CHAPTER-7

OPTIMAL ORDERING POLICIES FOR DETERIORATING ITEMS WITH CONTROLLABLE DETERIORATION RATE AND TIME DEPENDENT QUADRATIC DEMAND*

7.1 INTRODUCTION

Operations Research (OR) addresses the process of decision making in business enterprises and industries. It is known that the inventory management system is one of the important field of study in OR. The study of deteriorating items in inventory system has gained the attention of many researchers in this area of research. The study the inventory of deteriorating items was opened up by Within (1957). In his study, he considered the deterioration of fashion goods at the end of prescribed storage period. Ghare and Schrader (1963) extended the classical EOQ formula with exponential decay of inventory due to deterioration and gave a mathematical model of inventory of deteriorating items. The literature is replete with inventory models for deteriorating items on the basis of demand variations and various other conditions or constraints.

The rate of deterioration is faster in some products which causes loss to the retailer. The life span of such products can be increased using some preservatives. This rate of deterioration of items can be controlled using some preservation technology which reduces the deterioration rate thereby the retailer may increase the profit. The deterioration rate of items in the literature is viewed as an exogenous variable which is not controllable. In practice, the deterioration rate of products can be controlled and reduced through various efforts such as procedural changes and specialized equipment acquisition. The consideration of preservation technology is important due to rapid social changes and the fact that preservation technology can reduce the deterioration rate significantly. By the efforts of investing in preservation technology we can reduce the deterioration rate. Mishra (2013) studied an inventory model for deteriorating items with controllable deterioration rate for time dependent demand and holding cost.

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All these works were based on the assumption that the demand rate is either linear or exponential function of time. Hence, in this chapter, it is proposed to study the effect of preservation technology for deteriorating items while developing an inventory model with time dependent quadratic demand and time dependent holding cost. Section 7.2 contains the notations and assumption required developing the model. In section 7.3, the model and its solution are given. It also contains the numerical examples. Section 4 contains the conclusions of the model.

7.2 ASSUMPTION AND NOTATIONS

Following Mishra (2013), we use the following notations and assumptions to develop the inventory model of the system:

7.2.1 Notations

(i) $A$ denotes the ordering cost per order.
(ii) $C_1$ is the purchase cost per unit.
(iii) $h(t)$ is the inventory holding cost per unit per time unit.
(iv) $\pi_b$ denotes the backordered cost per unit short per time unit.
(v) $\pi_L$ is the cost of lost sales per unit.
(vi) Preservation technology (PT) cost is denoted by $\xi$ which reduces the deterioration rate in order to preserve the product, $\xi > 0$.
(vii) The deterioration rate is $\theta$.
(viii) $m(\xi)$ is the reduced deterioration rate due to preservation technology.
(ix) $\tau_p$ resultant deterioration rate, $\tau_p = \theta - m(\xi)$.
(x) $t_1$ the time at which the inventory level reaches zero, $t_1 \geq 0$.
(xi) $t_2$ the length of period during which shortages are allowed, $t_2 \geq 0$.
(xii) $T (= t_1 + t_2)$ the length of cycle time.
(xiii) The maximum inventory level during $[0, T]$ is denoted by $IM$.
(xiv) $IB$ denotes the maximum inventory level during shortage period.
(xv) $Q (= IM + IB)$ the order quantity during a cycle of length $T$.
(xvi) $I_1(t)$ the level of positive inventory at time $t$, $0 < t < t_1$.
(xvii) $I_2(t)$ the level of negative inventory at time $t$, $t_1 < t < t_1 + t_2$.
(xviii) $TC(t_1, t_2, \xi)$ the total cost per time unit.
7.2.2 Assumptions

(i) The selling rate \( D(t) \) at time \( t \) is assumed to be \( D(t) = a + bt + ct^2 \), \( a \geq 0, b \neq 0, c \neq 0 \).

Here ‘\( a \)’ is the initial rate of demand, ‘\( b \)’ is the initial rate of change of the demand and ‘\( c \)’ is the acceleration of demand rate.

(ii) Holding cost \( h(t) \) is linear function of time i.e., \( h(t) = \alpha + \beta t \), \( \alpha \geq 0, \beta \geq 0 \)

(iii) Shortages are allowed and partially backlogged.

(iv) The lead time is zero.

(v) The replenishment rate is infinite.

(vi) The planning horizon is finite.

