CHAPTER - IV

MULTI-SERVER GENERAL BULK SERVICE QUEUE
WITH DIFFERENT SERVICE RATES AND PHASETYPE
INTER ARRIVAL TIME DISTRIBUTION

4.1 Introduction

General bulk service queueing models have been considered by several authors starting from Neuts [66]. Neuts and Nadarajan [73] have obtained various measures for general bulk service queues. Sim and Templeton [84]. Medhi [55] and Borthakur [10] have contributed much to the development of this literature.

The random environment model under exponential assumptions was first considered by Eisen and Tainiter [23]. Yechiali [92] and Purdue [76] also treated exponential queueing models of the above type. Neuts [70] has analysed M/M/C queue with random environment which included the above models as special cases.

The models dealt in this chapter come under the category of Markovian models with random environment. The first model is pH/M (a,b)/C. The second one is M/M(a,b)/1. Both are with different service rates.
4.2 Model I: pH/M(a, b)/C with different service rates

4.2.1 Description of the Model

In this model interarrival time distribution is of phase - type with m-phases. There are ‘C’ homogeneous servers. The service is according to general bulk service rule introduced by Neuts [66]. The service rate varies as \( \mu_2 \) only when the batch is at its maximum size ‘b’. In all other cases it is same and equal to \( \mu_1 \). This model is analysed using matrix – geometric method. Steady state probability vector of number of customers in the queue and stability condition are obtained.

This model can be fitted into real life situations. For example consider a tourist taxi, when it is full it will run without stopping in between. When it is not full, it will stop in between to take in customers if any waiting to join. Thus the service rates differ.

4.2.2 Mathematical Formulation

In this pH/M (a, b)/c model with different service rates, the probability distribution \( F(.) \) of interarrival times is of phase - type with representation \( (a, T) \) where \( T \) is an irreducible matrix of order \( m \). The state space for the queueing model under consideration is given by \( \{(i, j, K, I) : i \geq 0, 0 \leq j \leq C, j \geq K, 1 \leq I \leq m\} \).
where ‘i’ denotes the number of customers waiting in the queue, ‘j’
denotes the number of busy servers and ‘K’ denotes the number of
servers who are serving a batch of size ‘b’ customers, among the busy
servers, while the arrival process is in phase I. This model can be
described as a continuous parameter Markov chain on the above state –
space. The intensity matrix Q has the block partitioned form as presented
below.

\[ Q = \begin{align*}
0 & & 1 & & 2 & & \cdots & & \cdots & & a - 1 & & a & & a + 1 & & \cdots \\
0 & & B_1 & & B_0 & & & & & & & & & & & & \\
1 & & B_1 & & B_0 & & & & & & & & & & & & \\
2 & & B_1 & & B_0 & & & & & & & & & & & & \\
\end{align*} \]

(4.1)

\[ \mathcal{I} = \text{the set of states } \{(i, j, K, I) : i \geq 0, \ 0 \leq j \leq C, j \geq K, 1 \leq I \leq m\} \].
To describe the submatrices the notations used are explained in 1.4 in Chapter 1.

The sub-matrix $B_i$ is given by

$$B_i = \begin{bmatrix}
T & \mu I & T - \mu I \\
\mu I & T - \mu I & 2\mu I \\
\mu I & \mu I & T - 2\mu I \\
2\mu I & T - \mu I & T - 2\mu I \\
\mu I & \mu I & 3\mu I \\
\mu I & 2\mu I & T - 3\mu I \\
\end{bmatrix}$$

$B_i$ is a matrix of order $\frac{(C+1)(C+2)}{2}$ and $I$ is the identity matrix of order $m$.

$B_0 = [T_0 \alpha]$, a diagonal matrix of order $\frac{(C+1)(C+2)}{2}$.

$A_0$ is a diagonal matrix of order $(C+1)$ given by

$A_0 = [T_0 \alpha]$.

$A_1$ is given by
\[ A_1 = \begin{bmatrix} T - C \mu_1 I, & T - (C-1) \mu_1 I - \mu_2 I, & T - (C-2) \mu_1 I - 2 \mu_2 I, & \ldots & T - C \mu_2 I \end{bmatrix} \]

\[ C_0 = \begin{bmatrix} 0 \\ A_0 \end{bmatrix} \]
\[ \frac{(C+1) \times (C+2)}{2} \times (C+1) \]

\[ A_2 \] is given by

\[ A_2 = \begin{bmatrix} C \mu_1 I & (C-1) \mu_1 I & (C-2) \mu_1 I & \cdots & \mu_1 I & 0 \\ 0 & \mu_2 I & 2 \mu_2 I & \cdots & \cdots & C \mu_2 I \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \]
\[ (C+1) \times (C+1) \]

\[ C_1 = \begin{bmatrix} 0 & A_2 \end{bmatrix} \]
\[ \left( (C+1)(C+2)/2 \right) \times (C+1) \]

\[ B_2 \] is given by

\[ B_2 = \begin{bmatrix} T_0 \alpha & & & & \\ T_0 \alpha & T_0 \alpha & & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & T_0 \alpha & \\ & & & T_0 \alpha & T_0 \alpha \end{bmatrix} \]
\[ \frac{(C+1) \times (C+2)}{2} \times (C+1) \times (C+2) \]

All the unmarked entries are zeros in the above matrices.
4.2.3 Steady State probability vector and stability condition

Let $\mathbf{X}$ be the steady state probability vector associated with $\mathbf{Q}$ such that

$$\mathbf{XQ} = 0 \text{ and } \mathbf{Xe} = 1 \quad (4.2)$$

where $\mathbf{0}$ is a column null vector of infinite order and $\mathbf{e}$ a column vector of infinite order with all its components equal to one.

