Chapter 6

G-Lets: Time-Frequency Domain Analysis

6.1 Introduction

The G-let time-amplitude domain analysis was shown to preserve all information of a signal in its G-let oscillations as amplitude gradients in Chapter 4. The structural symmetries were explored using a multiresolution of time-amplitudes. This chapter explains how the amplitude structural symmetries are shifted into the frequency domain in two different ways, one using a simple dilation transformation and another a differential operator. The instantaneous G-let frequencies obtained from the amplitude gradients of the signal, are used to build a parallel time-frequency analysis in the frequency domain. Every time-amplitude resolution in the time domain has an equivalent time-frequency resolution providing a multiresolution analysis of frequencies. The irreducible representations of a transformation group, enable G-let oscillations to provide space for strengthening the amplitude gradients. The G-let oscillations provide the space for a dilation operation to expose localized G-let frequencies. When a differential operator is used on the signal, G-let oscillations themselves expose the corresponding local G-let frequencies. This behavior gives G-let oscillations the power of tracking the transition of frequencies along a 1 – $D$ signal. For 2 – $D$ signals, G-let oscillations are dilated to get
frequencies in the same direction in which the oscillations were produced. G-let frequencies are presented in this chapter and the strength of the localization of the frequencies is tested by separating frequency subbands from a signal in Chapter 7 and by using them for feature extraction in Chapter 8.

Analyzing a signal in the frequency domain is easy for digital computing. Since different frequencies of a signal are its components, each frequency may be treated as a basis vector. Then the components may be generated by projections on to this basis. The challenge in signal processing has been to identify a basis closest to finding the natural set of a signal’s frequencies. It has been traditionally achieved by framing a basis that mainly represents a frequency as a phase angle (Clausen, 1989), a wave (Daubechies, 1988) or a kernel (Lenz, 1990). In all these methods the basis is not derived, it is framed and then tested in the signal. In another approach the basis is derived by assuming the signal to be in a tensor product space (Foote, 2000). The individual spaces that make this product are separated out and the signal is projected on them to obtain the components. Here again these bases cannot be taken as the frequency bases. The bases obtained thus represent a signal’s amplitudes. But in this work we show that it is possible to get frequencies from an amplitude resolution of the signal and the G-let transforms.

6.2 G-let Frequencies

The G-let frequency of a signal is defined to be:

**Definition 6.2.1.** G-let Frequency: The instantaneous frequency \( Freq(x) \) of a discrete signal \( S(x) \), is considered to be proportional to the gradient of the amplitudes at a point.

\[
Freq(x) \propto \frac{d(S(x))}{dx} \quad (6.2.1)
\]

**Definition 6.2.2** (Dilation). Changing the amplitude of G-let oscillations of a signal by a scalar value is called dilation.

Based on the above definitions, frequency analysis of G-lets may be performed in two ways:
• Using Dilation

• Without Dilation

It was shown in Chapter 4 that the oscillations introduced by G-lets in time domain analysis are proportional to the amplitude gradients in the neighborhood. A sharp rise or fall in amplitude among adjacent values or high amplitude difference is indicative of high frequency. Therefore this feature can be exploited to extract the frequencies. Within the limit of the highest rise or fall in the amplitudes, all other amplitude differences may be scaled proportionally using dilation. Interestingly this scaling itself exhibits the frequencies of the signal at their corresponding positions of occurrence. A high frequency portion of the signal has high amplitude and less bandwidth of the oscillations produced by the irreducible parts and a low frequency portion has the reverse i.e., low amplitude and larger bandwidth. Thus frequency filters are also generated in the same vector space \( V \). This is a simple time-frequency analysis using irreducible representations, and is called the G-let frequency analysis. The G-let frequency filters are also called as G-let time-frequency filters.

6.2.1 Using Dilation

The G-let oscillations generated by G-let transform contain two wave patterns which are superimposed on a signal in the time domain. These wave patterns expose the amplitude gradients of the signal. Since G-let frequencies are defined to be proportional to these amplitude gradients it is natural that the wave patterns are the ones that have to be used to capture signal frequencies. Since the G-let oscillations are a sum of both the odd and even wave patterns, a scaling or dilation operation on one of the wave patterns may be used to suppress smaller amplitude gradients. This will allow for the higher amplitude gradients to stand out in the oscillations. This is a very significant step in G-let analysis, because the G-let oscillations have been altered to show amplitude gradients proportionally. This new oscillations actually show the frequencies of the signal which are defined to be proportional to amplitude gradients. It is shown here that handling the amplitude gradients in this manner the G-let frequencies of a signal may be captured. Thus the G-let frequency is shown to be directly related to the dilated G-let oscillations which carry the information about amplitude gradients. As a result, a high frequency portion of a signal has sharp amplitude gradients which is seen as taller and thinner (i.e., less number
of oscillations) G-let oscillations. A low frequency signal has smoother amplitude gradients and hence is seen as shorter and wider (i.e., more number of oscillations) G-let oscillations. A signal with different frequencies shows different oscillations of this kind depending on the local frequency present in the signal. Interestingly, a powerful local frequency estimation seems to evolve out of G-let oscillations using a suitable dilation operation. Which wave pattern is chosen to perform the dilation operation is decided based on the G-let oscillations in the time domain. An immediate choice is to choose the wave pattern that contains the highest amplitude gradient since that is where the suppression of smaller gradients is convenient. All results of G-let analysis in this work are demonstrated using this choice for dilation.

