v_{i_3}$ are even, let $Q_1 = R_1 \circ P \circ R_2^{-1},$

$Q_2 = S_2 \circ Q \circ R_3^{-1}$ and $Q_3 = S_3 \circ R \circ S_1^{-1}$.

If $\text{deg } v_{i_1}, \text{deg } v_{i_2}$ and $\text{deg } v_{i_3}$ are odd, let $Q_1 = P_1 \circ P,$

$Q_2 = P_2 \circ Q$ and $Q_3 = P_3 \circ R.$

If $\text{deg } v_{i_1}, \text{deg } v_{i_2}$ are odd and $\text{deg } v_{i_3}$ is even, let $Q_1 = P_1 \circ$

$P \circ P_2^{-1}$, $Q_2 = R_3 \circ Q^{-1}$ and $Q_3 = S_3 \circ R.$

If $\text{deg } v_{i_1}, \text{deg } v_{i_2}$ are even and $\text{deg } v_{i_3}$ is odd, let $Q_1 = R_1 \circ$

$P \circ R_2^{-1}$, $Q_2 = S_2 \circ Q \circ P_3^{-1}$ and $Q_3 = R \circ S_1^{-1}.$

Then $\psi = (\bigcup_{j=1}^{s} \psi_{i_j} - \{P_1, P_2, P_3\}) \cup \{Q_1, Q_2, Q_3\}$ is a simple

path cover of $G$ such that every vertex of odd degree is an external
vertex of exactly one path in \( \psi \) and no vertex of even degree is an external vertex of any path in \( \psi \).

Hence \( \pi_s(G) = \frac{k}{2} \).

**Theorem 4.18.** For the wheel \( W_n = K_1 + C_{n-1} \), we have

\[
\pi_s(W_n) = \begin{cases} 
6 & \text{if } n = 4 \\
\left\lfloor \frac{n}{2} \right\rfloor + 3 & \text{if } n \geq 5
\end{cases}
\]

**Proof.** Let \( V(W_n) = \{v_0, v_1, \ldots, v_{n-1}\} \) and \( E(W_n) = \{v_0v_i : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-2\} \cup \{v_1v_{n-1}\} \).

If \( n = 4 \), then \( W_n = K_4 \) and hence \( \pi_s(W_n) = 6 \).

Now, suppose \( n \geq 5 \). Let \( r = \left\lfloor \frac{n}{2} \right\rfloor \)

If \( n \) is odd, let

\[
P_i = (v_i, v_0, v_{r+1}), \quad i = 1, 2, \ldots, r.
\]

\[
P_{r+1} = (v_1, v_2, \ldots, v_r),
\]

\[
P_{r+2} = (v_1, v_{2r}, v_{2r-1}, \ldots, v_{r+2}) \text{ and}
\]

\[
P_{r+3} = (v_r, v_{r+1}, v_{r+2}).
\]

If \( n \) is even, let

\[
P_i = (v_i, v_0, v_{r-1+i}), \quad i = 1, 2, \ldots, r - 1.
\]

\[
P_r = (v_0, v_{2r-1}),
\]

\[
P_{r+1} = (v_1, v_2, \ldots, v_{r-1}).
\]
\[ P_{r+2} = (v_1, v_{2r-1}, \ldots, v_r) \] and
\[ P_{r+3} = (v_{r-1}, v_r, v_{r+1}). \]

Then \( \psi = \{P_1, P_2, \ldots, P_{r+3}\} \) is a simple path cover of \( W_n \).

Hence \( \pi_s(W_n) \leq r + 3 = \left\lceil \frac{n}{2} \right\rceil + 3 \). Further, for any simple path cover \( \psi \) of \( W_n \) at least three vertices on \( C = (v_1, v_2, \ldots, v_{n-1}) \) are terminal vertices of paths in \( \psi \). Hence \( t \leq q - \frac{k}{2} - 3 \), so that
\[ \pi(G, W_n) = q - t \geq \frac{k}{2} + 3 = \left\lceil \frac{n}{2} \right\rceil + 3. \]

Thus \( \pi_s(W_n) = \left\lceil \frac{n}{2} \right\rceil + 3 \).

**Remark 4.19.** For the cycle \( C_4 \), \( \pi_s = 2 \) and \( \pi_s = 3 \). For the complete graph \( K_p \), \( \pi_s = \pi_s = q \). For the graph given in Figure 4.4, \( \pi_s = 4 \) and \( \pi_s = 3 \). Hence there is no relation between \( \pi_s \) and \( \pi_s \).

\[ \text{Figure 4.4} \]

**Remark 4.20.** Every simple acyclic graphoidal cover of a graph \( G \) is a simple path cover of \( G \) and every simple path cover of \( G \) is a path cover of \( G \). Hence it follows from Theorem 1.26 that
\[ \eta_{as} \geq \pi_s \geq \pi \geq \pi^*. \] These inequalities can be strict. For example, for the graph \( G \) given in Figure 4.5, \( \eta_{as} = 7, \pi_s = 6, \pi = 5 \) and \( \pi^* = 4. \)

**Figure 4.5**

**Theorem 4.21.** For any graph \( G \), \( \pi_s(G) \geq \lceil \frac{\Delta}{2} \rceil \). Further, the following are equivalent.

(i) \( \pi_s(G) = \lceil \frac{\Delta}{2} \rceil \).

