Chapter 4

G-Lets: Time-Amplitude Domain Analysis

4.1 Introduction

The central question of this thesis is to examine how one can analyze information contained in a signal by examining its structural symmetries. Such an analysis may be done in the time domain in terms of signal amplitudes or frequencies. As discussed in Chapter 3, the main advantage in performing the analysis in time-amplitude domain is that the mapping from a group structure to the signal under consideration may be done without any loss of information. Chapter 3 specifically discussed the different mapping techniques used in different signal processing algorithms from the standpoint of group theory. Since signal data is available as amplitudes as a function of time, constructing an analysis in the time-amplitude domain preserves the entire signal information. In this chapter, the new signal processing algorithm, G-lets, is presented in its most basic form, in the time-amplitude domain. Specifically, this chapter will focus on using group theory to explore the structural symmetries of a signal and construct a multiresolution analysis of the signal amplitudes in terms of amplitude gradients. This analysis will be shown to time localize signal features in terms of amplitudes. For images, as is known,
the corresponding localization is spatial. In order to illustrate the algorithm’s performance, dihedral groups are used throughout. Using other groups will highlight different-but-orthonormal structural symmetries of the signal. In G-let analysis using dihedral groups, a signal may be seen at different ‘rotation or reflection angles’. After every rotation or reflection, the signal’s characteristics are noted at a different resolution. A multiresolution analysis of the amplitudes of the signal have been made possible by this kaleidoscopic nature of transformations in group theory.

To begin with G-let analysis, a transformation group G (Hamermesh, 1970) needs to be chosen. Since only finite discrete signals are considered in this work, the signal exists in a finite vector space L, which shall henceforth be called as the signal space. For this signal, a finite set of G-let transforms are generated and the signal is transformed by the G-let transformations. G-let analysis of one-dimensional signals such as sine, square, sawtooth, Dirichlet, sinc, and chirp signals are shown in this chapter. To further illustrate the algorithm, a six-tuple one-dimensional signal is presented as an example, and all the G-let transforms and their corresponding G-let transform coefficients are shown, followed by the reconstruction of the signal. Action of G-let transforms on an ECG signal is presented, followed by the action on images. It is shown that the set of G-let transforms for an image generate a set of amplitude resolutions of the image, allowing a multiresolution analysis. The chapter ends with brief discussions on compression of images and the computational complexity of the algorithm.

### 4.2 Construction of G-let Transforms

According to the G-let framework discussed in Chapter 3, for G-let analysis, the signal module is chosen. Let the filter algebra be that of a dihedral group G. If the signal module has \( \text{length} = n \), then the dihedral group has an \( \text{order} = 2n \). The group has ‘n’ rotations \((R)\) and ‘n’ reflections \((S)\), the generating elements. The angle of rotations and reflections vary from \( \theta = 0^\circ/n \) to \( \theta = 360^\circ/n \). The transformations in the group are given by:

\[
G = \{ R, R^2, R^3, \cdots R^n, S, SR, SR^2, SR^3, \cdots SR^{n-1} \} \tag{4.2.1}
\]

There are three relations between the generating elements \( R \) and \( S \) of the above group. They
\[ S^2 = id; \quad (S \times R)^2 = id; \quad R^n = id; \quad (4.2.2) \]

where \( id = \) identity transformation.

The next task is to design the mapping between the signal module and the dihedral group filters. This is achieved by designing the action of the dihedral group on the signal module \( S \). Matrix representations are used to define this action. A G-let transform is derived from these matrix representations. Members of the dihedral group are transformations. Each transformation is represented by a single \( n \times n \) matrix. Each rotation is a \( n \)-dimensional rotation and reflection a \( n \)-dimensional transformation. From Schur’s orthogonality lemma (refer Chapter 2), it is known that such a representation may be decomposed into a direct sum of irreducible representations, which are again smaller dimensional rotations and reflections. Such a decomposition helps in creating local rotations and reflections in the signal module. The actual dimension of the irreducible representations \( d \) are related to the conjugacy classes of the dihedral group \( G \).

It is known that if the signal module length is ‘\( n \)’ and it is odd, then the number of conjugacy classes is \((n+3)/2\), whereas if ‘\( n \)’ is even, the number is \((n+6)/2\). In the dihedral group \( G \), identity transformation forms a conjugacy class by itself. The other conjugate members are formed by \( S \times R^k \times S = R^{-k} \). Hence the conjugacy classes are \( R^n, R, R^2, SR, SR^2, SR^3, SR^{n-1} \) for an odd dimension ‘\( n \)’ of the signal. For the odd dimension, all the reflections make one conjugacy class. The classes for an even dimension are \( R, R^n, R, R^2, SR, SR^{n/2} \) and \( SR^{n/2+1}, SR^{n-1} \). The reflections are grouped into two conjugacy classes. The conjugacy classes partition the dihedral group. For other transformation groups, the number and order of conjugacy classes are different. Since the squared sum of the dimensions of irreducible representations of the group is equal to the dimension of the signal module (refer Chapter 2), for dihedral groups, there are only one-dimensional and two-dimensional irreducible representations.

The \( i^{th} \) two-dimensional irreducible representation matrix \( (r_i) \) for rotation is given as

\[
 r_i = \begin{pmatrix}
 \cos i \times \theta & -\sin i \times \theta \\
 \sin i \times \theta & \cos i \times \theta \\
\end{pmatrix} \quad (4.2.3)
\]

and the reflection matrix \( (s_i) \) is taken to be any random reflection matrix.
Table 4.1 Number of Conjugacy Classes for Dihedral Groups

<table>
<thead>
<tr>
<th>S.No</th>
<th>Tuple (n)</th>
<th>Odd/Even</th>
<th>Number of Conjugacy Classes(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>Even</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>21</td>
<td>Odd</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>32</td>
<td>Even</td>
<td>19</td>
</tr>
<tr>
<td>4</td>
<td>51</td>
<td>Odd</td>
<td>27</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>Even</td>
<td>53</td>
</tr>
<tr>
<td>6</td>
<td>501</td>
<td>Odd</td>
<td>252</td>
</tr>
<tr>
<td>7</td>
<td>1000</td>
<td>Even</td>
<td>503</td>
</tr>
</tbody>
</table>

A G-let transform is built using these irreducible representations as diagonal blocks. There are only two types of irreducible representations here, out of which the one-dimensional representations are not useful for a local rotation or reflection. Therefore they are not used in a G-let transform. Only the two-dimensional irreducible representations are used and they are called G-let blocks for a dihedral group. The G-let blocks occupy the diagonal of the G-let transform. The total number of irreducible representations of the dihedral group G is equal to the number of conjugacy classes of the group.