Let $\mathbf{X}$ be partitioned as

$$\mathbf{X} = (\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_a, \mathbf{X}_{a+1}, \ldots)$$

where $\mathbf{X}_i$ is a $1 \times ((C+1)(C+2)/2)$ vector for $0 \leq i \leq a-1$

and $\mathbf{X}_i$ is a $1 \times (C+1)$ vector for $i \geq a$.

Following Neuts [69] we examine the existence of a solution of a form

$$\mathbf{X}_i = \mathbf{X}_a R_i^{i-a}, \quad i \geq a \quad (4.3)$$

From (4.2), it follows that

$$A_0 + RA_1 + R^{b+1}A_2 = 0 \quad (4.4)$$

According to Wallace [89] the square matrix $\mathbf{R}$ is of order $(C+1)$ and is the minimal non-negative solution to the matrix non-linear equation (4.4). The spectral radius of $\mathbf{R}$ is less than one. Following
Latouche and Neuts [49], the matrix $R$ can be computed by successive substitutions in the recurrence relation.

$$R(0) = 0$$

$$R(n+1) = -A_0A_1^{-1} - R^{b+1}(n)A_2A_1^{-1}$$

and is the limit of the monotonically increasing sequence of matrices $\{R_n, n \geq 0\}$.

Consider the infinitesimal generator. $A = A_0 + A_1 + A_2$ of order $(C+1)$. $A$ is irreducible. Let $\pi$ be its associated vector of steady state probabilities such that

$$\pi A = 0 \text{ and } \pi e = 1 \quad (4.5)$$

Following Neuts [69], the system is stable iff

$$\pi A_0 e < \pi A_2 e \quad (4.6)$$

using (4.5), it follows from (4.6) that

$$\lambda < bC\mu_1 \quad (4.7)$$

where $\lambda$ is the mean of the interarrival time distribution.

(4.7) is the equilibrium condition. The chain $Q$ is positive recurrent iff (4.7) is satisfied.
Finally to find \((X_0, X_1, X_2, \ldots, X_a)\) we define \(Q^*\) by

\[
Q^* = \begin{bmatrix}
B_1 & B_0 \\
0 & B_1 & B_0 \\
& \ddots & \ddots & \ddots \\
B_2 & 0 & 0 & B_1 & C_0 \\
C_1 + \sum_{i=1}^{b-a} R^i C_1 & R^{b-a+1} C_1 & \cdots & R^{b-1} C_1 & R^b A_2 + A_1
\end{bmatrix}
\]

Lemma: 4.2.3.1

\(Q^*\) is an infinitesimal generator.

Proof:

To prove \(Q^* \epsilon = 0\) only the last row of \(Q^*\) is to be considered, since the remaining rows of \(Q^*\) are identical to the rows of \(Q\).

Now (Last row of \(Q^*\)) \(\epsilon\)

\[
= \left( C_1 + \sum_{i=1}^{b-1} R^i C_1 + A_1 + R^b A_2 \right) \epsilon
\]

\[
= (1-R)^{-1} \left( 1-R^b \right) C_1 \epsilon + A_1 \epsilon + R^b A_2 \epsilon
\]

\[
= (1-R)^{-1} \left( -A_0 - RA_1 - R^{b+1} A_2 \right) \epsilon
\]
since $C_{1} \varepsilon + A_{1} \varepsilon + A_{0} \varepsilon = 0$

$$= 0 \text{ using } (A_{0} + A_{1} + A_{2}) \varepsilon = 0$$

and (4.4)

Hence the lemma.

The matrix $Q^*$ is also irreductible.

Now let $X^*$ be a solution of $X^* Q^* = 0$ where

$$X^* = (X_0, X_1, \ldots, X_a)$$

Then we have from $X^* Q^* = 0$

$$X_0 B_1 + X_{a-1} B_2 + X_a (1-R)^{-1} (1-R^{b-a+1}) C_1 = 0$$

$$X_0 B_0 + X_1 B_1 + X_a R^{b-a+1} C_1 = 0$$

$$\ldots \ldots \ldots$$

$$X_{a-1} C_0 + X_a \left( A_1 + R^b A_2 \right) = 0$$

The vectors $X_i (0 \leq i \leq a-1)$ can be calculated in terms of $X_a$ using the above set of equations and $X_a$ may be normalized by using

$$\sum_{i=0}^{a-1} X_i \varepsilon + X_a (1-R)^{-1} \varepsilon = 1$$
4.3 Model II: M/M(a, b)/1 with different service rates

In this section a particular case of the model in 4.2 is analysed. It is assumed that the arrivals occur singly in accordance with a Poisson process with parameter \( \lambda \) and there is only a single server in the system. The service rate is \( \mu_2 \) only when the batch is of maximum size \( b \). In all the other cases the service rate is \( \mu_1 \). This model is analysed using differential difference equations method. The waiting time distribution in the queue, the expected waiting time \( W_Q \) in the queue, and the expected queue length \( L_Q \) are obtained. \( W_Q \) and \( L_Q \) are found to satisfy Little’s formula.