(vii) The deterioration rate is constant.

(viii) During stock out period, the backlogging rate is variable and is dependent on the length of the waiting time for next replenishment. So that the backlogging rate for negative inventory is, \( B(t) = \frac{1}{1 + \delta(T-t)} \), \( \delta \) is backlogging parameter and \( (T-t) \) is waiting time \( (t_1 < t < T) \).

7.3 FORMULATION AND SOLUTION OF THE MODEL

The objective of the model is to identify the effect of the preservation technology on the total cost of the system for deteriorating items with time dependent quadratic demand rate. Let \( I(t) \) be the inventory level at any time ‘\( t \)’. The rate of change of inventory occurs due to demand and resultant deterioration rate \( (\tau_p) \) and due to demand and a fraction of demand is backlogged with backlogging rate is \( B(t) \) during \( [t_1, T] \). Hence, the inventory level at any time during \( [0, t_1] \) and during \( [t_1, T] \) is governed by the following differential equations

\[
\frac{dI_1(t)}{dt} + \tau_p I_1(t) = -(a + bt + ct^2), \quad 0 \leq t \leq t_1; \quad \tau_p = \theta - m(\xi). \quad (7.1)
\]

\[
\frac{dI_2(t)}{dt} = -(a + bt + ct^2) \frac{1 + \delta(T-t)}{1 + \delta(T-t)}, \quad t_1 \leq t \leq T; \quad T = t_1 + t_2 \quad (7.2)
\]

with boundary conditions

\( I_1(t) = I_2(t) = 0 \) at \( t = t_1 \) and \( I_1(t) = IM \) at \( t = 0 \)

Using the boundary conditions, the solutions of equation (7.1) & (7.2) are
\[ I_1(t) = a(t_1 - t) + (a(\theta - m(\xi))) + b\left(\frac{t_1^2}{2} - \frac{t^2}{2}\right) + (b(\theta - m(\xi))) + c\left(\frac{t_1^3}{3} - \frac{t^3}{3}\right) + \]
\[ c(\theta - m(\xi))\left(\frac{t_1^4}{4} - \frac{t^4}{4}\right) - ((\theta - m(\xi))t)\left[a(t_1 - t) + b\left(\frac{t_1^2}{2} - \frac{t^2}{2}\right) + c\left(\frac{t_1^3}{3} - \frac{t^3}{3}\right)\right] \]  
(7.3)

\[ I_2(t) = \ln\left(\frac{\delta t_1 - \delta t + \delta t_2 + 1}{\delta t_2 + 1}\right)\left(\frac{x_1}{\delta^3}\right) + \frac{(t - t_1)(c + \delta b + \delta c t_1 + \delta c t_2)}{\delta^2} + \frac{c}{2\delta}\left(t^2 - t_1^2\right) \]  
(7.4)

where
\[ x_1 = c\delta^3 t_1^2 + 2c\delta^3 t_1 t_2 + b\delta^3 t_1 + c\delta^3 t_2^2 + b\delta^3 t_2 + a\delta^2 + 2c\delta t_1 + 2c\delta t_2 + b\delta + c \]

Inventory Level (Q(t))

Thus the total cost per replenishment cycle consists of the following components:
The ordering cost is given by
\[ OC = A \]  
(7.5)

Inventory holding cost per cycle is given by
\[ IHC = \int_0^{t_1} h(t)I_1(t)dt = \int_0^{t_1} (\alpha + \beta t)I_1(t)dt \]
\[
\begin{align*}
\frac{a\beta t_1^3}{6} + \frac{a\alpha t_1^2}{2} + \frac{b\beta t_1^4}{8} + \frac{b\alpha t_1^3}{3} + \frac{c\beta t_1^5}{10} + \frac{c\alpha t_1^4}{4} \\
= -m(\xi) \left( \frac{a\alpha t_1^3}{6} + \frac{b\beta t_1^4}{8} + \frac{b\alpha t_1^3}{3} + \frac{c\beta t_1^5}{10} \right) \\
+ (\theta) \left( \frac{a\beta t_1^3}{24} + \frac{a\alpha t_1^2}{6} + \frac{b\beta t_1^4}{8} + \frac{b\alpha t_1^3}{3} + \frac{c\beta t_1^5}{10} \right)
\end{align*}
\]

(7.6)