The number of conjugacy classes or irreducible representations for different signal module dimensions ‘n’ is given in the Tab. 4.1.

A standard basis of the signal space L is chosen for the signal module as

\[
B = \sum_{i=1}^{n} \sum_{j=1}^{n} e_{ij} \quad \text{where} \quad e_i = \begin{cases} 
1, & i = j \\
0, & \text{otherwise}
\end{cases}
\]  

(4.2.4)

Based on the basis given above, the G-let blocks are defined for each two-dimensional irreducible representation. There is one G-let transform corresponding to each transformation of the dihedral group G. For different transformation groups other than the dihedral groups discussed here, the size of the G-let blocks may look as in Figure. 4.1 depending on their corresponding irreducible representations. Each diagonal block \( D_i(R) \) of the representation corresponds to one irreducible representation.

Thus the mapping between signal module and dihedral group filters is defined by G-let transforms. The signal is transformed by a G-let transform to get the G-let transform coefficients or henceforth G-let coefficients. For an \( i^{th} \) G-let transform, let \( R^i \) be the corresponding matrix representation for the given signal module \( \text{sig} \). Then, with \( * \) being the group action
of dihedral groups,

\[ G_i(sig) = R_i \ast Sig \]  \hspace{1cm} (4.2.5)

The algorithm for the generation of G-let transforms and the reconstruction is shown below.

**ALGORITHM G-let Decomposition (sig, D)**

- Calculate the dimension ‘n’ of the signal module ‘sig’
- Choose a transformation group, for example the Dihedral group D
- Generate matrices for G-let transforms G(sig) from 1 to 2n using ‘Generate_G-let_Transforms’ function given below. The \(i^{th}\) G-let transform is taken as \(G_i(sig) = R_i\)
- Transform the signal using G-let transforms
- Obtain Rotation G-let transform coefficients (\(G_1(sig), G_2(sig), G_3(sig) \ldots G_{n-1}(sig)\))
- Obtain Reflection G-let transform coefficients (\(G_{n+1}(sig), G_{n+2}(sig), G_{n+3}(sig) \ldots G_{2n-1}(sig)\))

**ALGORITHM Generate_G-let_Transforms()**

\[ angle = 2\pi/n \]
- Calculate number and dimension of irreducible representations
- Identify the conjugacy class of matrices
- Make the G-let transforms \(R\) using, two-dimensional irreducible representations irreducible representations placing them diagonally
- for rotations and reflections, in the case of Dihedral groups
- return \(R\)

The G-let transforms for a dihedral group, may be generated in two types. The type I matrices from 1 to \(n - 1\) are given by
\[ G - \text{let} \ 1 = \begin{pmatrix} 1 & \cos((\theta/n) \ast 1) & \sin((\theta/n) \ast 1) & 0 \\ \cos((\theta/n) \ast 1) & -\sin((\theta/n) \ast 1) & \cos((\theta/n) \ast 1) & \cos((\theta/n) \ast 1) & -\sin((\theta/n) \ast 1) \\ \sin((\theta/n) \ast 1) & \cos((\theta/n) \ast 1) & \cos((\theta/n) \ast 1) & \sin((\theta/n) \ast 1) & \cos((\theta/n) \ast 1) \\ 0 & \cos((\theta/n) \ast 1) & \sin((\theta/n) \ast 1) & \cos((\theta/n) \ast 1) \end{pmatrix} \] (4.2.6)

where \( \theta = 2\pi \) and the same G-let block repeats in the diagonal of the G-let transform. More G-let transforms of the same type are given as below from \( n + 1 \) to until \( 2n - 1 \).

\[ G - \text{let} \ 2 = \begin{pmatrix} 1 & \cos((\theta/n) \ast 2) & \sin((\theta/n) \ast 2) & 0 \\ \cos((\theta/n) \ast 2) & -\sin((\theta/n) \ast 2) & \cos((\theta/n) \ast 2) & \cos((\theta/n) \ast 2) & -\sin((\theta/n) \ast 2) \\ \sin((\theta/n) \ast 2) & \cos((\theta/n) \ast 2) & \cos((\theta/n) \ast 2) & \sin((\theta/n) \ast 2) & \cos((\theta/n) \ast 2) \\ 0 & \cos((\theta/n) \ast 2) & \sin((\theta/n) \ast 2) & \cos((\theta/n) \ast 2) \end{pmatrix} \] (4.2.7)

\[ G - \text{let} \ (n - 1) = \begin{pmatrix} 1 & \cos((\theta/n) \ast (n - 1)) & \sin((\theta/n) \ast (n - 1)) & 0 \\ \cos((\theta/n) \ast (n - 1)) & -\sin((\theta/n) \ast (n - 1)) & \cos((\theta/n) \ast (n - 1)) & \cos((\theta/n) \ast (n - 1)) & -\sin((\theta/n) \ast (n - 1)) \\ \sin((\theta/n) \ast (n - 1)) & \cos((\theta/n) \ast (n - 1)) & \cos((\theta/n) \ast (n - 1)) & \sin((\theta/n) \ast (n - 1)) & \cos((\theta/n) \ast (n - 1)) \\ 0 & \cos((\theta/n) \ast (n - 1)) & \sin((\theta/n) \ast (n - 1)) & \cos((\theta/n) \ast (n - 1)) \end{pmatrix} \] (4.2.8)

For Type II, the above G-let transforms are modified to include different G-let blocks within a single G-let transform,

\[ G - \text{let} \ 1 = \begin{pmatrix} 1 & \cos((\theta/n) \ast 1) & \sin((\theta/n) \ast 1) & 0 \\ \cos((\theta/n) \ast 1) & -\sin((\theta/n) \ast 1) & \cos((\theta/n) \ast 1) & \cos((\theta/n) \ast 2) & -\sin((\theta/n) \ast 2) \\ \sin((\theta/n) \ast 1) & \cos((\theta/n) \ast 1) & \cos((\theta/n) \ast 1) & \sin((\theta/n) \ast 2) & \cos((\theta/n) \ast 2) \\ 0 & \cos((\theta/n) \ast 1) & \sin((\theta/n) \ast 1) & \cos((\theta/n) \ast 1) \end{pmatrix} \] (4.2.9)

\[ G - \text{let} \ 2 = \begin{pmatrix} 1 & \cos((\theta/n) \ast 2) & \sin((\theta/n) \ast 2) & 0 \\ \cos((\theta/n) \ast 2) & -\sin((\theta/n) \ast 2) & \cos((\theta/n) \ast 2) & \cos((\theta/n) \ast 4) & -\sin((\theta/n) \ast 4) \\ \sin((\theta/n) \ast 2) & \cos((\theta/n) \ast 2) & \cos((\theta/n) \ast 2) & \sin((\theta/n) \ast 4) & \cos((\theta/n) \ast 4) \\ 0 & \cos((\theta/n) \ast 2) & \sin((\theta/n) \ast 2) & \cos((\theta/n) \ast 2) \end{pmatrix} \] (4.2.10)
and

\[
G - \text{let } 2 = \begin{pmatrix}
1 & 0 \\
\cos((\theta/n) \times (n/2)) & -\sin((\theta/n) \times (n/2)) \\
\sin((\theta/n) \times (n/2)) & \cos((\theta/n) \times (n/2)) \\
\cos((\theta/n) \times n) & -\sin((\theta/n) \times n) \\
\sin((\theta/n) \times n) & \cos((\theta/n) \times n) \\
0 & \ldots
\end{pmatrix}
\]  