4.3.1 Mathematical Formulation

For this model the state space is \( \{(n,0):0 \leq n \leq a-1\} \cup \{(n,K):n \geq 0, K = 1, 2\} \). The state \( (n, 0) \) denotes that the service channel is idle and there are \( n(0 \leq n \leq a-1) \) units waiting in the queue and the state \( (n, 1) \) denotes that the service channel is busy and the server is serving a batch of size ‘S’ \( (a \leq S \leq b-1) \) customers. The state \( (n, 2) \) denotes that the server is busy and serving a batch of size ‘b’. In both \( (n,1) \) and \( (n, 2) \) states \( n \geq 0 \) customers are waiting in the queue.
Let \( P_{n,0}(t) = \text{pr} \{ \text{that at time } t \text{ the system is in} \}
\]
\[ \text{the state } (n,0) \} , \quad 0 \leq n \leq a-1 \]
\[ P_{n,1}(t) = \text{pr} \{ \text{that at time } t \text{ the system is in} \}
\]
\[ \text{the state } (n,1) \} , \quad n \geq 0 \]
and \( P_{n,2}(t) = \text{pr} \{ \text{that at time } t \text{ the system is in} \}
\]
\[ \text{the state } (n,2) \} , \quad n \geq 0 \]

Now assuming that the steady state distribution exists,

Let

\[
\lim_{t \to \infty} P_{n,0}(t) = P_{n,0} \]
\[
\lim_{t \to \infty} P_{n,1}(t) = P_{n,1} \]
\[
\lim_{t \to \infty} P_{n,2}(t) = P_{n,2} \]

The transient state probability equations are

\[
P_{n,2}^t = - \left( \lambda + \mu_2 \right) P_{n,2}(t) + \lambda P_{n-1,2}(t) + \mu_2 P_{n+b,2}(t)
\]
\[+ \mu_1 P_{n+b,1}(t), \quad n \geq 1 \]
\[
P_{0,2}^t = - \left( \lambda + \mu_2 \right) P_{0,2}(t) + \mu_1 P_{b,1}(t) + \mu_2 P_{b,2}(t) \]
Steady state balance equations are obtained from transient state probability equations using the result

\[ \lim_{t \to \infty} P_{n,K}^r(t) = P_{n,K}, \quad K = 0, 1, 2 \quad \text{and} \]

\[ \lim_{t \to \infty} P_{n,K}^r(t) = 0 \]

The steady state Balance equations

\[ (\lambda + \mu_2)P_{n,2} = \lambda P_{n-1,2} + \mu_2 P_{n+1,2} + \mu_1 P_{n+1,1} \quad (4.8) \]

\[ (\lambda + \mu_2)P_{0,2} = \mu_1 P_{0,1} + \mu_2 P_{0,2} \quad (4.9) \]

\[ (\lambda + \mu_1)P_{n,1} = \lambda P_{n-1,1} \quad (4.10) \]

\[ (\lambda + \mu_1)P_{0,1} = \sum_{s=a}^{b-1} P_{s,2} + \mu_1 \sum_{s=a}^{b-1} P_{s,1} + \lambda P_{a-1,0} \quad (4.11) \]
\begin{equation}
\lambda P_{n,0} = \mu_1 P_{n,1} + \mu_2 P_{n,2} + \lambda P_{n-1,0} \quad 1 \leq n \leq a-1 \tag{4.12}
\end{equation}

\begin{equation}
\lambda P_{0,0} = \mu_1 P_{0,1} + \mu_1 P_{0,1} \tag{4.13}
\end{equation}

Considering (4.10) and writing \( E(P_{n,1}) = P_{n+1,1} \) equation (4.10) can be written as

\[ h(E)[P_{n,1}] = 0, \quad n = 0, 1, 2, ... \] such that the characteristic equation is

\[ h(Z) = (\lambda + \mu_1)Z - \lambda \]

Denote the root of \( h(Z) = 0 \) by \( r_1 \) then

\[ r_1 = \frac{\lambda}{\lambda + \mu_1}, \quad 0 < r_1 < 1 \]

Also we have \( \frac{\lambda}{\mu_1} = \frac{r_1}{1-r_1} \) \( \tag{4.14} \)

And the solution of (4.10) is

\[ P_{n,1} = P_{0,1} r_1^n, \quad n \geq 0 \] \( \tag{4.15} \)

Next considering equation (4.8) and writing (4.8) as

\[ 1(E)[P_{n,2}] = -\mu_1 P_{n+b+1,1}, \quad n \geq 0 \]
such that the characteristic equation is

\[ 1(Z) = \mu_2 Z^{b+1} - (\lambda + \mu_2)Z + \lambda = 0 \]  \quad (4.16)

Now taking \( f(Z) = - (\lambda + \mu_2)Z \) and \( g(Z) = \mu_2 Z^{b+1} \) and applying Rouche’s theorem

\[ 1(Z) = f(Z) + g(Z) \] will have only one zero inside \(|Z| = 1\). Denote this root of \( 1(Z) \) as \( r_2 \), \( 0 < r_2 < 1 \) and the other roots by \( s_1, s_2, s_3, \ldots, s_b \).

