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Research Notes and Memoranda
OF APPLIED GEOMETRY FOR
PREVENIENT NATURAL PHILOSOPHY

A Non Static Plane Symmetric Cosmological Model in Bimetric Theory of Gravitation

T.M. Karade, K.S. Adhav and S.D. Katore

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A NON STATIC PLANE SYMMETRIC COSMOLOGICAL MODEL IN BIMETRIC THEORY OF GRAVITATION

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Abstract

The Bera's non-static metric in Rosen's (1973) bimetric theory of gravitation is considered. The plane-symmetric non-static cosmological model has been derived in presence of fluid distribution obeying certain equation of state. Various kinematical parameters and some physical features of the model have been studied.

KEY WORDS: Rosen's Bimetric Theory, Plane Symmetry, Zel'dovich fluid.

1. INTRODUCTION

Rosen (1973) has modified the formalism of the general relativity theory by introducing into it, besides the metric tensor $g_{ij}$ associated with the line-element,

$$ds^2 = g_{ij} \, dx^i \, dx^j \quad ,$$

(1.1)

a second metric tensor corresponding to flat space-time described by the metric

$$d\sigma^2 = f_{ij} \, dx^i \, dx^j \quad ,$$

(1.2)

at each point of the space-time.

The first metric tensor $g_{ij}$ describes the geometry of a curved space-time and thereby the gravitational field. The second metric tensor $f_{ij}$ refers to the flat space-time, whose curvature tensor vanishes and describes the inertial forces associated with the acceleration of the frame of reference. One can regard $f_{ij}$ as giving the geometry that would exist if there were no matter. Using this formalism Rosen formulated a bimetric theory of gravitation which satisfies the principle of covariance and equivalence: the foundations of GR.

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(On Teacher fellowship under UGC, India)
The field equations of bimetric relativity derived from variational principle come out to be

\[ K^i_j = N^i_j - \frac{l}{2} \delta^i_j N = -8\pi k T^i_j, \]  

(1.3)

where

\[ N^i_j = \frac{1}{f} f^{ab} (g^{\alpha \beta} g_{\alpha j \beta}) \]

\[ N = N_i = \frac{\sqrt{f}}{f} \]

and

\[ g = \det(g_{\alpha \beta}) \]

\[ f = \det(f_{\alpha \beta}) \]

Here \( T^i_j \) is the usual energy-momentum tensor of the matter or other non-gravitational fields satisfying the conservation law

\[ T^i_j = 0 \]  

(1.4)

where (\( ; \)) denotes covariant differentiation with respect to \( g_{\alpha \beta} \) and a vertical bar (\( \lVert \)) stands for covariant differentiation with respect to \( f_{\alpha \beta} \). In case \( f_{\alpha \beta} \) is taken as a Lorentz metric: \( \text{diag}(-1, -1, -1, 1) \), \( f \)-differentiation reduces to the usual partial differentiation.

Though there finds considerable work, Rosen (1973, 1975), Yilmaz (1975), Liebscher (1975), Karade and Dhole (1980), Karade (1980, 1981), Reddy and Venkateshwarlu (1989), Reddy and Venkateswara Rao (1998), exhibiting several aspects of the bimetric theory of gravitation, we hold the view that the investigation is not yet complete and there is a scope of further work which may unravel some of the hidden secrets of the theory.

In this paper the plane-symmetric vacuum model and a non static plane-symmetric antistiff-fluid models are presented and studied. Some kinematical parameters and physical features are also discussed.

2. THE FIELD EQUATIONS

Consider the Bera's non-static metric

\[ ds^2 = e^{2h} (dt^2 - dr^2 - r^2 d\theta^2 - s^2 dz^2), \]  

(2.1)

where \( r, \theta, z \) are cylindrical polar co-ordinates and \( h \) and \( s \) are functions of time \( t \) only.

Bera has shown that the metric (2.1) is Riemannian (non-flat) and satisfies the field equations of gravitation for empty space.
The background flat space-time corresponding to metric (2.1) is

\[ ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - dz^2 \]  

(2.2)

Assuming that the space-time is filled with perfect fluid given by the energy-momentum tensor

\[ T^i_j = (\rho + p) v^i v_j - p \delta^i_j \]  

(2.3)

where \( v^i \) is the four-velocity vector of the fluid having \( p \) and \( \rho \) as proper pressure and energy-density of the fluid respectively.