(4.2.11)

G-let analysis may be performed either using type I or type II G-let transforms. In this work, only type I transforms are used to demonstrate the algorithm because each G-let block in this case has only one transformation angle \( \theta \). This makes it easier to show G-let analysis with respect to one transformation of the dihedral group.

The signal information is available as amplitudes as a function of time (or position in an image). This may also be visualized as amplitude gradients as a function of time. As shown in the following sections of this chapter, each transformation angle used by the corresponding G-let transform highlights different amplitude gradients differently. Therefore, the transformation of a signal by each G-let transform will create different amplitude resolutions of the signal described in the next section. The G-let transforms gradually filter from low pass to high pass, the amplitudes of the signal, producing a different amplitude resolution for each G-let transform. The G-let transform may also be called as G-let filter or G-let amplitude filter or G-let time-amplitude filter.

Type II matrix representations, on the other hand, contain G-let blocks of different transformation angles along the block diagonal of the G-let transform. Therefore, G-let transforms will act as local high- or low-pass filters, highlighting different sets of amplitude gradients at different time or spatial locations throughout the signal. If one knows a priori what features of a signal need to be highlighted at which time or spatial location, it is possible to design unique type II G-let transforms that directly extract those features from a signal. This has, however, not been explored in the current work. The focus of the rest of the chapter and the rest of the thesis is to examine what signal information may be obtained by constructing a multiresolution analysis using type I G-let transforms, how to convert the time-amplitude domain data to time-frequency domain data, and how to perform feature extraction on 1-D and 2-D signals.
using the data obtained from type I G-let transformation matrices. The G-let transform for both 1-D and 2-D signals are the same.

### 4.2.1 G-let Oscillations

**Definition 4.2.1 (G-let Oscillation):** A G-let oscillation is the repetitive variation, typically in time, of the signal amplitude, between two or more different values.

G-let oscillations arise from the design of the G-let transform. They are denoted as $O(s)$. The G-let transform contains local transformations of the size of the corresponding G-let blocks. The magnitude of oscillations depends on the choice/nature of the irreducible representations (G-let blocks) and the structural symmetries of the signal (signal features). The group action of dihedral groups on a signal module through G-let transforms induces G-let oscillations in the signal module. If one were to visualize the oscillations introduced by the G-let transform to be the base signal, then the signal module may be visualized as a carrier signal. On the signal module acting as the carrier signal, the oscillations $O(s)$ are superimposed by G-let transform, in effect highlighting the features of the signal module. This is in contrast to traditional signal processing methods, where the original signal is treated as the base signal and a signal processing algorithm suggests its frequencies to be the carrier.

This sub-section discusses how the type I or type II G-let transforms construct amplitude resolutions of a signal. When a signal is transformed using a G-let transform, a pattern in induced in the signal. The pattern is specific to the type of the G-let transform for G-let analysis. This pattern induced by G-let transform in the signal is called as *G-let oscillations, $O(s)$*. The most important aspect of G-let analysis in time-amplitude domain is the creation of this pattern in a signal module, *i.e.*, oscillations. For each dihedral transformation and each irreducible representation, the base oscillations introduced are different. Therefore, each G-let transform highlights the variations in the amplitude gradients of the signal module differently, thus creating a multiresolution analysis. Further, it may be noted that none of the approximations discussed in Chapter 3 appear in this analysis because it is performed in the time-amplitude domain rather than time-frequency domain.
The relationship between G-let oscillations and the G-let blocks of a G-let transform is defined by the choice of the transformation group for analyzing the signal module. The design of G-let transforms using the irreducible representations of a transformation group, determines the characteristics of G-let oscillations as stated in the theorem below.

**Theorem 4.2.1.** A linear transformation matrix $T(R)$ of a vector space $L_n$ has its diagonal blocks as irreducible representations $IR_1$, $IR_2$, $\cdots$ $IR_{(n+z)/2}$, and produces oscillations $O(s)$ with one oscillation per G-let block, $\mu_r$, when transforming a $n$-tuple signal.

$$O(s) = \mu_1 \oplus \mu_2 \oplus \cdots \oplus \mu_{(n+z)/2}$$

where

$$z = \begin{cases} 3 & \text{if } n \text{ is odd}, \\ 6 & \text{if } n \text{ is even} \end{cases} \quad (4.2.12)$$

The operation $\oplus$ in the above equation shows that the R.H.S is a linear combination of the irreducible representations $\mu_i$.

**Corollary 4.2.1.** In a dihedral G-let transform $D(R)$, the oscillations $O(s)$ happen at alternate values due to the non-trivial two-dimensional irreducible representations.

**Proof.** The irreducible representations are $r \times r$ matrices. Since matrices can be treated as a function mapping from one basis to another in a vector space, each block is a small function $g_r(x)$. A plot of this function may be termed as a base curve. An array of these functions forms the transform and hence a transform creates a repeating base curve or wave pattern like this:

$$T = g_1 \oplus g_1 \oplus \cdots \oplus g_{(n+z)/2} \quad (4.2.13)$$

The G-let is given by

$$G(x) = T \ast S(x) \quad (4.2.14)$$

where $S(x)$ is the signal and $\ast$ indicates the group action of $G$ on $S(x)$. The oscillations in the G-let will then simply depend on the dimension $r$ of the irreducible representations that form blocks.
**Illustration** The effect of transforming a signal by a G-let transform modifies the respective amplitudes of the repeating wave pattern. This can be tested using a simple signal. Let the signal be a straight line. Then a transformation by the G-let transform will show the natural wave pattern of the G-let transform. Considering a dihedral rotation $R(\theta)$, the number of irreducible representations is given by $\theta(n+z)/2$ for an $n$-tuple signal with $z = 3$ for odd and $z = 6$ for even $n$ respectively. Each irreducible representation is either one- or two-dimensional. Disregarding trivial one-dimensional identity representations, the transform has two-dimensional irreducible representations as the diagonal blocks. Therefore the transformation of a straight line $S$ by the transform is calculated to be $R(\theta) \times S'$ with

$$R(\theta) = \begin{pmatrix}
\cos \theta & -\sin \theta & \cdots & 0 \\
\sin \theta & \cos \theta \\
\vdots & \ddots & \ddots & \vdots \\
\cdots & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix} \quad (4.2.15)$$

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad (4.2.16)$$

$$R(\theta) \times S' = \begin{pmatrix}
\cos \theta - \sin \theta \\
\sin \theta + \cos \theta \\
\vdots \\
\cos \theta - \sin \theta \\
\sin \theta + \cos \theta
\end{pmatrix} = \begin{pmatrix} a & b & a & b & \cdots & a & b & a & b \end{pmatrix}' \quad (4.2.17)$$

where $a = \cos \theta - \sin \theta$ and $b = \sin \theta + \cos \theta$.