The amount of scaling or dilation is not arbitrary. The amplitude gradients calculated in the time domain is evaluated to obtain the dilation parameter. Once the dilation operation is performed, frequencies show up. The dilated G-let transform takes the signal to the frequency domain. The dilation operation is performed on the G-let oscillations to get the G-let frequencies. Therefore the G-let frequency transform cannot be determined without the time domain G-let transform. Further, it can be seen that any kind of approximation is not involved in the generation of a transform both in the time and frequency domain. This is the advantage of doing the processing in two steps. It is a time-frequency analysis method as shown in Fig. 6.1. The first block in the Fig. 6.1 refers to G-let oscillations induced by G-let transform on a signal in the time-amplitude domain.

![Diagram of Frequency Analysis Using G-lets (with Dilation)](a)

Figure 6.1 Frequency Analysis Using G-lets (with Dilation)

It is to be noted that though the spectral components of the irreducible representations are the characters, the characters are not directly used for G-let spectral analysis. The irreducible
representations are directly used to show the spectral components. Also, for every G-let transform in the time domain, there is a corresponding dilation, and hence a frequency transform in the frequency domain. For each of the frequency transform the G-let frequencies are seen differently as if through a kaleidoscope. For special G-let transform, the frequencies are also particularly remarkable. The highest frequency stands out clearly for a transformation of the signal by the G-let transform. The special G-let frequency transform is obtained automatically for the transformation angles $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$. For such G-let transforms, the low frequencies of the signal are heavily suppressed and high frequencies clearly stand out. This is because in the time domain, the G-let oscillations for special G-let transform already highlight only the sharp amplitude gradients. They are fine tuned by the dilation operation.

Dilated G-let oscillations suppress low frequencies and therefore high frequencies project out in the signal. The smooth portions of the signal are perceived to be the low frequency components, whereas the sharp regions of the signal are taken as high frequency components. For every G-let frequency filter, a low and high frequency separation of the signal may be done. Some G-let frequency filters give the highest frequency of the signal with all other frequencies heavily suppressed. In the process of separating the low and high frequency components, the time information of the frequencies is not lost. This gives rise to a simple time-frequency analysis method. As a result, a multiresolution analysis based on frequencies is obtained for the signal using G-lets. The signal processing steps involved in the frequency analysis of G-lets is summarized in a block diagram below in Fig. 6.2.

![Figure 6.2 Frequency Analysis of G-lets](image)
**Theorem 6.2.1.** *(Frequency and Irreducible Representations)* - G-let instantaneous frequencies are produced by the irreducible representations of transformation groups.

*Proof.* Please refer to theorem 4.2.2. \(\square\)

**Theorem 6.2.2** *(G-let Frequency).* The dilated G-let coefficients are proportional to signal frequencies.

*Proof.* Please refer to theorem 4.2.2. \(\square\)

**Illustration:** From Fig. 6.3(a) and Fig. 6.3(c), it can be seen that a low and high frequency signal can be distinguished by the bandwidth, number of oscillations and height of oscillations or scales. Low frequencies have more bandwidth, small scales and more oscillations. High frequencies are short in time, high scale but have fewer oscillations.

![Figure 6.3 Frequency Analysis of a Sharp Signal: (a) G-let Frequencies; Frequency Analysis of a Smooth Signal: (a) Signal (b) G-let Frequencies](image)

The first, third, sixth and tenth G-let transforms for a sharp (high frequency) and smooth (low frequency) signal are shown in the Fig. 6.3. In the sharp signal, the frequencies indicate that there are two curves, a sharp positive portion and a short negative portion. The highest frequency is at the descending end of the first portion. In the smooth signal, G-let frequencies
have high (absolute) values at both ends only with the highest frequency at the beginning of the signal. The frequency resolution of the signal ‘S’ is shown in the second column of Fig. 6.15 and Fig. 6.16 as G-lets in frequency domain.

**Theorem 6.2.3. (Time-Frequency Localization)** - G-let frequencies are time localized in the frequency domain. The frequencies obtained are instantaneous frequencies.

**Illustration:** The angle of transformation $\theta$ determines the oscillations $O(x)$ created by irreducible representations. Since the angle contains the frequency information of a signal $S(x)$, the effect is proportional to frequencies. If we take the angle itself, we lose the time information for a frequency. Therefore the angle is applied to the signal using G-let transform and its effect indicates the place and kind of frequency in a signal. The frequency ($Freq$) of the signal and the oscillations produced by irreducible representations in the G-let coefficients $G(x)$ are related as below. Since a G-let frequency is defined to be proportional to the amplitude gradients of the discrete signal $S(x)$,

$$Freq(x) \propto \frac{d(S(x))}{dx}$$

$$Freq(x) \propto \theta$$

The height and count of oscillations $O_{\text{height}}(x)$, $O_{\text{count}}(x)$ respectively are determined by $\theta$ and amplitude gradients $\frac{d(S(x))}{dx}$,

$$\theta \propto (O_{\text{height}}(x)) \tag{6.2.2}$$

$$\frac{d(S(x))}{dx} \propto O_{\text{count}}(x) \tag{6.2.3}$$

$$O(x) = f(O_{\text{height}}(x), O_{\text{count}}(x)) \tag{6.2.4}$$

Thus,

$$G(x) = f(\theta, \frac{d(S(x))}{dx}) \tag{6.2.5}$$

Therefore,

$$Freq(x) \propto G(x) \tag{6.2.6}$$
The basic $1 - D$ signals illustrated in Chapter 4 for G-let oscillations, G-let frequencies are shown in Fig. 6.4. It may be seen that one of the wave patterns is suppressed to produce the G-let frequencies of each signal.