The Back ordered cost per cycle is given by

\[
BC = \pi_b \left( \int_{t_1}^{t_1+\tau} -I_R(t) dt \right)
\]

\[
= \pi_b \left[ \ln \left( \frac{1}{\delta^2} + 1 \right) \left( \frac{c(t_1 + t_2)^2 + b(t_1 + t_2) + a}{\delta^2} + \frac{2c(t_1 + t_2) + b}{\delta^3} + \frac{c}{\delta^4} \right) \right]
\]

\[
+ \left\{ \frac{t_2 \delta t_2}{\delta^2} \left( b + c t_1 + \frac{3c t_2}{2} + \frac{2c}{\delta} - \frac{ct_2}{\delta} \right) \right. \\
+ \left. \left( \frac{t_2^2 \delta t_2}{2} + \frac{b}{2} + \frac{3c}{2\delta} - \frac{ct_2}{\delta} \right) \left( a + \frac{b}{\delta} + \frac{c}{\delta^2} \right) \right\}
\]

(7.7)

The Lost sales cost per cycle is given by

\[
LS = \pi_L \left( \int_{t_1}^{t_1+\tau} \left( -\frac{1}{1 + \delta(t_1 + t_2 - t)} \left( a + bt_1 + ct_1^2 \right) \right) dt \right)
\]

\[
= \pi_L \left[ \ln \left( \frac{1}{\delta^2} + 1 \right) \left( \frac{c(t_1 + t_2)^2 + b(t_1 + t_2) + a}{\delta^2} + \frac{2c(t_1 + t_2) + b}{\delta^3} + \frac{c}{\delta^4} \right) \right]
\]

\[
+ \left\{ \frac{(t_2) (c + \delta b + \delta c t_1 + \delta^2 t_2)}{\delta} + \frac{c((t_1 + t_2)^2 - t_1^2)}{2\delta} \right\}
\]

(7.8)

The Purchase cost per cycle = (purchase cost per unit) * (Order quantity in one cycle)

i.e., \(PC = C_r Q\), where \(Q = IM + IB\)

when \(t = 0\) the level of inventory is maximum and it is denoted by \(IM (= I_1 (0))\) then from the equation (7.3), we have

\[
IM = at_1 + \left( a(\theta - m\xi) + b \right) \left( \frac{t_1^2}{2} \right) + \left( b(\theta - m\xi) + c \right) \left( \frac{t_1^3}{3} \right) + \left( c(\theta - m\xi) \right) \left( \frac{t_1^4}{4} \right)
\]
The maximum backordered inventory is obtained at $t = t_1 + t_2$, then from the equation (7.7)

$$IB = -I_2(t_1 + t_2)$$

$$IB = (-1)^{\frac{1}{\delta + 1}} \left[ \ln \left( \frac{c(t_1 + t_2)^2 + b(t_1 + t_2) + a}{\delta} + \frac{2c(t_1 + t_2) + c}{\delta^2} + \frac{c}{\delta^3} \right) 
+ t_2(c + b\delta + c\delta t_1 + c\delta t_2) + \frac{c(t_1 + t_2)^2 - t_1^2}{2\delta} \right]$$

But

$$Q = IM + IB$$

$$Q = a t_1 + (a(\theta - m\xi) + b) \left( \frac{t_1^2}{2} \right) + (b(\theta - m\xi) + c) \left( \frac{t_1^3}{3} \right) + c(\theta - m\xi) \left( \frac{t_1^4}{4} \right)$$

$$= -\ln \left( \frac{c(t_1 + t_2)^2 + b(t_1 + t_2) + a}{\delta} + \frac{2c(t_1 + t_2) + c}{\delta^2} + \frac{c}{\delta^3} \right)$$

$$- \frac{t_2(c + b\delta + c\delta t_1 + c\delta t_2)}{\delta^2} - \frac{c(t_1 + t_2)^2 - t_1^2}{2\delta}$$

(7.9)

Thus the purchase cost per cycle is given by

$$PC = C_1Q$$

$$PC = a t_1 + (a(\theta - m\xi) + b) \left( \frac{t_1^2}{2} \right) + (b(\theta - m\xi) + c) \left( \frac{t_1^3}{3} \right) + c(\theta - m\xi) \left( \frac{t_1^4}{4} \right)$$

$$= -\ln \left( \frac{c(t_1 + t_2)^2 + b(t_1 + t_2) + a}{\delta} + \frac{2c(t_1 + t_2) + c}{\delta^2} + \frac{c}{\delta^3} \right)$$

$$- \frac{t_2(c + b\delta + c\delta t_1 + c\delta t_2)}{\delta^2} - \frac{c(t_1 + t_2)^2 - t_1^2}{2\delta}$$