The resultant vector is found to have same alternating values since signal $S$ is a straight line, one is a difference and the other a sum. This will generate a triangular wave with width $w = 2$ as the repeating wave pattern. For any signal $S$, its amplitudes modify the triangular wave of the rotation matrix as $R(\theta) \times S'$. This is shown in Fig. 4.2 and Fig. 4.3 giving G-let coefficients $G(x)$ as

$$G(x) = R(\theta) \times \sum x_i \delta_{ij} \quad (4.2.18)$$

where the signal $S = \sum x_i \delta_{ij}$ and $x_i$ denotes amplitudes of the signal and $\delta_i$ denote the unit vectors.
For a signal shown in Fig. 4.2(a), the G-let oscillations are demonstrated. How each G-let transformation of the dihedral group creates a wave pattern using an angle \( \theta \) to be imposed on the signal during a transformation is shown in Fig. 4.2. Each G-let transformation of the dihedral group generates two wave patterns depending on the dimension of the irreducible representations. Since the non-trivial irreducible representations are two-dimensional, there are two patterns in the transformation subject to odd and even positions in the signal. The G-let oscillations are a sum of these two wave patterns superimposed on the signal. For each G-let transformation at a specific angle \( \theta \), the wave pattern, the superimposition of the wave pattern on the signal, and the G-let oscillations are shown in the Fig. 4.2.

At the angle \( \theta = \pi/6 \) the odd and even wave patterns are very close in their magnitudes. The sum of the projections of the wave pattern on the signal shows the corresponding G-let oscillations. It is also seen that the odd wave pattern is positive and the even pattern is negative. When the angle increases to \( \theta = \pi/2 \), it is seen that the magnitudes of the odd wave pattern...
Figure 4.3 G-let Oscillations: (a) Wave Patterns for $\theta = \pi/2$ (b) Wave Pattern imposed on Signal (c) G-let Oscillations (d) Wave Patterns for $\theta = 2\pi/3$ (e) Wave Pattern imposed on Signal (f) G-let Oscillations (g) Wave Patterns for $\theta = 3\pi/4$ (h) Wave Pattern imposed on Signal (i) G-let Oscillations

decrease and stay negative, while that of even wave pattern increase and become high positive values. For an angle $\theta = 2\pi/3$, the odd wave pattern has negative high values and the even wave pattern has positive small values. For an angle $\theta = 3\pi/4$, only the even wave pattern exists with maximum positive values. The odd wave pattern in very close to zero (not exactly zero). This particular G-let transformation is found to be important because it shows the maximum G-let amplitudes. Such a G-let transformation that produces only one prominent wave pattern is called a special G-let transformation and the angle at which this happens is $\theta = 3\pi/4$. Other than this G-let transformations, only one wave pattern may also be seen at angles $\pi/4$, $5\pi/4$ and $7\pi/4$. These four G-let transformations play a special role in many aspects of G-let analysis which is discussed at relevant places throughout the current work.

The oscillations produced by G-let transforms actually act as filters for amplitude gradients.
Each G-let transform is thus an amplitude gradient filter. The G-let oscillations highlight a set of amplitude gradients in the signal, thereby generating a different (G-let amplitude gradient) resolution $A(t)$ of the signal in the time-amplitude domain.

**Theorem 4.2.2.** The G-let transform operating on a signal produces multiresolution analysis of local signal amplitudes and amplitude gradients.

**Proof.** Consider a two-tuple signal $[y_1, y_2]$ corresponding to the times $[t \Delta t]$. Mathematically they are related as $y_2 = y_1 + (\tan \alpha) \Delta t$, where $\tan \alpha$ is the slope of the signal at $y_1$ and $\Delta t$ is the sampling time. Applying a two-dimensional rotation transform $R_\theta$ on the signal,

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

the signal is mapped to the changed coordinates to obtain,

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} (-\sin \theta) \Delta t (\tan \alpha) \\ (\cos \theta) \Delta t (\tan \alpha) \end{pmatrix}$$

It is seen that the group action by a two-dimensional irreducible representation for rotation results in an outcome proportional to

- Nature and choice of irreducible representation (for example, dihedral group and choice of $\theta$)
- Local amplitude ($y_1$)
- Local amplitude gradient ($\tan \alpha$)
- Sampling rate ($\Delta t$)
4.2.2 G-let Amplitude Gradients

A G-let transform modifies the amplitudes of a signal as shown in Fig. 4.2. The amplitude of the signal at a point appears low or high in the G-let oscillations produced by G-let transforms. Comparing the amplitudes that are handled by odd wave pattern of all G-let transforms, it is found that the amplitude at any point first decreases gradually and then increases (or vice versa). Simultaneously an amplitude of the even wave patterns first increases slowly and then decreases (or vice versa). As shown in the G-let oscillations, it is the high amplitude gradient points that change very little. Therefore looking at the amplitudes of odd wave pattern across G-let transforms, some amplitudes change faster and some have slow changes. This generates what is called an amplitude gradient resolution. For special G-let transforms, this amplitude gradient resolution is the lowest because one of the wave patterns is almost zero and most of the information of the signal is carried by one wave pattern only.

Thus the amplitude gradient resolution in the time-amplitude domain varies from the highest resolution in the first G-let transform to lower resolutions by the consecutive G-let transforms(filters) in terms of one wave pattern. Instead, if the other wave pattern alone is considered across the G-let transforms, then the reverse happens. Starting from lowest resolution, amplitude gradient resolution becomes higher and better with every G-let filter. If the amplitude gradient resolution has a decreasing tendency, the details of the signal settle down in the amplitude gradients of the other wave pattern not in consideration. Since the G-let oscillations are defined by the irreducible representations and scaled by the angle of transformation, there exists a relationship between the angle of a G-let transform and the corresponding amplitude gradient resolution. The amplitude gradient resolution in the time-amplitude domain is portrayed using two signals in Fig. 4.4. The two signals have different amplitude gradients. The signal in Fig. 4.4(a) has a fast-changing amplitude gradient and that in Fig. 4.4(b) has a slow-changing amplitude gradient. The G-let oscillations show a different set of amplitude gradient resolutions for each signal in Fig. 4.4(c), Fig. 4.4(d), Fig. 4.4(e) and Fig. 4.4(f), and Fig. 4.4(g), Fig. 4.4(h), Fig. 4.4(i) and Fig. 4.4(j), respectively.