In this case we find that

\[ T^i_i = 0 \quad \text{\quad} (i \neq j) \]

We use comoving co-ordinates so that

\[ v^1 = v^2 = v^3 = 0 \quad \text{and} \quad v^4 = e^{-h} \]  

(2.4)

and the equation (2.3) gives

\[ T^1_1 = T^2_2 = T^3_3 = -p \quad \text{and} \quad T^4_4 = \rho \]  

(2.5)

The field equations (1.3) for the metric (2.1) with the help of equation (2.5) become

\[ K^i_j = K^2_2 = \dot{h} + \left[ \frac{s \ddot{s} - \dot{s}^2}{2s^2} \right] = -8 \pi k \rho \] \quad \text{\quad} \text{\quad} \quad \text{\quad} \text{\quad} (2.6)

\[ K^3_3 = \ddot{h} - \left[ \frac{s \ddot{s} - \dot{s}^2}{2s^2} \right] = -8 \pi k \rho \] \quad \text{\quad} \text{\quad} \quad \text{\quad} \text{\quad} \quad \text{\quad} (2.7)

\[ K^4_4 = \dot{h} + \left[ \frac{s \ddot{s} - \dot{s}^2}{2s^2} \right] = 8 \pi k \rho \] \quad \text{\quad} \text{\quad} \text{\quad} \text{\quad} \quad \text{\quad} \text{\quad} \quad \text{\quad} \text{\quad} \quad \text{\quad} \text{\quad} \text{\quad} \quad \text{\quad} \text{\quad} \text{\quad} \text{\quad} \quad \text{\quad} \text{\quad} \text{\quad} (2.8)

where dot (\( . \)) denotes ordinary differentiation with respect to time \( t \). Equations (2.6) and (2.8) yield

\[ p + \rho = 0 \] \quad \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} \text{\quad} (2.9)

which is an equation of state.

In view of equation (2.9), the field equations (2.6) - (2.8) reduce to (2.6) and (2.7) only.

The conservation equation (1.4) and the equation (2.9) gives

\[ p = \text{constant}, \quad \rho = \text{constant}. \]

Hence the fluid distribution in the model is homogeneous.
3. SOLUTIONS OF THE FIELD EQUATIONS

I. Vacuum Model.

In view of reality conditions \( \rho > 0 \), \( p > 0 \), equation (2.9) implies that

\[ \rho = 0, \quad p = 0. \]

When \( p = 0 = \rho \) (vacuum), Equations (2.6) and (2.7) give the solution

\[ e^{2t} = k_1 e^{2 k_1 t}, \quad s = k_2 e^{k_2 t}, \]

where \( k_1, k_2, k_3 \) and \( k_4 \) are constants of integration.

The corresponding vacuum model become

\[ ds^2 = k_1 e^{2 k_1 t} [dt^2 - dr^2 - r^2 d\theta^2 - k_2 e^{2 k_2 t} dz^2]. \]

This can be transformed through a proper choice of coordinates and suitable constants to

\[ ds^2 = e^{2T} [dT^2 - dr^2 - r^2 d\theta^2 - e^{2 k_2 t} dz^2]. \]  \( (3.1) \)

It is interesting to note that, the model is free from singularity. At \( T = 0 \), the model reduces to flat one.

II. Antistiff - Fluid Model

For the fluid distribution obeying (2.9), the field equations (2.6) and (2.7) give

\[ s = A_1 e^{At} \]

and hence

\[ h = c_0^2 e^{4t + At} \quad \text{and} \quad c_0^2 = 8 \pi \rho A_1 = \text{positive constant}. \]

To solve it, we put

\[ e^{4t + At} = L \]

and obtain

\[ L \frac{d^2 L}{dt^2} - \left( \frac{dL}{dt} \right)^2 = 4 c_0^2 L \]  \( (3.2) \)

On substituting \( \frac{dL}{dt} = \sqrt{M} \), equation (3.2) becomes

\[ \frac{dM}{dL} + \left( \frac{c_0^2}{L} \right) M = 8 c_0^2 L^2 \]

or

\[ M = 8 \ c_0^2 L^3 + A^2 L^2 \]

or

\[ \frac{dL}{L \sqrt{L + \left( \frac{A}{\sqrt{8} c_0} \right)^2}} = \sqrt{\frac{c_0}{3}} dt. \]
or 

\[ h = \frac{1}{2} \left[ \log D - \log \left( \frac{A^B}{c_0} e^{A^1} \right) \right], \]

where 

\[ D = \left[ \frac{A}{\sqrt{2} c_0 e^{2c_0}} \right] \] = constant and \( A, B \) are constants of integration.

The metric of the corresponding solution can be written in the form

\[ ds^2 = \left[ \frac{D}{e^{c_0} - e^{A^1}} \right] \left( dt^2 - dr^2 - r^2 d\theta^2 - A_1 \right) e^{2A^1} dz^2 \] (3.3)

This model can be transformed to

\[ ds^2 = \left[ T_0 - T \right]^{-1} \left\{ \frac{a}{T} dT^2 - dr^2 - r^2 d\theta^2 - T^3 dz^2 \right\}, \] (3.4)

where \( T_0 \) and \( a \) are arbitrary constants of integration.