The G-let frequencies of each of the signals in Fig. 6.4 have been obtained from special G-let frequency transform, where the highest frequency of each signal is highlighted. For the sine signal, the highest frequency is in the middle. For the square signal, it is at the two points where the square begins and ends. For the sawtooth signal it is in the end of the signal. For the dirichlet signal in Fig. 6.4(m), the highest frequency is available at the position 21 where the highest amplitude gradient appears in the signal.
Figure 6.4 G-let Frequencies of Basic 1-D Signals: (a) Sine  (b) G-let Oscillations  (c) Dilation  (d) G-let Frequencies (a) Sine (b) G-let Oscillations (c) Dilation (d) G-let Frequencies (e) Square (f) G-let Oscillations (g) Dilation (h) G-let Frequencies (i) Sawtooth  (j) G-let Oscillations  (k) Dilation  (l) G-let Frequencies (m) Dirichlet (n) G-let Oscillations  (o) Dilation  (p) G-let Frequencies
The G-let coefficients of a chirp signal are shown in Fig. 6.5(b), and its frequencies after dilation is shown in Fig. 6.5(c). From a closer view of the signal using Fig. 6.5(d) and Fig. 6.5(e), it may be noticed that the gradual change in frequencies is captured from the signal in direct proportion to scale and indirect proportion to bandwidth. The results of wavelets (DB-1) and Fourier transform are shown for comparison.

![Graphs showing different frequency representations.](image)

Figure 6.5 Frequencies of Chirp Signal: (a) Chirp Signal (b) G-let Signal (c) All Frequencies (d) First Few Frequencies (e) Closer View of Frequencies (f) Wavelet Frequencies (g) Fourier Frequencies Real Part (h) Fourier Frequencies Imaginary Part

**Corollary 6.2.1.** The difference $G_d(x)$ between consecutive high $G_h(x)$ and low $G_l(x)$ transform coefficients is proportional to the difference $S_d$ in the amplitude between consecutive points of the signal in the same position $x$ (indicative of the frequency at $x$).

$$G_d(x) \propto S_d(x) \quad (6.2.7)$$
Corollary 6.2.2. Depending on the choice of the transformation \( T(R) \), maximum frequency \( f_{\text{max}} \) in a signal results in either maximum \( G_{\text{dmax}} \) or minimum \( G_{\text{dmin}} \) difference between the consecutive high and low transform coefficients \( G(x) \).

Illustration: A linear transformation matrix transforms a signal. For the dihedral rotation matrix \( R \), when the angle of transformation changes, the amplitude of the base oscillations vary as well. For instance, for the example shown in illustration 1.1, as \( \theta \) increases from zero, the amplitudes of the triangular wave increase until \( \theta = \pi/4 \). Thus, each value of \( \theta \) creates a different amplitude resolution of the signal. At the rotation angle \( \pi/4 \), the value of both cosine and sine function are equal to \( \sqrt{2}/2 \). Therefore the triangular wave is zeros at the low values (difference) and double (sum) at the peaks. Here, as well as at angles \( 3\pi/4, 5\pi/4 \) and \( 7\pi/4 \), the highest amplitude difference among neighboring points can be captured as shown:

\[
R(\pi/4) = \begin{pmatrix}
0 \\
1.4142 \\
.4142 \\
0 \\
1.4142
\end{pmatrix},
R(3\pi/4) = \begin{pmatrix}
-1.4142 \\
0 \\
-1.4142 \\
0 \\
-1.4142
\end{pmatrix}
\] (6.2.8)

\[
R(5\pi/4) = \begin{pmatrix}
0 \\
-1.4142 \\
1.4142 \\
0 \\
-1.4142
\end{pmatrix},
R(7\pi/4) = \begin{pmatrix}
1.4142 \\
0 \\
1.4142 \\
0 \\
1.4142
\end{pmatrix}
\] (6.2.9)

These angles are always available for G-let transform at \( (\theta \times n/360) \) where \( \theta \) is any one of the angles mentioned earlier. For example, if \( n = 256 \), then \( (\pi/4 \times n/360) \times (180/\pi) = 32 \) whose transformation angle in dihedral groups is given by \( (360/n) \times 32 = 45 \) or \( \pi/4 \) and the other angles are \( \theta = 135, 225 \) and \( 315 \). Two cases illustrate how the amplitude difference between consecutive points in a signal is captured by G-lets. Consider an example of consecutive points being \( a1 = [0.1, 61] \). In this case, the matrix \( R(\pi/4) \) transforms \( a1 \) to \( a1' = [-43.0628, 43.2042] \) whose absolute values are almost equal. Whereas, for \( a2 = [20, 61] \), the difference is less and it is transformed to \( a2' = [28.9914, 57.2756] \). The real benefit of this character of G-let transforms is
reaped when the transformed signal \((i.e.,\) the G-let coefficients\) is subject to a dilation operation as discussed below. 

**Theorem 6.2.1.** A dilation operation \(L(x)\) may be defined to further transform the signal so as to highlight the highest frequency \(f_{\text{max}}\) of the signal, while suppressing the lower frequencies \(f_{\text{min}}\).

\[
f_{\text{max}}(orf_{\text{min}}) = L(x)T(R)S
\]

**Corollary 6.2.3.** Different dilations \(L_m(x)\) may be constructed for different amplitude resolutions in such a manner that the highest frequency \(f_{\text{max}}\) of the signal stands out in each resolution.

\[
f_{\text{max}} = L_m(x)A(t)
\]

**Corollary 6.2.4.** After dilation, at a specific amplitude resolution \(A(t)\) \((e.g.,\) specific transformation angles for a dihedral group\), the maximum frequency \(f_{\text{max}}\) stands out more clearly than at other resolutions.