(ii) \( \eta_{as} = \Delta - 1 \).

(iii) \( G \) is homeomorphic to a star.

**Proof.** Since \( \pi_s \geq \pi \), the inequality follows from Theorem 1.24.

Suppose \( \pi_s(G) = \lceil \frac{\Delta}{2} \rceil \).

Let \( \psi = \{P_1, P_2, \ldots, P_r\} \), where \( r = \lceil \frac{\Delta}{2} \rceil \) be a minimum simple path cover of \( G \). Let \( v \) be a vertex of \( G \) with \( \text{deg} \ v = \Delta \). Then \( v \) lies on each \( P_i \) and \( v \) is an internal vertex of all the paths.
in $\psi$ except possibly for at most one path. Hence $V(P_i) \cap V(P_j) = \{v\}$, for all $i \neq j$, so that $G$ is homeomorphic to a star.

Obviously, if $G$ is homeomorphic to a star, then $\pi_s(G) = \lceil \frac{n}{2} \rceil$.

Thus (i) and (iii) are equivalent. Equivalence of (ii) and (iii) follows from Theorem 3.16.

**Theorem 4.22.** For any graph $G$, $\pi_s(G) \geq \binom{\omega}{2}$, where $\omega$ is the clique number of $G$.

**Proof.** Similar to that of Theorem 3.17.

In the following theorem we characterize cubic graphs for which $\pi_s = \binom{\omega}{2}$.

**Theorem 4.23.** Let $G$ be a cubic graph. Then $\pi_s(G) = \binom{\omega}{2}$ if and only if $G = K_4$.

**Proof.** Let $G$ be a cubic graph with $\pi_s(G) = \binom{\omega}{2}$.

Clearly $\omega = 3$ or 4.

Suppose $\omega = 3$. Then it follows from Corollary 4.14 that $\pi_s(G) \geq \frac{\omega}{2}$, so that $p = 6$. Hence $G$ is isomorphic to the graph $H$ given in Figure 4.6.

![Figure 4.6](image-url)
Now, it can be shown that $\pi_s(H) = 6 \neq \binom{6}{2}$.

Thus $\omega = 4$ and consequently $G = K_4$.

If $\Delta \leq 3$, then every simple path cover of $G$ is a simple acyclic graphoidal cover of $G$ and hence $\eta_{as}(G) = \pi_s(G)$. However, the converse is not true. For the complete graph $K_p(p \geq 5)$, $\pi_s = \eta_{as}$ whereas $\Delta \geq 4$. We now prove that the converse is true for trees and unicyclic graphs.

**Theorem 4.24.** Let $G$ be a tree. Then $\eta_{as}(G) = \pi_s(G)$ if and only if $\Delta \leq 3$.

**Proof.** Let $G$ be a tree with $\eta_{as}(G) = \pi_s(G)$.

Suppose $\Delta \geq 4$. Let $v$ be a vertex of $G$ with $\text{deg} \: v \geq 4$.

Let $\psi$ be a minimum simple acyclic graphoidal cover of $G$. Let $P_1$ and $P_2$ be two paths in $\psi$ having $v$ as a terminal vertex. Let $Q = P_1 \circ P_2^{-1}$. Since $G$ is a tree, $Q$ is an induced path and hence $\psi_1 = (\psi \setminus \{P_1, P_2\}) \cup \{Q\}$ is a simple path cover of $G$ with $|\psi_1| = |\psi| - 1 = \eta_{as} - 1$ so that $\pi_s(G) \leq \eta_{as}(G) - 1$, which is a contradiction.

Hence $\Delta \leq 3$. 
Theorem 4.25. Let $G$ be a unicyclic graph. Then $\eta_{as}(G) = \pi_s(G)$ if and only if $\Delta \leq 3$.

Proof. Let $G$ be a unicyclic graph with $\eta_{as}(G) = \pi_s(G)$. Let $k$ denote the number of vertices of odd degree and $n$ be the number of pendant vertices of $G$.

It follows from Theorem 3.10 and Theorem 4.17 that $k = 2n$. Now, suppose $\Delta > 3$. Then

$$2q = \sum_{v \in V(G)} \deg v + \sum_{v \in V(G)} \deg v + \sum_{v \in V(G)} \deg v$$

$$> n + 3(k - n) + 2(p - k)$$

$$= 2p,$$

which is a contradiction.

Hence $\Delta \leq 3$.

Conclusion and Scope

In this chapter we have obtained some new bounds for the path covering number $\pi$ which was introduced by Harary [22]. We have also introduced the concept of simple path cover and simple path covering number $\pi_s$ of a graph $G$ and several results concerning this parameter have been presented. The following are some interesting problems for further investigation.

(i) Characterize graphs for which $\pi = \left\lfloor \frac{q}{p-1} \right\rfloor$.

(ii) Characterize graphs of even size $q$ for which $\pi = \frac{q}{2}$. 
(iii) Characterize graphs for which \( \pi_s = \frac{k}{2} \), where \( k \) is the number of vertices of odd degree.

(iv) Characterize graphs for which \( \pi_s = \left( \frac{\omega}{2} \right) \).

(v) Characterize graphs for which \( \pi^* = \pi = \pi_s \).

(vi) Characterize graphs for which \( \pi_s = \eta_{\text{as}} \).