(7.10)

The total cost per unit time is given by

$$TC(t_1, t_2) = \frac{1}{t_1 + t_2} \left[ \text{Ordering cost} + \text{Carrying cost} + \text{Backordering cost} + \text{ lost sale cost} + \text{purchasecost} \right]$$

$$= \frac{1}{t_1 + t_2} \left[ \text{OC} + \text{IHC} + \text{BC} + \text{LS} + \text{PC} \right]$$
Differentiating equation (11) with respect to $t_1$, $t_2$ and $\xi$, we get

$$
\frac{\partial TC}{\partial t_1}, \frac{\partial TC}{\partial t_2} and \frac{\partial TC}{\partial \xi}
$$

The optimal values of $t_1$, $t_2$ and $\xi$ can be obtained by solving the following equations in order to minimize the total cost $TC(t_1, t_2, \xi)$ per unit time:

$$
\frac{\partial TC}{\partial t_1} = 0, \quad \frac{\partial TC}{\partial t_2} = 0 and \quad \frac{\partial TC}{\partial \xi} = 0
$$
provided the determinant of principal minor of hessian matrix (H-Matrix) of $TC (t_1, t_2, \xi)$ is positive definite. i.e., $\det (H_1) >0, \det (H_2) >0, \det (H_3) > 0$, where $H_1, H_2, H_3$ are the principal minors of the H-Matrix.

The Hessian matrix of the total cost $TC (t_1, t_2, \xi)$ is given by

$$H = \begin{bmatrix}
\frac{\partial^2 TC}{\partial t_1^2} & \frac{\partial^2 TC}{\partial t_1 \partial t_2} & \frac{\partial^2 TC}{\partial t_1 \partial \xi} \\
\frac{\partial^2 TC}{\partial t_2 \partial t_1} & \frac{\partial^2 TC}{\partial t_2^2} & \frac{\partial^2 TC}{\partial t_2 \partial \xi} \\
\frac{\partial^2 TC}{\partial \xi \partial t_1} & \frac{\partial^2 TC}{\partial \xi \partial t_2} & \frac{\partial^2 TC}{\partial \xi^2}
\end{bmatrix}$$

### 7.3.1 Numerical Example

To illustrate the model, an inventory system is considered with the following parameter in proper units:

- $A = 250$
- $a = 30$
- $b = 10$
- $c = 0.05$
- $\theta = 0.01$
- $C_1 = 20$
- $\alpha = 2$
- $\beta = 0.1$
- $\pi_b = 16$
- $\pi_L = 12$
- $\delta = 4$
- $\xi = 0.5$

and $m(\xi) = \theta(1 - e^{-0.01\xi})$.

The output of the program by using the MATHCAD is $t_1 = 0.2936, t_2 = 1.7884, \xi = 5.1065$ and $TC(t_1, t_2, \xi) = 1071.3$. i.e. the value of $t_1$ at which the inventory level become zero is 0.2936 unit time, shortage period is 1.7884 unit time and the optimal value of preservation technology cost is 5.1065 per unit.

We have compared this solution with time dependent linear demand i.e., $c=0$. In the above numerical example, we have taken $c=0$. Thus we obtain $t_1 = 0.2898, t_2 = 1.8151, \xi = 5.1984$. $TC(t_1, t_2, \xi) = 1069.4$. It is also noted that a marginal variation is found in case of preservation technology cost and the total cost of the system in both the models (i.e., time dependent linear demand model and time dependent quadratic demand model). Thus it may be concluded that the two models exhibit similar characteristics when the deterioration rate is controllable by preservation technology.
7.4 CONCLUSION
In this chapter we developed deterministic inventory model for deteriorating items with shortages using preservation technology for deteriorating items. The deterministic demand rate is assumed to be a quadratic function of time. The holding cost is assumed as linear function of time. It is noted that the cost of preservation technology increases when the rate of deterioration increases from 0.01 to 0.05. It is also observed that the use of preservation technology to control the deterioration rate is completely independent of time dependent demand patterns.