**Corollary 6.2.5.** For dihedral transformations, considering an \(n\)-tuple signal, such oscillations are created by matrices \((\theta \times (n/360))\) with the transformation angles \(\theta\) corresponding to \(\pi/4\), \(3\pi/4\), \(5\pi/4\) and \(7\pi/4\) respectively.

**Proof.** The dilation \((L_m(x))\) is a process of scaling down the transform coefficients at the points where there are peaks in the G-let coefficients. It is given by

\[
L_m(x) = p_1(x) \oplus p_2(x) \oplus \cdots \oplus p_{(n+2)/2}(x)
\]
where
\[
p_i(x) = \begin{cases} 
1, & \text{if } x \text{ is odd (or even)} \\
p, & \text{if } x \text{ is even (or odd)} 
\end{cases} \quad (6.2.13)
\]

with \( p = \) the dilation parameter. The dilation parameter may be chosen in such a manner as to highlight or suppress one or more frequencies in the signal. To highlight the highest frequency, the dilation parameter is defined as inversely proportional to the absolute value of the ratio of the highest G-let coefficient and its smallest immediate non-zero neighbor. In other words, if the highest absolute G-let coefficient \( G_{\text{max}} \) occurs at the discrete even position \( j \),

\[
p \propto \frac{G_{j-1}}{G_{\text{max},j}} = \frac{G_{j-1}}{G_{\text{max},j}} \quad (\text{say}) \quad (6.2.14)
\]

To highlight any other frequency, the corresponding absolute G-let coefficient may be chosen in defining the ratio. The proof below is provided for the dilation operation highlighting the highest frequency, but it may easily be extended to highlight any frequency. The result after dilation is given from \( T(x) \) and \( L_m(x) \) by

\[
DG(x) = G(x)L_m(x) \quad (6.2.15)
\]

\[
DG(x) = G_1(x) \ast \frac{G_{j-1}}{G_{\text{max},j}} \oplus G_2(x) \ast \frac{G_{j-1}}{G_{\text{max},j}} \oplus \cdots \oplus G_j(x) \ast \frac{G_{j-1}}{G_{\text{max},j}} \oplus \cdots \oplus G_m(x) \ast \frac{G_{j-1}}{G_{\text{max},j}} \quad (6.2.16)
\]

Since the \( j^{th} \) coefficient is the highest, only that coefficient stands out while the rest are suppressed. Thus, a dilation operation has been constructed to highlight the maximum frequency of the signal.

Also from the resultant equation, it is seen that the dilation operation is localized. Hence the G-let transform has the ability to retain the position of each frequency in the G-let coefficients.
unlike a Fourier transform. For a signal \( S \), using the rotation transformation in dihedral groups we get \( DG(x) \) to be,

\[
DG(x) = R(\theta) \times \sum x_i \delta_{ij} \times \sum p_i(x) \delta_{ij}
\]  
(6.2.17)

The G-let coefficients now has been separated into a dilated signal \( DG(x) \) and an oscillatory signal \( O(x) \), which reconstruct the G-let coefficients. From G-let coefficients the signal can be retrieved either from one G-let transform coefficients or from \( n - 1 \) G-let coefficients.

\[
S(x) = G(x)'; \quad \text{or} \quad S(x) = G_1(x) \oplus G_2(x) \oplus \cdots \oplus G_{n-1}(x)
\]  
(6.2.18)

\[
G_i(x) = DG_i(x) \oplus O_i(x)
\]  
(6.2.19)

For special G-let transform, alternate values are very close to zero. The dilation operation \( L_m(x) \) chosen in the current work, matches another triangular wave pattern (from dihedral G-lets) with fixed odd and even points. The dilation parameter ‘\( p \)’ can be chosen to act upon the G-let oscillations on either the odd or even wave pattern. The dilated signal carries the local frequency information of the signal. The absolute amplitudes of \( DG(x) \) indicate frequencies and the positive or negative sign represents a descending or ascending portion of the signal respectively, indicating points of high frequencies. The oscillations that are suppressed and removed during dilation make an oscillatory signal \( O(x) \) which has alternate values as zeros. Instead of one frequency or the current highest frequency at a time, a range of frequencies may also be explored by considering a range of high values in the dilated signal. A multicomponent signal with varying frequencies can thus be comfortably handled to figure out its constituent frequencies by this method.

**Illustration:** Consider the signal in Fig. 6.8(a). Obtaining the G-let coefficients and dilating them corresponding to \( \theta = \pi/4 \) yields the frequencies of the signal shown in Fig. 6.8(b), where the highest frequency at \( x = 42 \) matches the maximum amplitude difference in the signal. It is of interest to note that the actual amplitude of the signal at \( x = 42 \) is irrelevant in identifying its frequency. In fact, the highest amplitude in the signal corresponds to \( x = 21 \) as shown in Fig. 6.8(a).