To further illustrate the effect of G-let oscillations on the different amplitude gradients, two extreme types of 30-tuple signals are considered. The first signal is \( S_1 = [1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1] \). The wave patterns for the signal at angle \( \theta = 180\)
and \( \theta = 5\pi/4 \) are given in Fig. 4.5. In this signal the only interesting position is the highest amplitude at position 15. This point is seen as the maximum in the first G-let amplitude resolution and minimum in the second amplitude resolution. The second signal is \( S_2 = [1 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8 2.9 3.0 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8 3.9 4.0 4.1 4.2 4.3 4.4 4.5 4.6 4.7 4.8 4.9 5.0] \), where the amplitude variation is exactly the same throughout the signal. The wave patterns for the same G-let transforms \( \theta \) as above are given in Fig. 4.6. It may be noticed that since \( n = 30 \) for both the signals, the odd and even wave patterns at each angle is the same for both the signals. Further, the amplitude gradients are reflected in the G-let oscillations of both the signals. The highest amplitude gradient in the first signal is preserved in both its G-let oscillations. For the second signal, since the amplitude differences are the same throughout the signal the G-let oscillations do not highlight any particular point in the signal. \( \square \)
4.3 G-lets Analysis of 1-D Signals

4.3.1 Basic 1-D Signals

Some of the basic 1-D signals, each with $n = 26$, and their $1^{st}$ and $7^{th}$ G-let transform coefficients are demonstrated separately. The sine signal and its G-let coefficients are shown in Fig. 4.7. It is seen that G-let oscillations in the $1^{st}$ G-let is seen only in smooth amplitude gradient portions of the signal. The angle $\theta$ in this case is given by $(360/n)*1 = 13.8462$. In the $7^{th}$ G-let transform across the signal increases along with the height of the oscillations. The angle $\theta$ in this case is given by $(360/n)*1 = 96.9231$. The highest amplitude gradient of the signal is position 13, which is the middle of the sine signal. The G-let oscillations do not affect this point in the signal. The same is the case with the beginning and the end of the signal which have higher amplitude gradients than the other portions of the signal. In the $10^{th}$ G-let transform shown in Fig. 4.13(a), the G-let oscillations have the maximum height since the $\theta$ is approximately close to the angle $3\pi/4$ which is induced by the wave pattern seen in 4.5(d).

For a square signal the G-let oscillations are shown in Fig. 4.8. The amplitude gradients are maximum in the middle and the end of the signal period. Accordingly, the G-let oscillations are found to begin in the no amplitude gradient region in the $1^{st}$ G-let transform and increase in the same places in the $7^{th}$ G-let transform. The middle flat portion of the signal is not
oscillating. The square signal is specifically interesting, since there is a single discontinuity in the signal. This discontinuity is clearly captured by G-let oscillations. The sawtooth signal and G-let coefficients are shown in Fig. 4.9. The amplitude gradient has a high point in the center of the signal. The G-let oscillations are found to increase for every G-let transform away from this point. The increase is in the height of the oscillations. A dirichlet signal along with its G-lets are shown in Fig. 4.10. The signal has a greater amplitude gradient in the 2nd position. Therefore the G-let oscillations start at the smaller amplitude gradients seen in the mid region of the signal. It may be noticed that the amplitude is at its high in this center point of the signal. But the oscillations begin here since the amplitude gradient is smooth in this region.

An example for periodic and aperiodic signals are the sinc and chirp signals. The sinc signal is shown in Fig. 4.11 and a chirp signal is shown in Fig. 4.12. In the case of the sinc signal, there is a gradual change of amplitudes uniformly throughout the signal. The amplitude changes are sharp in the beginning of the curve and the end. At these portions the amplitude gradients are high. Therefore the G-let oscillations are taller in the large curve in the middle portion of the signal in the 1st G-let tranform. In the 7th G-let tranform the oscillations grow taller in the middle and are short in the head and tail ends of the curve.

For the chirp signal, the trend of the amplitude changes is different in different parts of the signal. Initially the amplitude changes are gradual but they are faster towards and from the middle of the signal. The linear continuous change in the amplitude gradients is reflected in
the G-let oscillations. It may be seen in the 1st G-let transform that there are very few and short oscillations in the second half of the signal which has high gradients. In the 7th G-let transform, the height of the oscillations have increased. There are portions of the signal where the oscillations are not significant. There are two points in the signal where the trend of the amplitude gradients change from slow to fast, and from fast to faster. At these points only the oscillations break. The height of the oscillations increase in these areas. The changing amplitude gradients are shown by the height of oscillations in a signal. Though this is the nature of the G-let oscillations, it is not a trivial result. The power of these oscillations may immediately be apparent if one were to use them to extract time-frequency information in the signal. This is performed in Chapter 5. For a fast changing monocomponent non-stationary signal, the trend in amplitude changes is captured by the G-let oscillations. The result has significant implications in analyzing all signals of this type, especially as frequency subbands. A signal may be decomposed into the regions of similar amplitude gradients. Especially using the G-let transforms at an angle $\theta$ which produces the wave pattern shown in Fig. 4.5(d), the G-let oscillations of the chirp signal is given in Fig. 4.12(a). In this special case, the amplitude gradient breakpoints are clearly identified and the oscillations show the fast-changing attitude by an increase in their height across the signal.