From (4.16) we have

\[ \mu_2 r_2^{b+1} - (\lambda + \mu_2) r_2 + \lambda = 0 \quad \text{if} \quad |s_i| \geq 1, \forall i \]

i.e. \[ \frac{\lambda}{\mu_2} = \frac{r_2 (1 - r_2^b)}{(1 - r_2)} \]  \quad (4.17)

Now the solution of (4.8) is

\[ p_{n,2} = A_1 r_2^n - \frac{\mu_1 \nu_{n+b+1,1}}{1(E)}, \quad n \geq 0 \]
Evaluating and using the initial conditions we have

\[ P_{n,2} = \left[ P_{0,2} - \frac{\mu_1 P_{0,1} r_1^b}{(\mu_2 - \mu_1) - \mu_2 r_1^b} \right] r_2^n + \left[ \frac{\mu_1 P_{0,1} r_1^b}{(\mu_2 - \mu_1) - \mu_2 r_1^b} \right] r_1^n \]

\[ = M r_2^n + L r_1^n, \quad n \geq 0 \]  \hspace{1cm} (4.18)

where \( M = P_{0,2} - L \)

and \( L = \frac{\mu_1 P_{0,1} r_1^b}{(\mu_2 - \mu_1) - \mu_2 r_1^b} \)

From (4.9), (4.11) and (4.13) we have

\[ \lambda P_{a-1,0} = \lambda P_0 - \mu_1 \sum_{S=a}^b P_{S,1} - \mu_2 \sum_{S=a}^b P_{S,2} \]

where \( \lambda P_0 = \lambda P_{0,0} + \lambda P_{0,1} + \lambda P_{0,2} \)

Again putting \( n = 1, 2, \ldots a-1 \) in (4.12), we get recursively

\[ P_{m,0} = P_0 - \frac{\mu_1}{\lambda} \sum_{S=m+1}^b P_{S,1} - \frac{\mu_2}{\lambda} \sum_{S=m+1}^b P_{S,2} \]  \hspace{1cm} (4.19)
Now using (4.15) and (4.18) in (4.13) we have

\[ P_{m,0} = P_0 - \frac{L(\mu_2 - \mu_1)}{\lambda r_1 b} \left( \frac{r_1^{m+1} - r_1^{b+1}}{1 - r_1} \right) \]

\[ - \frac{M \mu_2}{\lambda} \frac{r_2^{m+1} - r_2^{b+1}}{1 - r_2} \]  \hspace{1cm} (4.20) \hspace{1cm} (m = 1, 2, 3, \ldots a-1)

Since total probabilities is equal to 1 we have,

\[ \sum_{m=0}^{a-1} P_{m,0} + \sum_{n=0}^{\infty} P_{n,1} + \sum_{n=0}^{\infty} P_{n,2} = 1 \]  \hspace{1cm} (4.21)

\[ P_{0,0}, P_{0,1}, \text{ and } P_{0,2} \text{ can be obtained from (4.9), (4.13), (4.20), (4.21), (4.15), and (4.18)} \]

Finally (4.15), (4.18) and (4.21) give the steady state probabilities of the system.

4.3.2 Distribution of the waiting time for the system M/M (a, b) / 1 with Different Rates of Service

Assume that the system is in steady state. Let the random variable \( \cdot T \cdot \) denote the waiting time in the queue for an arriving unit and \( \nu(t) \) be the probability density function of \( \cdot T \cdot \).
An arriving unit may find the system in any one of the following states. If there are just \((a-1)\) units waiting and if the server is idle then he need not wait. The corresponding probability is \(P_{a-1,0}\). The cases in which he has to wait are the following.

(i) \((Kb + m, 2)\), \(a-1 \leq m \leq b-1\)
(ii) \((Kb + m, 2)\), \(0 \leq m \leq a-2\)
(iii) \((m, 1)\), \(a-1 \leq m \leq b-1\)
(iv) \((Kb + m, 1)\), \(a-1 \leq m \leq b-1\)
(v) \((Kb + m, 1)\), \(0 \leq m \leq a-2\)
(vi) \((m, 1)\), \(0 \leq m \leq a-2\)
(vii) \((m, 0)\), \(0 \leq m \leq a-2\)

In case of (i) the arriving unit has to wait for \((K + 1)\) service completions at the rate of \(\mu_2\).

In case of (ii) the arriving unit has to wait till either the services of \((K + 1)\) batches are completed at the rate of \(\mu_2\) or \((a - 1 - m)\) units arrive, whichever occurs latter.

In case of (iii) the arriving unit has to wait for the service completion of one batch at the rate of \(\mu_1\).
In case of (iv) the arriving unit has to wait for the service completion of one batch at the rate of $\mu_1$ and $K$ batches at the rate of $\mu_2$.