The dilation representation matrix for a dihedral group is chosen to be \( D(z) \) with \( p \) as the dilation parameter given by
A simple frequency analysis method emerges out of the dilation performed on G-let transform coefficients. The procedure is applicable to both stationary and non-stationary signals, since no future information is assumed. A suitable dilation operation, though simple to implement, is very crucial in obtaining the frequencies. The dilation operation may be customized for different signals. If a signal contains a lot of high frequencies, then two dilations can be combined to sort out the frequencies. A specific dilation will be used for even wave pattern and another for the odd wave pattern on the G-let oscillations, for signals of the multicomponent type.

### 6.2.2 Without Dilation

Since instantaneous frequency is defined to be proportional to the amplitude gradients, it is also possible to obtain the frequencies of a signal by constructing the time-domain G-lets of the gradient of the signal. This approach is in Fig. 6.6.

![Figure 6.6 Frequency Analysis using G-lets (without Dilation)](image)

G-let frequencies can also be obtained without using the dilation operation. A differential operator is applied on a signal. For this new signal G-let transform are applied taking the signal directly to the frequency domain. For example, consider a sine signal.

Let a signal $S(x)$ be

$$S(x) = \sin(\omega t)$$  \hspace{1cm} (6.2.21)
Then by differentiation of the signal and its G-let transform,

\[ G(S(x)) = R(\theta) \times S(x) \]
\[ \frac{d(S(x))}{dt} = \omega \cos(\omega t) \]
\[ G(\omega \cos(\omega t)) = \omega G(\cos(\omega t)) \]

it is seen that what stands out is the frequencies \( \omega \). Using a dilation factor \( d_f \) on \( G(S(x)) \) it was seen that frequencies are obtained. If the dilation factor is taken to be \( \omega \), it means that the transformation G-let oscillations by the \( \cos(\omega t) \) gives the frequencies of \( S(x) \). This can be summarized in the block diagram of Fig. 6.6. The dilated and undilated frequencies are demonstrated by the signal \( S(x) \) and \( G(x) \).

![Figure 6.7 Demonstration of G-let Frequencies (without Dilation): (a) Sine Signal (b) G-let Frequencies of (a) using Dilation (c) Cosine Signal (d) G-let Frequencies of (c) without Dilation](image)

The sine signal \( S(x) \) in Fig. 6.7(a) is the original signal. The G-let frequencies of this signal using dilation are shown in Fig. 6.7(b). When the sine signal is differentiated, a cosine signal is obtained shown in Fig. 6.7(c). The G-let oscillations of the cosine signal are demonstrated in Fig. 6.7(d). The highest amplitude gradient of the sine signal is at position 13. The G-let frequencies of both the signals exactly the show the highest amplitude gradient at this point correctly. Thus G-let frequency analysis may be performed even without the dilation operation. This is possible because of the differential which emphasizes the amplitude gradients.
of the signal before G-let oscillations are produced. For signals where the dilation parameter is difficult to decide, specifically for multicomponent signals, this may be useful. This study is left for future work.

### 6.2.3 G-let Frequencies of 1-D Signals

The dilation parameter affects different types of signals differently. For example, an ECG signal and a chirp signal should use different dilations. The basic idea is to manipulate the G-let oscillations to focus on a particular frequency. The nature of oscillations depends on the available frequencies in the signal. For illustration, an ECG signal (which was synthesized) and its frequencies are shown in Fig. 6.9.

![Figure 6.8 Frequency Analysis of a 1-D Signal: (a) Signal (b) G-let Frequencies](image)

Figure 6.8 Frequency Analysis of a 1-D Signal: (a) Signal (b) G-let Frequencies

![Figure 6.9 Frequency Analysis of ECG Signal: (a) Signal (b) G-let Frequencies (c) Highest Frequency (d) Second Highest Frequency (e) Third Highest Frequency](image)

Figure 6.9 Frequency Analysis of ECG Signal: (a) Signal (b) G-let Frequencies (c) Highest Frequency (d) Second Highest Frequency (e) Third Highest Frequency
The highest frequency of the dilated signal is obtained simply by

\[ \pm f_{max} = \max(\text{abs}(DG(x))) \]  \hspace{1cm} (6.2.22)

For two adjacent high frequencies, the neighborhood local frequencies overlap but the point of separation can be identified as demonstrated in Fig. 6.10.

Figure 6.10 Frequency Analysis of 1 – D Signal with Two Adjacent Frequencies: (a) Signal (b) G-let Frequencies (c) Highest Frequency of the Signal (d) Second Highest Frequency of the Signal

In the Fig. 6.10 there are two similar frequencies adjacent to each other. The signal has \( n = 60 \), therefore the G-let transform that has to be examined is calculated as \((135 \times (n/360)) = 22.5\). The 23\(^{rd}\) G-let transform is used and the dilation uses the parameter \( p = p/(200 \times p) \). The smooth signal achieved has got high frequencies in two places marking descending slopes with a positive value. The sharper frequency has got the highest value. Let us consider the highest frequency first. Since it indicates a descent, its left side oscillations are inspected. There is a series of low frequencies with a lower frequency marking the beginning of that particular signal portion and related frequencies. Cutting immediately to the left of this lowest frequency, it is seen that all the frequencies in the second curve of the signal are obtained. This is shown in the Fig. 6.10(a) and Fig. 6.10(b) along with the corresponding frequencies Fig. 6.10(c) and Fig. 6.10(d) respectively.
6.2.4 G-let Frequencies of 2-D Signals

For an image the oscillations are induced either horizontally or vertically. Consider the example of producing G-let oscillations row wise (vertical). The high frequency in each row is calculated and, the highest among them is the highest frequency of the entire image. The frequencies retrieved from a ‘Lena’ image are shown in the Fig. 6.11.