### 4.3.2 G-let transforms for a 6-tuple

The G-let transform generated for a signal may be shown using a small signal using dihedral groups. Consider a 6-tuple $\text{Sig} = [0.2 \ 0.3 \ 0.1 \ 1.2 \ 0.1 \ 0.6]$. Since $n = 6$ (even), the number of irreducible representations is $(n + 6)/2 = 6$. There are four one-dimensional representations and two two-dimensional representations which are given by the following matrices generated from the dihedral group $D_6$. They are produced from type I representation matrices. There are 6 rotation G-let transforms and 6 reflection G-let transforms. The sixth and twelfth transforms are identity and its reflection, they are not shown here. The G-let oscillations are produced by these G-let transforms for the signal. The G-let transforms shown here include both the one and two-dimensional irreducible representations. Since the signal has only $n = 6$, the one-dimensional irreducible representations are ignored in order to generate G-let oscillations. The one-dimensional irreducible representations anyways do not produce any change in the signal. Again for a signal which is odd dimensional, one one-dimensional irreducible representation is
unavoidable. Either the signal is padded with a suitable value to make it an even signal or the one-dimensional irreducible representation may be put to use to match the dimension of the signal which is transformed by the G-let transforms.

\[
R_1 = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5000 & -0.8660 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.8660 & 0.5000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5000 & -0.8660 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.8660 & 0.5000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5000 & -0.8660 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.8660 & 0.5000
\end{bmatrix}
\] (4.3.1)

\[
R_2 = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.5000 & -0.8660 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.8660 & -0.5000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.5000 & -0.8660 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.8660 & -0.5000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.5000 & -0.8660 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8660 & -0.5000
\end{bmatrix}
\] (4.3.2)

\[
R_3 = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1.0000 & -0.0000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0000 & -1.0000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1.0000 & -0.0000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0000 & -1.0000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1.0000 & -0.0000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0000 & -1.0000
\end{bmatrix}
\] (4.3.3)

\[
R_4 = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.5000 & 0.8660 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.8660 & -0.5000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.5000 & 0.8660 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.8660 & -0.5000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.5000 & 0.8660 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.8660 & -0.5000
\end{bmatrix}
\] (4.3.4)

\[
R_5 = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5000 & 0.8660 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.8660 & 0.5000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5000 & 0.8660 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.8660 & 0.5000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5000 & 0.8660 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8660
\end{bmatrix}
\] (4.3.5)
\[
R_7 = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.5000 & 0.5000 & -0.8660 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.8660 & -0.5000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] (4.3.6)

\[
R_8 = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.5000 & -0.8660 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.8660 & 0.5000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] (4.3.7)

\[
R_9 = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.0000 & -1.0000 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.0000 & 1.0000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] (4.3.8)

\[
R_{10} = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.5000 & 0.8660 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5000 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.5000 & 0.8660 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] (4.3.9)

\[
R_{11} = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.5000 & 0.8660 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5000 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.5000 & 0.8660 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] (4.3.10)

From the G-let coefficients of Table 4.2, the signal ‘Sig’ may be reconstructed as discussed in the reconstruction section later in this chapter.
Table 4.2 G-let Coefficients of Signal ‘Sig’ with n = 6

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>0.2000</td>
<td>-0.1598</td>
<td>-0.3598</td>
<td>-0.2000</td>
<td>0.1598</td>
<td>0.3598</td>
<td>-0.1598</td>
<td>-0.3598</td>
<td>-0.2000</td>
<td>0.1598</td>
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</tr>
<tr>
<td>0.3000</td>
<td>0.3232</td>
<td>0.0232</td>
<td>-0.3000</td>
<td>-0.3232</td>
<td>-0.0232</td>
<td>0.3232</td>
<td>0.0232</td>
<td>-0.3000</td>
<td>-0.3232</td>
<td>-0.0232</td>
</tr>
<tr>
<td>0.1000</td>
<td>-0.9892</td>
<td>-1.0892</td>
<td>-0.1000</td>
<td>0.9892</td>
<td>1.0892</td>
<td>-0.9892</td>
<td>-1.0892</td>
<td>-0.1000</td>
<td>0.9892</td>
<td>1.0892</td>
</tr>
<tr>
<td>1.2000</td>
<td>0.6866</td>
<td>-0.5134</td>
<td>-1.2000</td>
<td>-0.6866</td>
<td>0.5134</td>
<td>0.6866</td>
<td>-0.5134</td>
<td>-1.2000</td>
<td>-0.6866</td>
<td>0.5134</td>
</tr>
<tr>
<td>0.1000</td>
<td>-0.4696</td>
<td>-0.5696</td>
<td>-0.1000</td>
<td>0.4696</td>
<td>0.5696</td>
<td>-0.4696</td>
<td>-0.5696</td>
<td>-0.1000</td>
<td>0.4696</td>
<td>0.5696</td>
</tr>
<tr>
<td>0.6000</td>
<td>0.3866</td>
<td>-0.2134</td>
<td>-0.6000</td>
<td>-0.3866</td>
<td>0.2134</td>
<td>0.3866</td>
<td>-0.2134</td>
<td>-0.6000</td>
<td>-0.3866</td>
<td>0.2134</td>
</tr>
</tbody>
</table>

4.3.3 ECG Signal

The G-let oscillations for a signal contains different amplitude gradient components is demonstrated, for example, an ECG signal. The ECG signal has dimension $1 \times 336$. Therefore $n = 336$. The conjugacy classes are given by $(336 + 6)/2 = 171$. There are 169 G-lets from all the classes. In other words, there are 167 two-dimensional irreducible representations and two one-dimensional representations adding up to $n = 334 + 2$, the actual dimension of the signal. The signal and some of its G-let coefficients are shown in Fig. 4.14. The G-let oscillations of each of these G-lets are also shown in Fig. 4.14. The 5th and 100th G-let transforms of the ECG signal are shown in Fig. 4.14(b) and Fig. 4.14(c), respectively. Thus different amplitude gradient resolutions for the signal is produced. The characteristics of the signal include some sharp amplitude gradient portions and some smooth amplitude gradient portions. The G-let oscillations are seen to be sensitive to such gradients. The height of oscillations for a smooth gradient is different from that for a sharp gradient for each G-let transform.

4.4 G-let Analysis of 2-D Signals

Consider the standard image ‘Lena’. The image has a dimension $n = 256$. The number of conjugacy classes is $(256 + 6)/2 = 131$. Four among these are identity representations. We get a count of 127 G-let transforms from all the classes. The ‘Lena’ image and its amplitude gradient resolution are shown in the figures Fig. 4.15(a), Fig. 4.15(b), Fig. 4.16(a), Fig. 4.16(b), Fig. 4.17(a) and Fig. 4.17(b). The lines seen in the image results are due to the two-dimensional irreducible representations used by the G-let transforms.
4.4.1 G-let Amplitude Multiresolution Analysis

The amplitude gradient resolution for 1-D and 2-D signals allow for a multiresolution analysis of the signals. The amplitude gradient resolution is generated without any loss of information in the signal. As shown in Fig. 4.15, Fig. 4.16, Fig. 4.17, Fig. 4.18 and Fig. 4.19, some of the amplitude gradient resolutions are interesting because of the high amplitude gradient regions being highlighted. The high amplitude gradient points are not changed by G-let oscillations and hence these points appear as an outline of the face in the image in the low resolution image in Fig. 4.17(a) and Fig. 4.19(a).