In case of (v) the arriving unit has to wait till either the service completion of one batch at the rate of $\mu_1$ and $K$ batches at the rate of $\mu_2$ or $(a-1-m)$ units arrive whichever occurs latter.

In case of (vi) the arriving unit has to wait for $(a-1-m)$ units to arrive or for the service completion of one batch at the rate of $\mu_1$ whichever occurs latter.

In case of (vii) the arriving customer has to wait for the arrival of $(a-1-m)$ customers.

The probability density function $v(t)$ of $T$ is given by

$$v(t) = \sum_{K=0}^{\infty} \sum_{m=0}^{a-1} P_{Kb+m,2} f(\mu_2, K+1, t)$$

$$+ \sum_{K=0}^{\infty} \sum_{m=0}^{a-2} P_{Kb+m,2} f(\lambda, a-1-m, t) \Gamma_f(\mu_2, K+1)$$

$$+ \Gamma_f(\lambda, a-1-m, t) f(\mu_2, K+1, t)$$
where \( f(\alpha, K, t) \) is the probability density function of gamma distribution with parameters \( \alpha, K \).

i.e. \( f(\alpha, K, t) = \alpha^K t^{K-1} e^{-\alpha t} / (K-1)! \), \( t > 0, K = 1, 2, ... \)

and \( \Gamma_x(\alpha, K) \) is the incomplete gamma function

i.e. \( \Gamma_x(\alpha, K) = \int_0^x f(\alpha, K, t) \, dt \)

\[
= 1 - \sum_{S=0}^{K-1} \frac{e^{-\alpha x} (\alpha x)^S}{S!}
\]
is the probability density function of the random variable $Z$ given by

$$Z = \max \{ \text{gamma variate with parameters } \lambda, a-1-m, \text{ gamma variate with parameters } \mu_2, K + 1 \}$$

$$\Gamma_t(\lambda, a-1-m) \Gamma(\mu_2, \mu_1, K-1,t) + f(\lambda, a-1-m,t) \int_0^t f(\mu_2, \mu_1, K-1,t) dt$$

is the probability density function of the random variable $X$ given by

$$X = \max \{ \text{of the durations of } \{ a-1-m \text{ arrivals; one service completion at the rate } \mu_1 \text{ and } K \text{ service completion at } \mu_2 \} \}$$

The rate $\mu_2$}

and $f(p,q,r,t) = \int_0^t pq e^{-p(t-h)} e^{-qh} \left( p(t-h) \right)^r r! dh$

Now Denote $\sum_{K=0}^{m-1} Z^K / K! = e(m, Z)$

and $e^{-Z}(e(m,z)) = E(m,z), m \geq 1$

Then we have

$$\sum_{m=0}^{a-2} f(\lambda, a-1-m,t) = \lambda e^{-\lambda t} e(a-1, \lambda t)$$

and $\sum_{m=0}^{a-2} r^m f(\lambda, a-1-m,t) = \lambda e^{-\lambda t} r^{a-2} e(a-1, \frac{\lambda t}{r})$
using the above notations and results the right hand side of (4.22) is simplified as follows.

The first member on the right hand side of (r.h.s) (4.22) is

\[
\sum_{K=0}^{\infty} \sum_{m=a-1}^{b-1} P_{Kb+m,2} \left( f\left( \mu_2, K+1, t \right) \right)
\]

\[
= \sum_{K=0}^{\infty} \sum_{m=a-1}^{b-1} \left[ M r_2^{b-K} + L r_1^{Kb+m} \right] \left[ \mu_2 \left( \mu_2 t \right)^K e^{-\mu_2 t} / K! \right].
\]

\[
= M \mu_2 e^{-\mu_2 t} \left(1 - r_2^b \right) \frac{r_2^{a-1} - r_2^b}{(1-r_2)}
\]

\[
+ L \mu_2 e^{-\mu_2 t} \left(1 - r_1^b \right) \frac{r_1^{a-1} - r_1^b}{(1-r_1)}
\]

(4.23)

The second member on the r.h.s. of (4.22) is

\[
= \sum_{K=0}^{\infty} \sum_{m=0}^{a-2} P_{Kb+m,2} \left[ f\left( \lambda, a-1-m, t \right) \Gamma t \left( \mu_2, K+1 \right) \right]
\]

\[
+ \Gamma t \left( \lambda, a-1-m \right) f\left( \mu_2, K+1, t \right)]
\]

\[
= \sum_{K=0}^{\infty} \sum_{m=0}^{a-2} \left[ M r_2^{Kb+m} + L r_1^{Kb+m} \right] \left\{ \frac{\lambda^{a-1-m} t^{a-2-m} e^{-\lambda t}}{(a-2-m)!} \right\}
\]