This is non-static homogeneous model. It has singularity at \( T = T_0 \), i.e., at \( t = (B/c_0) \) and the beginning of the universe corresponds to \( T = 0 \). The model has no singularity at \( T = 0 \), since \( T = e^{A^1} \). The fluid distribution in the model is given by an equation of state \( p + \rho = 0 \), which can be written as \( p = (-\rho) \), which resembles to that of stiff-fluid with negative energy - density. We term this fluid 'as anti-F- fluid'.

The spatial volume, the scalar expansion, the shear scalar and the deceleration parameters for the model (3.4) have the following expressions,

Spatial volume \( \left( V^3 \right) = \left[ \frac{r^2 T^2}{(T - T_0)^3} \right] \),

Expansion scalar \( \left( \Theta \right) = \left[ \frac{T + 2T_0}{2 \sqrt{a(T_0 - T)}} \right] \),

Shear Scalar \( \left( \sigma^2 \right) = \frac{7}{72} \left[ \frac{(T + 2T_0)^2}{a(T_0 - T)} \right] \),

Deceleration parameter \( (q) \) is given by [Feinstein et al (1995)]

\[ q = - 3 \Theta^{-2} \left[ \Theta_0 U^u + \frac{1}{3} \Theta^2 \right], \]

which in this case has the expression

\[ q = - \left[ \frac{3T(4T_0 - T)}{(T + 2T_0)^2} + 1 \right]. \]
For the model (3.4), the spatial volume tends to zero, as \( T \to \infty \) and hence the model essentially gives an empty space for large value of \( T \). The expansion scalar and the shear scalar are functions of time \( T \) only and becomes infinite for large value of \( T \). The deceleration parameter will act as an indicator of the existence of inflation. If \( q > 0 \), the model decelerates in the standard way, while \( q < 0 \) indicates inflation. For the model (3.4) the deceleration parameter \( q \) depends on \( T \) only.

4. SOME PHYSICAL FEATURES

For the nonstatic homogeneous model (3.4) the flow vector \( v^i \) of the distribution is given by

\[
v_1 = v_2 = v_3 = v_4 = v^2 = v^3 = 0,
\]

\[
v_a = \frac{1}{T} \left( \frac{T_0 - T}{a} \right)^{1/2} \quad \text{and} \quad v^4 = T \left( \frac{T_0 - T}{a} \right)^{1/2}.
\]

(4.1)

The flow vector \( v^i \) satisfies the equations of the geodesics \( v^i v_i = 0 \). Hence the lines of flow are geodesics.

The motion of a test particle in the model is governed by the geodesics:

\[
r'' - r \Theta^2 + \left( \frac{r' T}{T_0 - T} \right) = 0,
\]

\[
\Theta'' + \frac{2 r' \Theta'}{r} + \frac{\Theta' T'}{T_0 - T} = 0
\]

\[
z'' + \left[ \frac{2}{T} + \frac{1}{T_0 - T} \right] z' T' = 0,
\]

\[
T'' + \frac{1}{T_0 - T} \left[ \frac{T^2}{2a} \left( r^2 + r \Theta^2 + T^2 z^2 \right) + \frac{T^2}{2} \right] + \frac{T^2 z^2}{a} - \frac{T^2}{T} = 0,
\]

(4.2)

where \( T' = \frac{dT}{ds} \) etc.

If a particle is initially at rest, that is, if

\[
r' = \Theta' = z' = 0,
\]
we get

\[ T' = \frac{dT}{ds} = \frac{T}{\sqrt{a}} (T_0 - T)^{1/2} \]  \hspace{1cm} (4.3)

These equations conclude that, for all such particles, the component of spatial acceleration would vanish namely,

\[ r'' = \theta'' = z'' = 0 \]

and the particle would remain permanently at rest.

The track of a light pulse in the model \(3.4\) is obtained by setting \(ds^2 = 0\), that is,

\[ \frac{T^2}{a} \left( \frac{dr}{dT} \right)^2 + \frac{T^2 a^2}{r^2} \left( \frac{d\theta}{dT} \right)^2 + \frac{T^2 a^2}{z^2} \left( \frac{dz}{dT} \right)^2 = 1 \]  \hspace{1cm} (4.4)

and for the case when the velocity is along the \(z\)-axis equation \(4.4\) gives

\[ \frac{dz}{dT} = \pm \frac{\sqrt{a}}{T^2} = \pm \phi (T). \]  \hspace{1cm} (4.5)

Therefore, the light pulse leaving a particle at \((0, \theta, z)\) at time \(T_1\) would arrive at the origin at a later time \(T_2\) given by

\[ \int_{T_1}^{T_2} \phi (T) dT = \frac{z}{\phi}. \]

Following the method outlined by Tolman (1962), the redshift in the model \(3.4\) is given by

\[ \frac{\lambda + \delta \lambda}{\lambda} = \frac{T_2^2}{T \sqrt{T_0 - T}} \left[ \frac{\sqrt{a} T_1^2 - U}{a T^3 (T_0 - T)^2 - U^2} \right] \]  \hspace{1cm} (4.6)

where \(U\) is the velocity of the source at the time of emission and \(U_z\) denotes the \(z\)-component of the velocity.