The G-let coefficients gradually reduce in details to edges and further down to points in the image which have very less depth in terms of 3D information. Before the image completely loses its points, it is found that the front layer of the image can be identified, and as the details gradually appear again, the outline of the image is obtained. The continuity in the points is a trend towards the growing outline of an object in the image whose limits are decided using a threshold. Depending on the threshold, the image may be separated as layers from the front to the back. In each layer, the limits of the outline are boxed and all the G-let coefficients are used to reconstruct that portion of the original image within the box. Every layer may be put together to get the collective image back.

G-let coefficients of some more images have been shown in Fig. 4.20, Fig. 4.21, Fig. 4.22, Fig. 4.23, Fig. 4.24, Fig. 4.25, and Fig. 4.26. Two G-let coefficients are shown for each image, one of them a resolution at $\theta = \pi/4$ and the other a different amplitude gradient resolution. It is repeatedly noticed in all the images that, for special amplitude gradient resolutions, the lowest resolution is obtained. At the special amplitude gradient resolution, the edges of the objects in the image, which indicate sharp amplitude gradient points in the image are seen.

4.5 Retrieving Signal from G-let Coefficients

Since the G-let coefficients are G-let transform coefficients, it is possible to rebuild the signal after any G-let transform individually or from multiple G-let transforms by a linear combination. It is also possible to build the signal from rotation G-let coefficients alone with dihedral filters. It is also possible to build the signal from reflection G-let coefficients alone with dihedral filters.
The signal can be reconstructed from every G-let transform, since each transform may be written as a linear combination of the G-let blocks. A linear sum of the G-let coefficients of ‘n’ rotation (or reflection) matrices also allow a perfect reconstruction of the discrete signal ‘sig’ as in the equation 4.5.2.

The signal is reconstructed from G-let coefficients from the two different transforms by:

\[
\text{sig} = (G_1(\text{sig}) \oplus G_2(\text{sig}) \oplus G_3(\text{sig}) \cdots \oplus G_{n-1}(\text{sig})) \times (-In) \quad (4.5.1)
\]

\[
\text{sig} = (G_{n+1}(\text{sig}) \oplus G_{n+2}(\text{sig}) \oplus G_{n+3}(\text{sig}) \cdots \oplus G_{n-1}(\text{sig})) \times (-In) \quad (4.5.2)
\]

Another possibility is that with just half the members of a conjugacy class that has 2 members, reconstruction may be done. For conjugacy classes containing reflections, one cannot randomly choose a member since they are more than two. It is possible that among the members ‘m’ of the reflection class, one can find the difference between the \(m^{th}\) G-let coefficients and the sum of the other \(m-1\) G-let coefficients in the class and use this difference to represent the specific conjugacy class of reflections. So the G-let transform one from each rotation class, along with the difference between two successive G-let tranforms in the reflection class, will give the original signal. Therefore effectively only one G-let tranform from each conjugacy class is needed. The original signal can be represented by simply only so many amplitude gradients or frequencies giving a lossless compression of approximately 50% in terms of reconstructing the signal from all G-let coefficients.

Also, during the reconstruction of a signal from G-let coefficients in time-amplitude domain, signal discontinuities are reproduced exactly and no Gibbs effect is observed.

**Theorem 4.5.1.** Signal discontinuities are preserved by oscillations.

**Corollary 4.5.1.** There is no Gibbs effect because discontinuities are not distorted.

**Illustration:** Signal discontinuities are found to be very short breaks or high frequency points. In dihedral groups, the G-let oscillations occur at alternate points in the signal. Short breaks in a signal will be seen as very few oscillations, typically just two. Such oscillations may be separated as high frequencies which may be easily separated. Since a finite signal is taken and only finite basis vectors or vector space \(L_n\) or finite matrix representations of vector space \(L_{n \times n}\)
Table 4.3 Rotation G-let Coefficients of Signal in Fig. 6.4(g)

<table>
<thead>
<tr>
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<td>0.3117</td>
<td>-0.1113</td>
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<td>-0.4505</td>
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<td>0.5000</td>
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<td>-0.4701</td>
<td>-1.0862</td>
<td>-0.8844</td>
<td>-0.0166</td>
<td>0.8637</td>
<td>1.0936</td>
<td>0.5000</td>
</tr>
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<td>1.0144</td>
<td>0.2649</td>
<td>-0.6840</td>
<td>-1.1179</td>
<td>-0.7100</td>
<td>0.2326</td>
<td>1.0000</td>
</tr>
<tr>
<td>1.0000</td>
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<td>-1.1974</td>
<td>-1.3349</td>
<td>-0.4671</td>
<td>0.7524</td>
<td>1.4053</td>
<td>1.0000</td>
</tr>
<tr>
<td>1.0000</td>
<td>1.4053</td>
<td>0.7524</td>
<td>-0.4671</td>
<td>-1.3349</td>
<td>-1.1974</td>
<td>-0.1583</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.5000</td>
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<td>-0.6674</td>
<td>-0.2335</td>
<td>0.3762</td>
<td>0.7027</td>
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</tr>
<tr>
<td>0.5000</td>
<td>0.7027</td>
<td>0.3762</td>
<td>-0.2335</td>
<td>-0.6674</td>
<td>-0.5987</td>
<td>-0.0792</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

is considered, reconstruction of a signal from finite G-let coefficients does not introduce any loss or approximation in the resultant signal. Thus there is no Gibbs effect. For the signal in Fig. 6.4(g), the G-let coefficients and the corresponding reconstruction is shown in Table. 4.3. Thus the reconstructed signal is obtained from G-let coefficients of the signal in Fig. 6.4(g) with \( n = 7 \) by

\[
Rec_S(x) = (Glet1 \oplus Glet2 \oplus Glet3 \oplus Glet4 \oplus Glet5 \oplus Glet6) \times (-1) \tag{4.5.3}
\]

### 4.6 Computational Complexity

In the time-amplitude domain, the size of G-let transforms is \( n \times n \) if the dimension of the signal is ‘n’. But only the diagonal G-let blocks are non-zero. Hence a sparse matrix multiplication contributes to a computational cost \( CC = \sum_{i=1}^{n} size_i (irreducible \ representation) \).

For dihedral groups, the size of each irreducible representation is \( 2 \times 2 = 4 \). Since the number of irreducible representations is given by \( (n+z)/2 \) where \( z = 3 \) for an odd tuple and \( z = 6 \) for even tuple, the computational cost is calculated as \( CC = (4^*(n+z)/2) \approx O(2n) \).