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\[
\int_0^t \left( \mu_2(K + 1, t) dt \right) + \frac{\mu_2(\mu_2 t) e^{-\mu_2 t}}{K!} \left[ 1 - \sum_{s=0}^{a-2-m} \left( \frac{\lambda t}{s} \right)^s e^{-\lambda s t} \right]
\]

\[
= \left[ M \mu_2 e^{-\mu_2 r_2} \right] \left[ 1 - \frac{1 - e^{-\mu_2 (1-r_2^b)}}{1-r_2} \right]
\]

\[
+ \left[ L \mu_2 e^{-\mu_2 r_1} \right] \left[ 1 - \frac{1 - e^{-\mu_2 (1-r_1^b)}}{(1-r_1^b)(1-r_1)} \right]
\]

\[
+ \left[ M \mu_2 e^{-\mu_2 (1-r_2^b)} \right] \left[ \frac{1-r_2 a-1}{1-r_2} - \frac{E(a-1, \lambda t)}{1-r_2} \right]
\]

\[
\quad + \frac{r_2 a-1 e^{-\lambda t} e(a-1, \lambda t / r_2)}{1-r_2}
\]

\[
+ \left[ L \mu_2 e^{-\mu_2 (1-r_1^b)} \right] \left[ \frac{1-r_1 a-1}{1-r_1} - \frac{E(a-1, \lambda t)}{1-r_1} \right]
\]

\[
\quad + \frac{r_1 a-1 e^{-\lambda t} e(a-1, \lambda t / r_1)}{1-r_1}
\]

\[ (4.24) \]
The third member on the r.h.s. of (4.22) is

\[
= \sum_{m=a-1}^{b-1} P_{m,1}^m f(\mu_1, 1, t) = \sum_{m=a-1}^{b-1} P_{0,1}^m \mu_1 e^{\mu_1 t}
\]

\[
\cdot \left[ P_{0,1}^m e^{-\mu_1 t} \right] \frac{r_1^{a-1} - r_1^b}{1 - r_1}
\]

(4.25)

The fourth member on the r.h.s. of (4.22) is

\[
= \sum_{K=1}^{\infty} \sum_{m=a-1}^{b-1} P_{Kb+m,1}^m f(\mu_2, \mu_1, K-1, t)
\]

\[
= \sum_{K=1}^{\infty} \sum_{m=a-1}^{b-1} P_{0,1}^m P_{Kb+m,1}^m \left[ \mu_2 e^{\mu_2 t} - \mu_1 e^{\mu_1 t} \right] \frac{r_1^{a-1} - r_1^b}{1 - r_1}
\]

(4.26)

The fifth member on the r.h.s. of (4.22) is

\[
= \sum_{K=1}^{\infty} \sum_{m=0}^{a-2} P_{Kb+m,1} \left[ f(\mu_2, \mu_1, K-1, t) \Gamma_t(\lambda, a-1-m) + f(\lambda, a-1-m, t) \right]
\]

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\[
\sum_{K=1}^{\infty} \sum_{m=0}^{a-2} P_{01} P_{10}^{K} \left[ 1 - \frac{a-2-m}{S!} (\lambda t)^S e^{-\lambda t} \right] \times
\]

\[
\left[ \int_0^t \mu_2 \mu_1 e^{-\mu_1 h} - \mu_2 (t-h) \left( \frac{\mu_2 (t-h)}{(K-1)!} \right)^{K-1} dh \right]
\]

\[
+ \sum_{K=1}^{\infty} \sum_{m=0}^{a-2} P_{01} P_{10}^{K} \frac{\lambda (\lambda t)^{a-2-m} e^{-\lambda t}}{(a-2-m)!}
\]

\[
\int_0^t \int_0^{\mu_2 (t-h)} \mu \mu e^{-\mu_1 h} - \mu_2 (t-h) \left( \frac{\mu_2 (t-h)}{k-1} \right)^{k-1} dt dt
\]

\[
= \mu_1 \mu_2 P_{01} P_{10}^{b} \left[ \frac{1-r_1^{a-1}}{1-r_1} - \frac{E(a-1, \lambda t)}{1-r_1} + \frac{r_1^{a-1} e^{-\lambda t} e(a-1, \lambda t / r_1)}{1-r_1} \right]
\]

\[
\frac{e^{-\mu_2 (1-r_1^b)}}{(\mu_1 - \mu_2) + \mu_2 r_1^b}
\]

\[
+ \left[ r_1^b P_{01} \mu_2 \mu_2 \frac{r_1^{a-1}}{(1-r_1)} e\left( a-1, \frac{\lambda t}{r_1} \right) e^{-\lambda t} \right]
\]

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\[
\left[ \frac{1-e^{-[(1-r_i)\mu_2]} \frac{1-e^{-\mu_1 t}}{\mu_2(1-r_i)} \left( \frac{1}{\mu_2 r_i^b - (\mu_2 - \mu_1)} - \frac{1}{\mu_1 (\mu_2 r_i^b - (\mu_2 - \mu_1))} \right) }{1-e^{-[(1-r_i)\mu_2]} \frac{1-e^{-\mu_1 t}}{\mu_2(1-r_i)} \left( \frac{1}{\mu_2 r_i^b - (\mu_2 - \mu_1)} - \frac{1}{\mu_1 (\mu_2 r_i^b - (\mu_2 - \mu_1))} \right) } \right] (4.27)
\]