5. CONCLUSION

For the Bera's nonstatic model \((2.1)\), Reddy and Imaiah (1985) derived a non-static Zeldovich fluid distribution in GR. Our investigation shows that, the only possible matter field agreeable to BR is an 'antistiff' fluid 'distribution. Hence, in this theory, the plane-symmetric homogeneous model having time singularity is obtained and studied.
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Plane symmetric space-times in bimetric relativity theory

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It is established that some of the plane symmetric homogeneous models in Rosen’s metric relativity are singular.

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\$; words: bimetric relativity theory, plane symmetry$

1 Introduction

The bimetric theory of gravitation due to Rosen [1–3] defines at each point of the $ce$-time two metric tensors — the Riemannian metric $g_{ij}$ and the background flat $ce$-time metric $f_{ij}$. The tensor $g_{ij}$ describes the geometry of a curved space-time and $f_{ij}$ refers to inertial forces.

The field equations of the theory are

\[ N_{ij} - \frac{1}{2} g_{ij} N = -8\pi k T_{ij}, \]  \tag{1.1}

where

\[ N^i_j = \frac{1}{2} f^{ab} (g^{hi} g^{kj} f_{ij})_{ab}, \quad N = N^i_i, \quad k = \sqrt{\frac{g}{f}}, \]

\[ g = \det(g_{ij}), \quad f = \det(f_{ij}). \]

$T_{ij}$ is the usual energy-momentum tensor of the matter or other non-gravitational fields satisfying the conservation law

\[ T_{ij}^{;i} = 0, \]  \tag{1.2}

the semicolon (;) denotes covariant differentiation with respect to $g_{ij}$, and

the vertical bar (\]) stands for covariant differentiation with respect to $f_{ij}$. Though there exists considerable work, Rosen [1–3], Yilmaz [4], Liebscher [5], Karade and Dhoble [6], Karade [7–9], Reddy and Venkateswarlu [10], Reddy and Venkateswar Rao [11], exhibiting several aspects of the bimetric theory of gravitation, we hold the view that the investigation is not yet complete and there is a scope of further work which may unravel some of the hidden secrets of the theory. Therefore we have taken up the study of bimetric relativity and demonstrated that the bimetric relativity is not completely free from singularity as regard the plane-symmetric space-times filled with perfect fluid and this is the content of Sections 2 and 3.

2 The field equations

Consider the plane-symmetric line element

$$ds^2 = -e^{2\alpha}dz^2 - e^{2\beta}(dy^2 + dz^2) + e^{2\gamma}dt^2,$$

where $\alpha$, $\beta$, and $\gamma$ are functions of $x$ and $t$.

The background flat space-time corresponding to (2.1) is

$$ds^2 = -dz^2 - dy^2 - dz^2 + dt^2.$$ 

Assume that the space-time is filled with perfect fluid given by the matter tensor

$$T_{ij} = (p + \rho)U_iU_j - pg_{ij},$$

such that $g_{ij}U^iU^j = 1$, where $U^i$ is the four velocity vector of the fluid with $p$ as the pressure and $\rho$ as the energy density of the fluid, respectively. Let

The field equations (1.1) become

$$(\alpha_{11} - \alpha_{44}) - 2(\beta_{11} - \beta_{44}) - (\gamma_{11} - \gamma_{44}) = -16\pi e^{\alpha+2\beta+\gamma},$$

$$(\alpha_{11} - \alpha_{44}) + (\gamma_{11} - \gamma_{44}) = 16\pi e^{\alpha+2\beta+\gamma},$$

$$(\alpha_{11} - \alpha_{44}) + 2(\beta_{11} - \beta_{44}) - (\gamma_{11} - \gamma_{44}) = -16\pi e^{\alpha+2\beta+\gamma}.$$  

The suffixes 1 and 4 represent partial differentiation with respect to $x$ and $t$ respectively. The conservation equation (1.2) becomes

$$p_1 + (p + \rho)\gamma_1 = 0$$

and

$$\rho_4 + (p + \rho)(\alpha_4 + 2\beta_4) = 0.$$  

The three equations (2.4)–(2.6) contain five unknowns $\alpha, \beta, \gamma, p$ and $\rho$ and hence the system is indeterminate as it stands. We can introduce two more conditions: either by an ad hoc assumption corresponding to some physical situation or an arbitrary mathematical supposition. However, both the procedures have some drawbacks. The physical situation may lead to differential equations which will be difficult to integrate and mathematical supposition may lead to a non physical situation.
3 Solutions of the field equations

In this section we solve the field equations (2.4)-(2.8) corresponding to some cases of mathematical and physical significance.