![Image of Lena](image1)

(a)

![Graph of Lena](image2)

(b)

![Graph of Lena](image3)

(c)

![Graph of Lena](image4)

(d)

Figure 6.11 Frequency Analysis of ‘Lena’ Image: (a) G-let Frequencies (b) Highest Frequency (c) Second Highest Frequency (d) Third Highest Frequency

Three consecutive G-let frequencies starting from the highest frequency of the image in Fig. 6.11 are shown with their horizontal neighborhood in Fig. 6.11(b), Fig. 6.11(c) and Fig. 6.11(d) respectively. G-let frequencies using dilation are shown for more images like ‘Butterfly’ and ‘Tiger’ as in Fig. 6.12 and Fig. 6.13 respectively. In both of these images, the highest G-let frequencies are shown to closely outline the edges of the object present in the scene. Edges are basically part of the high frequency areas.
Figure 6.12 Frequency Analysis of G-lets - 'Butterfly' Image: (a) Original Image (b) G-let Frequencies

Figure 6.13 Frequency Analysis of G-lets - 'Tiger' Image: (a) Original Image (b) G-let Frequencies

6.2.5 G-let Frequency Multiresolution Analysis

G-let transform slowly filter from low frequencies to high frequencies in parallel to the G-let time-amplitude filters. This may be stated in the following theorem.

Theorem 6.2.4. (Multiresolution Analysis) - For every amplitude resolution of G-lets, there is a corresponding frequency resolution.

Illustration: Rapid localized changes are a sign of high frequencies and are seen in steep amplitude gradients. Low frequencies are characterized by smooth and gradual changes in gradients. The latter is affected by G-let oscillations more in terms of the height and spread of oscillations. Whenever transformation angle increases the oscillations subdue mild gradients or
low frequencies. At special resolutions only steep gradients are left out, all others are pushed into strong oscillations. Slowly from lower to higher frequencies the diminishing of frequencies occurs thus giving rise to a corresponding frequency resolution. Consider a simple signal $S = [1 1 1 5 1 1 2 3 2 1]$ for demonstration. The entire amplitude resolution of the signal is shown in the first column of Fig. 6.15 and Fig. 6.16 as G-lets in time domain. The oscillations of a signal begin to affect the low frequencies of a signal in the first G-let transform and then high frequencies are affected slowly across consequent G-lets. To separate the frequencies, the regions of the signal affected by oscillations are extracted from G-let coefficients.

Such a frequency multiresolution analysis coupled with G-let oscillations in the time domain mark an opportunity for feature extraction using G-let frequencies. At different G-let frequency resolutions, the features may be perceived differently by a multi-view in the frequency domain. Segmenting a feature in the frequency domain is supported by the multi-view so that the most suitable outline of a feature is selected in an appropriate frequency resolution. In the special G-let frequency transforms, generated by special G-let transform using the prominent presence of only one wave pattern, a feature is represented by the highest frequencies. In the other frequency bases, more frequencies are present in and around a feature. Thus using a G-let frequency multiresolution analysis, the amount of details (in low frequencies) in the image may be chosen using the corresponding frequency resolution. This is demonstrated in the upcoming feature extraction Chapter 8.

![Figure 6.14 Frequency Multiresolution Analysis of G-lets](a)
Figure 6.15 Amplitude Resolution of Signal 'S': G-lets - (a) (b) (c) (d) (e); Frequency Resolution of Signal 'S': G-let Frequencies - (f) (g) (h) (i) (j); (Using Rotations)

Figure 6.16 Amplitude Resolution of Signal 'S': G-lets - (a) (b) (c) (d) (e); Frequency Resolution of Signal 'S': G-let Frequencies - (f) (g) (h) (i) (j); (Using Reflections)
6.2.6 Computational Complexity

The G-let frequencies are obtained by G-let transformation and a dilation matrix. If no dilation is used, the signal’s first derivative is first found out and the derivative is projected on the G-let transform. In both the cases the computations account to $O(2n)$.

6.2.7 Comparison With Fourier and Wavelet Frequency Analysis

The transformations used in wavelets are scaling, dilation and translation. Since translations cannot be finite, scaling and dilation can form a transformation group. Wavelet frequencies are both temporally and spatially localized because they look for similar structures in a signal at a particular time. Within the rules of group theory, the same effect may be achieved with transformation groups. Geometric wavelets segregate fine and coarse details of the input signal using a plane similar to the choice of a transformation group and the corresponding dilation operation done in G-lets. The computational complexity involved in repeating the process for the sub-regions to create a dictionary of multi-scale analysis of data can be compensated by the use of the G-let transforms mapped to the transformation group.

Finally, for a simple signal shown in Fig. 6.17, $\text{sig} = [0.4818 \ 0.8443 \ 0.9980 \ 0.9048 \ 0.5878 \ 0.1253 \ -0.3681 \ -0.7705 \ -0.9823 \ -0.9511 \ -0.6845 \ -0.2487]$, frequencies obtained from G-lets using
Table 6.1 Frequencies for Signal in Fig. 6.17