The sixth member on the r.h.s. of (4.22) is

\[
= \sum_{m=0}^{a-1} P_{m,1} \left[ \Gamma_t(\lambda, a-1-m) \Gamma_t(\mu_1, 1) + (\lambda, a-1-m, t) \Gamma_t(\mu_1, 1) \right]
\]

\[
= \sum_{m=0}^{a-1} P_{0,1} \mu_1^m \left\{ \left( 1 - \sum_{S=0}^{a-2-m} \frac{(\lambda t)^S}{S!} e^{-\lambda t} \right) \mu_1 e^{-\mu_1 t} \right. \\
+ \frac{\lambda (\lambda t)^{a-2-m}}{(a-2-m)!} e^{-\lambda t} \left[ 1 - e^{-\mu_1 t} \right] \right\}
\]

\[
= P_{0,1} \left[ \frac{1-r_i^{a-1}}{1-r_i} \left( \mu_1 e^{-\mu_1 t} \right) - P_{0,1} E(a-1, \lambda t) \mu_1 e^{-\mu_1 t} \right]
\]

\[
+ \frac{P_{0,1} \mu_1 r_i^{a-1}}{(1-r_i)} e^{-\lambda t} e^{\left( a-1, \frac{\lambda t}{r_i} \right)} \right\} (4.28)
\]

The seventh member on the r.h.s. of (4.22) is

\[
= \sum_{m=0}^{a-2} P_{m,0} f(\lambda, a-1-m, t)
\]
\[= \sum_{m=0}^{a-2} \left\{ P_0 - \frac{L(\mu_2 - \mu_1)}{1 - r_1} \left[ \frac{r_1^{m+1} - r_1^{b+1}}{1 - r_1} \right] \right\} \]

\[-\left( \frac{M \mu_2}{\lambda} \right) \left[ \frac{r_2^{m+1} - r_2^{b+1}}{1 - r_2} \right] \frac{\lambda (\lambda t)^{a-2-m} e^{-\lambda t}}{(a-2-m)!} \]

\[= \left[ \lambda P_0 + \frac{M \mu_2 - r_2^{b+1}}{(1 - r_2)} + \frac{L(\mu_2 - \mu_1) r_1}{(1 - r_1)} \right] E(a-1, \lambda t) \]

\[- \left( \frac{M \mu_2}{\lambda} \right) \left[ \frac{r_2^{a-1}}{(1 - r_2)} \right] e^{a-1, \lambda t r_2} \left( e^{\lambda t} \right) \]

\[- \frac{L(\mu_2 - \mu_1)}{r_1^{b}} \left[ \frac{r_1^{a-1}}{1 - r_1} \right] e^{a-1, \lambda t r_1} e^{-\lambda t} \]

(4.29)

Adding (4.23) to (4.29) and cancelling like terms of opposite signs. We have \( \nu(t) \)

\[\nu(t) = \left[ \lambda P_0 + \frac{M \mu_2 - r_2^{b+1}}{(1 - r_2)} + \frac{L(\mu_2 - \mu_1)}{r_1^{b}} \right] \frac{r_1}{(1 - r_1)} \]
\[
- M \mu_2 e^{-\mu_2 t} \frac{1-r_2 \lfloor 1-r_2 \lfloor b \rfloor \rfloor}{(1-r_2)} - e^{-\mu_1 t} \frac{L(\mu_2 - \mu_1)}{r_1 b} E(a-1, \lambda t)
\]

\[
+ M \mu_2 e^{-\mu_2 t} \frac{1-r_2 \lfloor 1-r_2 \lfloor b \rfloor \rfloor}{1-r_2}
\]

\[
+ \frac{1-r_1}{1-r_1} \left( \frac{L(\mu_2 - \mu_1)}{r_1 b} \right) e^{-\mu_1 t}
\]

Using the results (i), (ii) and (iii) given below \( \nu(t) \) \( dt \) can be evaluated.

**RESULTS**

(i) \( \int_0^\infty E(a-1, \lambda t) \, dt = \frac{(a-1)}{\lambda} \)

(ii) \( \int_0^\infty E(a-1, \lambda t) e^{-\mu_2 t} \frac{1-r_2 \lfloor 1-r_2 \lfloor b \rfloor \rfloor}{1-r_2} \; dt = \frac{r_2}{\lambda} \frac{1-r_2 a-1}{1-r_2} \)

(iii) \( \int_0^\infty E(a-1, \lambda t) \, e^{-\mu_1 t} \, dt = \frac{r_1}{\lambda} \frac{1-r_1 a-1}{1-r_1} \)

(iv) \( \sum_{m=0}^{a-1} P_{m,0} + \sum_{m=0}^\infty P_{m,1} + \sum_{m=0}^\infty P_{m,2} = 1 \)
Substituting $P_{m.0}$, $P_{m.1}$ and $P_{m.2}$ from (4.20), (4.15) and (4.18)

(iv) $= P_0 a + \frac{L(\mu_2 - \mu_1)}{\lambda r_1} \left( \frac{a r_1}{1 - r_1} \right) + \frac{M \mu_2 r_2^{b+1}}{1 - r_2}$

\[- \left[ \frac{L(\mu_2 - \mu_1 r_1 (1 - r_1^a)}{r_1^b (1 - r_1^2) \lambda} + \frac{M \mu_2 r_2 r_2^a}{(1 - r_2^2)^2 \lambda} \right] \]

\[+ \left[ \frac{P_{0.1}}{1 - r_1} + \frac{M}{1 - r_2} + \frac{L}{1 - r_1} \right] = 1 \]