3.1 Vacuum solutions

For vacuum $T_{ij} = 0$ and then the resulting field equations

$$\alpha_{11} - \alpha_{44} = \beta_{11} - \beta_{44} = \gamma_{11} - \gamma_{44} = 0$$

imply that the functions $\alpha$, $\beta$ and $\gamma$ are harmonic in $x - t$. Thus "the vacuum plane symmetric solutions of bimetric relativity can be expressed in terms of harmonic functions" and the result is in agreement with that of Karade [8].

3.2 Static homogeneous models

Due to homogeneity the pressure $p$ and density $\rho$ have the same value everywhere (Tolman [12]) and hence we treat them as constant. The conservation equation (2.7) results into three possibilities:

(i) $\gamma_1 = 0$,
(ii) $p + \rho = 0$,
(iii) $\gamma_1 = 6$ and $p + \rho = 0$.

Case (i)

Let $\gamma_1 = 0$. Then without loss of generality we take $\gamma = 0$.

The field equations (2.4)-(2.6) give

$$\begin{align*}
\alpha_{11} - 2\beta_{11} &= -16\pi pe^{\alpha+2\beta}, \\
\alpha_{11} &= 16\pi pe^{\alpha+2\beta}, \\
\alpha_{11} + 2\beta_{11} &= -16\pi pe^{\alpha+2\beta}.
\end{align*}$$

These field equations give

$$\begin{align*}
\alpha_{11} &= \beta_{11} = 16\pi pe^{\alpha+2\beta}, \\
3\alpha_{11} &= -16\pi pe^{\alpha+2\beta},
\end{align*}$$

which in turn result in

$$3p + \rho = 0$$

and

$$\alpha_{11} = \beta_{11}, \quad \text{i.e.,} \quad \alpha = \beta + Az + A_0,$$

where $A$ and $A_0$ are arbitrary constants of integration.

Then from Eqs. (3.4) and (3.7) we write

$$\beta_{11} = C^2 e^{3\beta + Az}, \quad C^2 = 16\pi pe^{A_0} = \text{positive constant}.$$
To solve it, we put
\[ e^{3A + Ax} = u \]
and obtain
\[ u \frac{d^2 u}{dx^2} - \left( \frac{du}{dx} \right)^2 = 3C^2 u^3. \]
Substituting \( du/dx = \sqrt{u} \), the above equation becomes
\[ \frac{dw}{du} + \left( \frac{-2}{u} \right) w = 6C^2 u^2 \]
which has the solution
\[ w = 6C^2 u^3 + A^2 u^2 \]
or
\[ \left( \frac{du}{dx} \right)^2 = 6C^2 u^3 + A^2 u^2 \]
or
\[ \frac{du}{u\left(1 + \left( \frac{A}{\sqrt{6C}} \right) \right)} = \sqrt{6C} dx. \]
After integrating and simplifying the result, we obtain
\[ \beta = \frac{2}{3} \left[ \log B_0 - \log(e^{AB/C} - e^{Ax}) \right], \]
where \( B_0 = [(2A/\sqrt{6C})e^{AB/2C}] = \) constant and \( B \) is a constant of integration.

Then the line element (2.1) becomes
\[ ds^2 = - \left[ e^{AB/C} - e^{Ax} \right]^{4/3} (e^{2Ax} \alpha_1 dx^2 + dy^2 + dz^2) + \alpha_1^2, \]
where \( \alpha_1 \) is an arbitrary constant of integration.

After a proper choice of coordinates and constants the above line element can be written as
\[ ds^2 = -[X_0 - X]^{-4/3} \left\{ \frac{1}{X^2} dX^2 + dy^2 + dz^2 \right\} + d\tau^2, \quad (3.9) \]
where \( X_0 \) is an arbitrary constant of integration.

If we take the arbitrary constant of integration equal to zero in (3.7), we get \( \alpha = \beta \). Then the solution of (3.8) gives the space-time:
\[ ds^2 = -[X_0 - X]^{-4/3} \left\{ \frac{1}{X^2} dX^2 + dy^2 + dz^2 \right\} + d\tau^2. \quad (3.10) \]
Similarly the field equations (2.4) - (2.6) with the conditions (ii) and (iii) yield respectively
\[ ds^2 = -[X_1 - X]^{-1} \left\{ \frac{1}{X} dX^2 + dy^2 + dz^2 \right\} + d\tau^2 \quad (3.11) \]
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and

$$ds^2 = -e^{2X}[dX^2 + dY^2 + dZ^2] + dT^2,$$  \(3.12\)

where \(X_1\) is an arbitrary constant of integration. The detailed calculations of the models (3.11) and (3.12) are respectively given in case (I) and case (II) of Appendix A.