### 4.7 Compression

Using special G-let transforms which have only one prominent wave pattern either odd or even, a signal may be represented by just only the amplitudes of prominent wave pattern. This means the signals may be represented by just half the values for dihedral transformation groups. In case of \( 2 - D \) signals, the same may be done vertically and horizontally so that the resultant
signal is quarter the size of the original. The lowest resolution or the compressed form of a 1-D signal is shown using an ECG signal and that of a 2-D signal using ‘Lena’ image shown in the following Fig. 4.28 and Fig. 4.29 respectively. The ECG signal has $n = 1500$. As a result of the compression the information in the other wave pattern is lost. But this is very minimal. Reconstructing the signals from the compressed signals may be done by generating the coefficients of the other wave pattern very close to zero in only one direction for 1-D signals and two directions for $2 - D$ signals, and applying the corresponding inverse G-let transforms. Compression of a signal using G-let transforms is stated in the theorem below.

**Theorem 4.7.1.** *If a set of special amplitude gradient resolutions are chosen, then it is possible to compress G-let coefficients.*

*Proof.* From illustration 4.1, it is seen that for the angles $\pi/4$ and $3\pi/4$, one of the wave patterns, either positive or negative, are very close to zero. If these values are removed, the coefficients reduce to half their size. Since the G-let transform takes the signal module to an orthonormal basis of the vector space, the signal may be directly reconstructed from G-let coefficients by an inverse transform, by reinserting values into the same alternate places. The values should be very close to zero. Similar compression may be achieved at other angles by suitable G-let transforms, so as to make the alternate values close to zero. \( \square \)

### 4.8 Conclusions

A time-amplitude domain analysis of signals known as G-lets has been proposed in this chapter. This algorithm employs the members of any finite transformation group and their irreducible representations to construct operations on the signal (G-let transforms). Each G-let transform superimposes the signal on a base oscillatory signal that highlights a set of amplitude gradients in the signal. Thus, G-let transforms are shown to form a multiresolution analysis of the signal in terms of its amplitude gradients localized in time or spatial location, thus forming a set of high-pass filters. Using simple $1 - D$ signals, ECG signals, and images, this observation is
illustrated. It is shown that perfect reconstruction of the signal is possible with approximately 50% of the G-let coefficients. The algorithm has a low computational complexity of \( O(2^n) \). A simple method for compressing signals is also provided.

In the next chapter, the multiresolution analysis created for signals will be utilized to extract features of signals in the time-amplitude domain with specific focus on edge detection. Defining instantaneous frequency as the local amplitude gradient, an algorithm will be presented in the subsequent chapter (Chapter 6) to construct a time-frequency analysis of signals. The special resolution G-let transforms will be used to capture the highest frequency using a suitably defined dilation operation. The rest of the frequencies will be captured similarly. The ability of this time-frequency analysis to separate frequency subbands in monocomponent and multi-component signals will be discussed in Chapter 7. Finally, Chapter 8 will utilize the proposed time-frequency analysis to extract signal features in the time-frequency domain.
Figure 4.7 Sine Signal and its G-Let tranforms: (a) Original Sine signal (b) 1<sup>st</sup> G-let tranform (c) 7<sup>th</sup> G-let tranform

Figure 4.8 Square Signal and its G-Let Tranforms: (a) Original Square signal (b) 1<sup>st</sup> G-let tranform (c) 7<sup>th</sup> G-let tranform

Figure 4.9 Sawtooth Signal and its G-Let Tranforms: (a) Original Sawtooth signal (b) 1<sup>st</sup> G-let tranform (c) 7<sup>th</sup> G-let tranform

Figure 4.10 Dirichlet Signal and its G-Let Tranforms: (a) Original Dirichlet signal (b) 1<sup>st</sup> G-let tranform (c) 7<sup>th</sup> G-let tranform
Figure 4.11 Sinc Signal and its G-Let Tranforms: (a) Original Sinc signal (b) 1\textsuperscript{st} G-let tranform (c) 7\textsuperscript{th} G-let tranform

Figure 4.12 Chirp Signal and its G-Let Tranforms: (a) Original Chirp signal (b) 1\textsuperscript{st} G-let tranform (c) 7\textsuperscript{th} G-let tranform

Figure 4.13 Special G-let Tranforms: (a) Sine 10\textsuperscript{th} G-let tranform (b) Square 10\textsuperscript{th} G-let tranform (c) Chirp 10\textsuperscript{th} G-let tranform
Figure 4.14 ECG Signal and its G-Let Transforms: (a) Original ECG signal (b) 5th G-let transform (c) 100th G-let transform (d) 300th G-let transform

Figure 4.15 ‘Lena’ image - Lena Rotation G-let transforms: (a) Original Image (b) 10th G-let transform

Figure 4.16 Lena Rotation G-let transforms (a) 25th G-let transform (b) 89th G-let transform
Figure 4.17 Lena Rotation G-let transforms (a) $74^{th}$ G-let transform (b) $128^{th}$ G-let transform

Figure 4.18 Lena Reflection G-lets: (a) $14^{th}$ G-let (b) $41^{st}$ G-let

Figure 4.19 Lena Reflection G-lets (a) $66^{th}$ G-let (b) $74^{th}$ G-let
Figure 4.20 G-let transforms of ‘Wiberg’ Image: (a) Original Image  (b) $106^{th}$ G-let transform  (c) $146^{th}$ G-let transform

Figure 4.21 G-let transforms of ‘Building’ Image: (a) Original Image  (b) $140^{th}$ G-let transform  (c) $170^{th}$ G-let transform

Figure 4.22 G-let transforms of ‘Thophilis’ Image: (a) Original Image  (b) $110^{th}$ G-let transform  (c) $125^{th}$ G-let transform
Figure 4.23 G-let transforms of ‘Parakeet’ Image: (a) Original Image (b) 135\textsuperscript{th} G-let transform (c) 162\textsuperscript{nd} G-let transform

Figure 4.24 G-let transforms of ‘Lighthouse’ Image: (a) Original Image (b) 115\textsuperscript{th} G-let transform (c) 126\textsuperscript{th} G-let transform

Figure 4.25 G-let transforms of ‘Olipac’ Image: (a) Original Image (b) 16\textsuperscript{th} G-let transform (c) 189\textsuperscript{th} G-let transform
Figure 4.26 G-let transforms of ‘Amma’ Image: (a) Original Image  (b) 50\textsuperscript{th} G-let transform  (b) 96\textsuperscript{th} G-let transform

Figure 4.27 Signal to Demonstrate Reconstruction using G-lets
Figure 4.28 Compression of Long ECG Signal: (a) Signal (b) 1500th $G$-let (c) Compressed ECG Signal

Figure 4.29 Compression of ‘Lena’ Image: (a) G-let (b) Image reduced to half (c) G-let of Half Image (d) Image reduced to quarter