Now $\int_0^\infty \nu(t) \, dt = P_{a-1,0}$

$= a P_0 + \frac{L(\mu_2 - \mu_1)}{\lambda (1 - r_1)} + \frac{M \mu_2^{a+1} r_2^{b+1}}{\lambda (1 - r_2)}$

\[- \frac{M \mu_2 r_2 (1 - r_1^a)}{(1 - r_2^2)^2 \lambda} \left[ \frac{L(\mu_2 - \mu_1 r_1 (1 - r_1^a)}{r_1^b (1 - r_1) r_1^b \lambda} \right] \]

\[+ \frac{P_{0.1}}{(1 - r_1)} + \frac{M}{(1 - r_2)} + \frac{L}{(1 - r_1)} \]

$= 1$ by result (iv).

Thus $\int_0^\infty \nu(t) \, dt = 1 - P_{a-1,0}$
4.3.3 The Expected waiting time in the queue

\[ W_Q = E(T) = \int_0^\infty t \, \nu(t) \, dt \]

\[ = \int_0^\infty E(a-1, \lambda t) \lambda \left[ P_0 + \frac{M \mu_2 r_2^{b+1}}{\lambda (1-r_2)} + \frac{\eta_1 L(\mu_2 - \mu_1)}{\lambda (1-r_1)} \right] \, dt \]

\[ + \int_0^\infty E(a-1, \lambda t) \lambda \left[ \frac{M \mu_2 e^{-\mu_2 t(1-r_2)}}{1-r_2} \right] \, dt \]

\[ - \frac{e^{-\mu_1 t} L(\mu_2 - \mu_1)}{(1-r_1) r_1} \, dt \]

\[ + \int_0^\infty M \frac{1-r_2^b}{1-r_2} \frac{-\mu_2 e^{-\mu_2 t(1-r_2)}}{1-r_1} \, dt \]

\[ + \int_0^\infty \frac{L(\mu_2 - \mu_1)}{r_1^b} \frac{1-r_1^b}{1-r_1} e^{-\mu_1 t} \, dt \]
Now using the results given below \( W_Q = E(T) \), is evaluated.

Results

(i) \( \int_0^\infty tE(a-1, \lambda t) \, dt = \frac{a(a-1)}{2\lambda^2} \)

(ii) \( \int_0^\infty tE(a-1, \lambda t) e^{-\mu_2 t(1-r_2 b)} \, dt = \frac{r_2^a (1-r_2 a) - ar_2 a+1 (1-r_2)}{\lambda^2 (1-r_2)^2} \)

(iii) \( \int_0^\infty tE(a-1, \lambda t) e^{-\mu_1 t} \, dt = \frac{r_1^2 (1-r_1 a) - ar_1 a+1 (1-r_1)}{\lambda^2 (1-r_1)^2} \)

(iv) \( \int_0^\infty te^{-\mu_2 t(1-r_2 b)} \, dt = \frac{1}{\mu_2^2 (1-r_2 b)^2} \)

(v) \( \int_0^\infty te^{-\mu_1 t} \, dt = \frac{1}{\mu_1^2} \)

\( W_Q = E(T) \)

\[
= \left[ \lambda P_0 + M \mu_2 \left( \frac{r_2^{b+1}}{1-r_2} \right) + L(\mu_2 - \mu_1) \frac{r_1}{1-r_1} \right] \frac{a(a-1)}{2\lambda^2} \\
- \left[ \frac{M \mu_2}{1-r_2} \right] \left[ \frac{r_2^a (1-r_2 a) - ar_2 a+1 (1-r_2)}{\lambda^2 (1-r_2^2)} \right]
\]
4.3.3 Expected Number of Members in the Queue

\[ L_Q = a-1 \sum_{n=0}^{\infty} n P_{n,0} + \sum_{n=0}^{\infty} n P_{n,1} + \sum_{n=0}^{\infty} n P_{n,2} \]

\[ = a-1 \sum_{n=0}^{\infty} n \left\{ P_0 - \frac{M \mu_2}{\lambda} \left( \frac{r_2^{n+1} - r_2^{b+1}}{1-r_2} \right) \right\} \]

\[ - \frac{L \left( \mu_2 - \mu_1 \right)}{\lambda n_1^b} \left[ \frac{r_1^{n+1} - r_1^{b+1}}{1-r_1} \right] \}

\[ + \sum_{n=0}^{\infty} n \left[ M r_2^n + L r_1^n \right] + \sum_{n=0}^{\infty} n P_{0,1} r_1^n \]

\[ = \left[ P_0 + \frac{M \mu_2}{\lambda} \frac{r_2^{b+1}}{1-r_2} + \frac{L \left( \mu_2 - \mu_1 \right) r_1}{\lambda (1-r_1)} \right] \frac{a(a-1)}{2} \]
From (4.30 and (4.31)

\[ L_Q = \lambda \ W_Q \]

Thus little’s formula holds good for the model M/M(a,b)/1 with
different service rates.