In the model (3.9), \(X = X_0\) forms the singular plane. To investigate the nature of the singularities we compute \(g\) and the scalar curvature \(N\) as follows:

$$g = -[X_0 - X]^{-4}, \quad N = -2[X_0 - X]^{-2}.$$  

Here \(N\) is analogous to the scalar curvature \(R = g^{ij}R_{ij}\) in general relativity. Since these scalars are singular at \(X = X_0\), we term the singularity a physical one and it cannot be removed by any nonsingular coordinate transformation. If one regards (3.9) as the solution in general relativity, then all the components of the curvature tensor \(R_{ijl}\) and \(R\) are also singular at \(X = X_0\). However, this cannot be utilized for resolving the nature of the singularity because we are in bimetric gravitation theory.

Advancing the same argument, we claim that the singular planes \(X = X_0\) and \(X = X_1\) are again the physical singularities for the models (3.10) and (3.11), respectively.

In (3.9), the equation of state for the fluid distribution \(3p + \rho = 0\) can be written as \(p = \frac{1}{3}(-\rho)\), which resembles the radiation with negative energy density. We term this ‘antiradiation’ or antiradiating fluid.

In the line element (3.10), it seems that the model has singularity at \(X = 0\), but it is not true, since \(X = e^{\frac{4}{x}}\). The model preserves the ‘antiradiation’ fluid distribution as cited above.

In the metric (3.11), the equation of state for the fluid distribution \(p + \rho = 0\) can be written as \(p = (-\rho)\) which resembles stiff fluid with negative energy density. We term this fluid ‘antistiff fluid’.

The line element (3.12) is isotropic and is free from singularity. At \(X = 0\), it reduces to the flat one. The fluid distribution in the model is an antistiff fluid as is evident from the equation of state \(p + \rho = 0\) itself.

3.3 Purely nonstatic homogeneous models

Let \(\alpha, \beta\) and \(\gamma\) be functions of time only. The conservation equations (2.7) and (2.8) reduce to

$$(p + \rho)(\alpha_4 + 2\beta_4) = 0,$$  \(3.13\)

which leads to three possibilities:

(a) \(p + \rho = 0\),

(b) \(\alpha_4 + 2\beta_4 = 0\),

(c) \(p + \rho = 0\) and \(\alpha_4 + 2\beta_4 = 0\).
The field equations (2.4)–(2.6) with the condition (a) result into the space-time (with proper choice of coordinates and constants) as

$$ds^2 = [T_0 - T]^{-1} \left\{ -TdX^2 - dY^2 - dZ^2 + \frac{1}{T}dT^2 \right\},$$

(3.14)

where $T_0$ is an arbitrary constant of integration. For the detailed calculations one may refer to case (I) of Appendix B.

Similarly the field equations (2.4)–(2.6) with the conditions (b) and (c) respectively yield

$$ds^2 = -e^{-4\alpha T}dX^2 - e^{2\alpha T}(dY^2 + dZ^2) + \csc^4[C_4T]dT^2$$

(3.15)

and

$$ds^2 = -e^{-4\alpha T}dX^2 - e^{2\alpha T}(dY^2 + dZ^2) + e^{B_7 T}dT^2.$$  

(3.16)

The model (3.14) is singular at $t = (K/K_0)$ [see model (B.5) in Appendix B] and the beginning of the universe corresponds to $K = 0$. The fluid distribution in the model is an antistiff fluid as is evident from the condition $p + \rho = 0$ itself. The model has no singularity at $T = 0$ since $T = e^{tT}$.

The model (3.15) has initial singularity at $T = 0$. It is filled with stiff fluid distribution. At $T = 0$, the model (3.16) becomes flat. The equation of state $p + \rho = 0$ gives an antistiff fluid distribution in the model.

### 3.4 Pseudo-static models

Assume that $\alpha, \beta$ are functions of time and $\gamma$ the function of $x$ alone. From the structure of the conservation equations, we consider two cases:

1. $p = \text{constant}, \rho$ be a function of $x$ and $t$ and
2. $\rho = \text{constant}, p$ be a function of $x$ and $t$.

The assumption (1) results into

$$ds^2 = -e^{[K_1 t + K_2]x} \{ K_4 e^{2b_1 x}dx^2 + K_5 (dy^2 + dz^2) \} + dt^2.$$  

(3.17)

The pressure being constant and

$$p + \rho \propto \left[ \frac{1}{e^{(3K_1/2)t + K_2 t}} \right],$$

the density in the model decreases as time increases and after sufficiently long time it gives $p + \rho = 0$. At $t = 0$, the density in the model becomes constant everywhere and the model reduces to the flat one.