<table>
<thead>
<tr>
<th>Frequencies</th>
<th>G-let(Dilation)</th>
<th>G-let(No Dilation)</th>
<th>Wavelet</th>
<th>Fourier</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0050</td>
<td>-0.2374</td>
<td>0.9377</td>
<td>-0.0632</td>
<td></td>
</tr>
<tr>
<td>-0.0050</td>
<td>-0.0302</td>
<td>1.3455</td>
<td>2.2784  + 5.6672i</td>
<td></td>
</tr>
<tr>
<td>0.0066</td>
<td>0.1772</td>
<td>0.5043</td>
<td>0.1756  + 0.2682i</td>
<td></td>
</tr>
<tr>
<td>0.4119</td>
<td>0.5571</td>
<td>-0.8051</td>
<td>0.1419  + 0.1330i</td>
<td></td>
</tr>
<tr>
<td>0.0021</td>
<td>0.0632</td>
<td>-1.3671</td>
<td>0.1327  + 0.0733i</td>
<td></td>
</tr>
<tr>
<td>0.4464</td>
<td>0.6273</td>
<td>-0.6599</td>
<td>0.1293  + 0.0334i</td>
<td></td>
</tr>
<tr>
<td>-0.0044</td>
<td>-0.1094</td>
<td>-0.2564</td>
<td>0.1284</td>
<td></td>
</tr>
<tr>
<td>0.0665</td>
<td>0.1151</td>
<td>0.0659</td>
<td>0.1293  - 0.0334i</td>
<td></td>
</tr>
<tr>
<td>-0.0067</td>
<td>-0.1805</td>
<td>0.3270</td>
<td>0.1327  - 0.0733i</td>
<td></td>
</tr>
<tr>
<td>-0.3752</td>
<td>-0.5039</td>
<td>0.2845</td>
<td>0.1419  - 0.1330i</td>
<td></td>
</tr>
<tr>
<td>-0.0029</td>
<td>-0.0840</td>
<td>-0.0221</td>
<td>0.1756  - 0.2682i</td>
<td></td>
</tr>
<tr>
<td>-0.4685</td>
<td>-0.3082</td>
<td>2.2784  - 5.6672i</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

dilation, G-lets without dilation, wavelets, and Fourier analysis are shown in Table 6.1. The highest amplitude difference in the original signal ‘sig’ is found to be at position 7. The highest G-let frequency is found to be at position 6 in both cases, with and without dilation. In wavelets the highest frequency point is found at 2nd position and the negative highest is found to be at 5th position. In Fourier the highest frequency is at 2nd and 12th positions. The parameters involved in G-let analysis are the choice of the transformation group, a set of filters for signal space V and the dilation factor. The basis can be taken as the standard basis to create the representation matrices. It is the other two factors that determine the performance of G-lets. In the current work, dihedral groups that have rotations and reflections as transformations have been tested to do G-let analysis. The only deciding aspect in this case is the dilation factor. This factor is not a choice, since it is calculated from G-let coefficients in the time domain. Thereby no assumptions or approximations are involved in G-let analysis.