The assumption (2) gives

$$ds^2 = -e^{-4[d_1 t + d_2]x}dx^2 - e^{2[d_1 t + d_2]}(dy^2 + dz^2) + e^{2[d_3(x/2) + d_4]x}dt^2.$$  

(3.18)

The detailed calculations of (3.17) and (3.18) are given in Appendix C. Here also

$$\rho + p \propto \left[ \frac{1}{e^{[d_3(x/2) + d_4]x}} \right].$$

The pressure decreases as $x$ increases and gives relation $p + \rho = 0$ as $x$ tends to infinity. At $x = 0$, the pressure in the model becomes constant everywhere and the model reduces to the flat one.

4 Conclusion

In the derivation of his bimetric gravitation theory, the aim of Professor Rosen was to get rid of some of the unsatisfactory features of the general relativity, in particular physical singularities. Our investigation reveals that the bimetric gravitation theory does admit singularities which are of physical nature. Thus the difficulty faced in gravitation theory of relativity as regards the physical singularities also exists in bimetric theory. Hence the basic aim in building the bimetric relativity theory was not fully fulfilled.

Appendix A

In the static homogeneous case the equations (2.4)–(2.6) become

$$\alpha_{11} - 2\beta_{11} = -16\pi p e^{\alpha + 2\beta + \gamma}, \quad (A.1)$$
$$\alpha_{11} + \gamma_{11} = 16\pi pe^{\alpha + 2\beta + \gamma}, \quad (A.2)$$
$$\alpha_{11} + 2\beta_{11} - \gamma_{11} = -16\pi pe^{\alpha + 2\beta + \gamma}. \quad (A.3)$$

Case (I)

Let $p + \rho = 0$. Then Eqs. (3.19)–(3.21) give

$$\alpha_{11} = \beta_{11} = \gamma_{11} \quad (A.4)$$

or

$$\alpha = \beta + \frac{1}{2} L x + L_0 \quad \text{and} \quad \gamma = \beta + \frac{1}{2} L x + L_1,$$

where $L$, $L_0$ and $L_1$ are constants of integration. In view of Eq. (A.4), we can minimize the number of constants satisfying suitable conditions.

Then Eq. (A.1) reduces to

$$\beta_{11} = N^2 e^{\alpha + \beta + L x}, \quad N^2 = 8\pi p e^{L_0 + L_1} = \text{positive constant.} \quad (A.5)$$

Equation (A.5) is of the type (3.8) and hence

$$\beta = \frac{1}{2} \log M_0 - \log \left( e^{L M / N} - e^{L x} \right),$$

where $M_0 = (2L/\sqrt{8\pi}) e^{LM/2N} = \text{constant}$ and $M$ is a constant of integration.

Hence (2.1) reduces to

$$ds^2 = M_0 e^{LM/2N} \left( -e^{L x} a_2 dx^2 - a_3 dz^2 + e^{L x} a_3 dz^2 \right),$$

where $a_2$ and $a_3$ are arbitrary constants of integration. This model after a proper choice of coordinates and constants becomes (3.11).
Case (II)

Let \( \gamma_1 = 0 \) and \( p + \rho = 0 \). Then the Eqs. (A.1)–(A.3) give

\[
\alpha_{11} = \beta_{11} = -8\pi(p + \rho)e^{a + 2\beta + \gamma},
\]

which after integration give

\[
\alpha = A_1 x + A_2, \quad \beta = A_3 x + A_4 \quad \text{and} \quad \gamma = \text{constant} = C_0,
\]

where \( A_1, A_2, A_3, \) and \( A_4 \) are arbitrary constants of integration.

Then the line element (2.1) takes the form

\[
ds^2 = -e^{2(A_1 x + A_2)} dx^2 - e^{2(A_3 x + A_4)}(dy^2 + dz^2) + A_5 dt^2,
\]

where \( A_5 = e^{2C_0} \) is an arbitrary constant. This metric can be transformed to the model (3.12).

Appendix B

In the case of purely nonstatic homogeneous models, Eqs. (2.4)–(2.6) reduce to

\[
-\alpha_{44} + 2\beta_{44} + \gamma_{44} = -16\pi\rho e^{a + 2\beta + \gamma}, \tag{B.1}
\]

\[
\alpha_{44} + \gamma_{44} = -16\pi\rho e^{a + 2\beta + \gamma}, \tag{B.2}
\]

\[
-\alpha_{44} - 2\beta_{44} + \gamma_{44} = -16\pi\rho e^{a + 2\beta + \gamma}. \tag{B.3}
\]

Case (I)

Let \( p + \rho = 0 \). Then the Equations (B.1)–(B.3) give

\[
\alpha_{44} = \beta_{44} = \gamma_{44}
\]

or

\[
\alpha = \beta + \frac{1}{2}Rt + R_0 \quad \text{and} \quad \gamma = \beta + \frac{1}{2}Rt + R_1.
\]

The equation (B.3) reduces to

\[
\beta_{44} = K_0^2 e^{4\beta + Rt}, \tag{B.4}
\]

where \( K_0^2 = 8\pi\rho e^{R_0 + R_1} \) is positive constant.

The Eq. (B.4) is of the type (3.8) and hence

\[
\beta = \frac{1}{2}[\log K_1 - \log(e^{R K_0} - e^{Rt})],
\]

where \( K_1(2K/\sqrt{8K_0})e^{RK/2K_0} = \) constant with \( R, K \) being constants of integration.

Then the line element (2.1) becomes

\[
ds^2 = \frac{K_1}{e^{RK/K_0} - e^{Rt}}(-e^{Rt}a_4 dx^2 - dz^2 + e^{Rt}a_5 dt^2), \tag{B.5}
\]

where \( a_4 \) and \( a_5 \) are arbitrary constants of integration. This line element can be transformed to the model (3.14).

Case (II)

Let \( \alpha_4 + 2 \beta_4 = 0 \) or \( \alpha = -2 \beta \), assuming the constant of integration to be zero. The field equations (B.1)-(B.3) yield

\[
p = \rho, \quad \alpha = -2(at + b) \quad \text{and} \quad \beta = at + b,
\]

where \( a \) and \( b \) are arbitrary constants of integration.

The equation (B.1) reduces to

\[
\gamma_{44} = C_1 e^\gamma,
\]

where \( C_1 = -16\pi \rho = \text{constant}, \) or

\[
\frac{d\gamma}{\sqrt{C_2^2 + 2C_1 e^\gamma}} = dt,
\]

which after integration gives

\[
\gamma = \frac{1}{2} \log \left[ \left( \frac{C_2}{2C_1} \right)^2 \csc^4 \left( \frac{\sqrt{-C_2^2}}{2} [t + C_3] \right) \right],
\]

where \( C_2 \) and \( C_3 \) are arbitrary constants of integration.

Then we get

\[
ds^2 = -e^{2(at+b)}dx^2 - e^{2(at+b)}(dy^2 + dz^2) + \left( \frac{C_2}{2C_1} \right)^2 \csc^4 \left( \frac{\sqrt{-C_2^2}}{2} (t + C_3) \right) dt^2.
\]

After making proper choice of coordinates and constants this metric can be transformed to the line element (3.15).

Appendix C

In pseudo-static case the field equations (2.4)-(2.6) reduce to

\[
-\alpha_{44} + 2\beta_{44} - \gamma_{11} = -16\pi \rho e^{\alpha + 2\beta + \gamma},
\]

\[
-\alpha_{44} + \gamma_{11} = 16\pi \rho e^{\alpha + 2\beta + \gamma},
\]

\[
-\alpha_{44} + 2\beta_{44} + \gamma_{11} = 16\pi \rho e^{\alpha + 2\beta + \gamma},
\]

For the assumption (1), Eqs. (2.7), (2.8) and (C.1)-(C.3) give

\[
p + \rho = f(x) e^{-(\alpha + 2\beta)}
\]

and

\[
\alpha_{44} = \beta_{44} = 8\pi (p + \rho) e^{\alpha + 2\beta},
\]

which imply \( f(x) = \text{constant} = m, \) say, and

\[
\alpha = \frac{1}{2}h_1 t^2 + h_2 b + h_3, \quad \gamma = \text{constant} = 0,
\]

\[
\beta = \frac{1}{2}h_4 t^2 + h_5 b + h_6.
\]
where $K_1 = 8\pi m = \text{constant}$ and $b_1$, $b_2$, $K_2$, $K_3$ are arbitrary constants of integration. Then the metric (2.1) reduces to the model (3.17).

While for assumption (2), Eqs. (2.7), (2.8) and (C.1)--(C.3) yield

$$p + \rho = g(t)e^{-\gamma},$$

(C.4)

hence

$$\alpha = -2(d_1 t + d_2), \quad \beta = d_1 t + d_2,$$

where $d_1$ and $d_2$ are arbitrary constants of integration.

Then Eqs. (C.2) and (C.3) reduce to

$$\gamma_{11} = 8\pi(p + \rho)e^\gamma.$$  \hspace{1cm} (C.5)

Equations (C.4) and (C.5) yield

$$g(t) = \text{constant} = \alpha_0$$

and hence

$$\gamma = \frac{1}{2}d_3 x^2 + d_4 x + d_6,$$

where $d_3 = 16\pi d_0 = \text{constant}$ and $d_4$ and $d_6$ are constants of integration.

Therefore, the line element (2.1) gives the space-time (3.18).

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