6.3 Existing Frequency Analysis Methods

How instantaneous frequencies are defined in popular works of literature is summarized in Table 6.2.
<table>
<thead>
<tr>
<th>References</th>
<th>Algorithm</th>
<th>Application</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bayram, M., and R. G. Baraniuk, 1996</td>
<td>Thomson’s multiple window spectrum estimation scheme extended to the time-frequency and time-scale planes for extracting chirping line components is presented</td>
<td>Spectral Analysis</td>
<td>Time-varying spectral analysis is explored using extensions of Thomson’s spectral analysis</td>
</tr>
<tr>
<td>Hess-Nielsen, N., and M. V. Wickerhauser, 1996</td>
<td>An overview of time-frequency analysis and some of its key problems are presented</td>
<td>Study of Time-Frequency Methods</td>
<td>A comparison of time-frequency analysis methods has been explored</td>
</tr>
<tr>
<td>Cohen, L., and C. Lee, 1989</td>
<td>The derivative of the phase of a signal is shown to be equal to the conditional mean frequency of a positive time-frequency distribution, and is claimed to be equal to the instantaneous frequency of the signal</td>
<td>Instantaneous Frequency</td>
<td>Instantaneous frequency is redefined</td>
</tr>
<tr>
<td>O’Neill et. al., 1999</td>
<td>An alternative definition for the Wigner distribution is proposed</td>
<td>Classification of Discrete signals</td>
<td>Wigner Distribution has been studied in detail and explored</td>
</tr>
<tr>
<td>Boashash, 1992</td>
<td>The concept of instantaneous frequency (IF), its definitions, and the correspondence between the various mathematical models formulated for representation of IF are discussed</td>
<td>Classifying Signals</td>
<td>Various methods of defining and calculating instantaneous frequency are explained</td>
</tr>
<tr>
<td>Hlawatsch, F., and G. F. Boudreaux-Bartels, 1992</td>
<td>A tutorial review of both linear and quadratic representations is given</td>
<td>Understanding different methods</td>
<td>A comparative study of time frequency representations is presented</td>
</tr>
<tr>
<td>Stankovic, 1994</td>
<td>Analysis of the representation of instantaneous frequency and group delay using time-frequency transforms or distributions of energy density domain is performed</td>
<td>Instantaneous Frequency</td>
<td>Wavelet and Wigner Transforms are combined to calculate instantaneous frequency</td>
</tr>
<tr>
<td>Liu, J. C., and T. Lin, 1993</td>
<td>The short-time Hartley transform (STHT), a time-varying Hartley representation of a signal, is introduced. Specially for the representation of signals whose properties change markedly as a function of time</td>
<td>Non stationary signals</td>
<td>Hartley Transform is proposed</td>
</tr>
<tr>
<td>References</td>
<td>Algorithm</td>
<td>Application</td>
<td>Remarks</td>
</tr>
<tr>
<td>------------</td>
<td>----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
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<td>------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Pinnegar, C. R., and L. Mansinha, 2004a</td>
<td>The S-transform &quot;wavelet&quot; is obtained by multiplying a real window with the complex Fourier sinusoid</td>
<td>Non stationary signals</td>
<td>Wavelet and Fourier transform are combined to perform frequency analysis</td>
</tr>
<tr>
<td>Pinnegar, C. R., and L. Mansinha, 2004b</td>
<td>Since the Hartley kernel has no time localization, a short-time Hartley transform has been proposed and defined in analogy with the short-time Fourier transform</td>
<td>Time-dependent Signals</td>
<td>The method is an extension of Fourier transform</td>
</tr>
<tr>
<td>Stankovic, 1996b</td>
<td>A time-frequency distribution that produces high concentration at the instantaneous frequency for an arbitrary signal is proposed</td>
<td>Asymptotic Signals</td>
<td>The method is an extension of Wigner distribution</td>
</tr>
<tr>
<td>Almeida, 1994</td>
<td>Decomposition of a signal is shown in terms of chirp signals using Fourier transform</td>
<td>Signal Decomposition</td>
<td>This method is a generalization of Fourier transform used to convert a signal into smaller components</td>
</tr>
<tr>
<td>Angrisani et. al, 2005</td>
<td>Instantaneous frequency estimation is proposed using Warblet transform</td>
<td>Multicomponent Signals</td>
<td>Warblet transform is used to calculate instantaneous frequencies</td>
</tr>
<tr>
<td>Akan, A., and L. F. Chaparro, 1996</td>
<td>A discrete rotational Gabor transform is proposed</td>
<td>Frequency Analysis</td>
<td>Fourier and Gabor Transform are combined to perform a time-frequency analysis</td>
</tr>
<tr>
<td>Akan, A., and Y. Cekic, 2003</td>
<td>Basis functions of the proposed expansion are obtained via fractional Fourier basis</td>
<td>Chirp type Signals</td>
<td>Fractional Gabor Expansion is used for frequency analysis</td>
</tr>
<tr>
<td>?</td>
<td>The estimation of the instantaneous frequency (IF) of a harmonic complex-valued signal with an additive noise using the Wigner distribution is considered</td>
<td>Noisy and Complex Signal</td>
<td>Wigner Distribution is used for defining instantaneous frequency</td>
</tr>
<tr>
<td>References</td>
<td>Algorithm</td>
<td>Application</td>
<td>Remarks</td>
</tr>
<tr>
<td>--------------------------------</td>
<td>---------------------------------------------------------------------------</td>
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<td>------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Stockwell et. al., 1996</td>
<td>The S transform is introduced as an extension of the ideas of the continuous wavelet transform (CWT) based on a moving and scalable localizing Gaussian window</td>
<td>Multiresolution Analysis</td>
<td>The S transform is an extension of wavelet transform</td>
</tr>
<tr>
<td>Zhang et. al., 2004</td>
<td>The harmonic transform as an impulse-train spectrum for signals that are comprised of time-varying harmonics is proposed</td>
<td>Frequency Analysis</td>
<td>Harmonic Transform is used for spectrum analysis</td>
</tr>
<tr>
<td>Stankovic, 1997</td>
<td>Local polynomial time-frequency transforms, introduced by Katkovnik, are extended to Wigner distribution</td>
<td>Multicomponent Signals</td>
<td>Polynomial Transform and Wigner Distribution are used for multicomponent analysis</td>
</tr>
<tr>
<td>Cohen, L., and C. Lee, 1989</td>
<td>It is shown that the derivative of the phase of a signal, is a reasonable choice for instantaneous frequency</td>
<td>Frequency Analysis</td>
<td>Instantaneous frequency of a time–frequency distribution is defined</td>
</tr>
<tr>
<td>Krishman, 2005</td>
<td>An approach to extract the IF from its adaptive time-frequency distribution is proposed</td>
<td>Nonstationary Signals</td>
<td>Wigner distribution is used for instantaneous frequency extraction</td>
</tr>
<tr>
<td>Hussain, Z. M., and B. Boashash, 2002</td>
<td>Estimation of the instantaneous frequency (IF) of non-stationary mono and multicomponent FM signals with additive Gaussian noise is presented</td>
<td>Nonstationary, Multicomponent Signals</td>
<td>Quadratic Time-Frequency Distributions are used for FM signal analysis</td>
</tr>
<tr>
<td>Wu, D., and J. M. Morris, 1994</td>
<td>A radial Butterworth kernel (R BK) with two adjustable parameters for the Cohen’s (1966) class of distributions (CCD) is presented in the ambiguity function domain</td>
<td>Non stationary signals</td>
<td>A kernel is used for Cohen’s distribution for frequency analysis</td>
</tr>
</tbody>
</table>
6.4 Conclusions

A time-frequency analysis using G-lets was proposed in this chapter. The G-let oscillations produced in the time-amplitude domain are used to generate G-let frequencies either by a dilation operator or a differential operator. As a result, time localized G-let frequencies are obtained in the frequency domain which is demonstrated using a Chirp signal, ECG signal and images. It is shown that the highest G-let time-frequency filter, called the special resolution filter, provides a clear distinction between low and high frequencies. This algorithm adds another ‘n’ to the computational complexity of G-lets in time-amplitude domain $O(2n)$, making it $O(3n)$. In the next chapter, it is shown how the localization of G-let frequencies are used to separate the individual components of monocomponent and multicomponent signals.