1. Introduction

In this chapter explicit expressions and some recurrence relations satisfied by single and joint moment generating functions of generalized and lower generalized order statistics drawn from Gompertz and generalized exponential distributions have been derived, respectively. The results for order statistics, record and lower record values are deduced as special cases. Further, conditional expectation of function of generalized and lower generalized order statistics have been utilized to obtain the characterizations of the Gompertz and generalized exponential distributions. For related results one may refer to Ahsanullah and Raqab (1999), Raqab and Ahsanullah (2000, 2003), Saran and Singh (2003), Al-Hussaini et al. (2005, 2007) and Khan et al. (2010) among others.

2. Gompertz distribution

A random variable $X$ is said to have a Gompertz distribution if its probability density function ($pdf$) is given by

$$f(x) = \beta e^{\alpha x} \exp \left( -\frac{\beta}{\alpha} (e^{\alpha x} - 1) \right), \quad x \geq 0, \; \alpha, \beta > 0$$

and the distribution function ($df$) is

$$F(x) = \exp \left( -\frac{\beta}{\alpha} (e^{\alpha x} - 1) \right), \quad x \geq 0, \; \alpha, \beta > 0,$$

where

Part of the results of this chapter appeared in Khan et al. (2010, 2011b)
\[ F(x) = 1 - F(x). \]

Note that for Gompertz distribution defined in (5.2.1)
\[ F(x) = \frac{1}{\beta} e^{-\alpha x} f(x). \] (5.2.3)

The relation in (5.2.3) will be used to derive some recurrence relations for the moment generating functions of \( gos \) from Gompertz distribution.

### 2.1 Relations for single moment generating function

As given in Chapter I, the \( pdf \) of \( r \)-th \( gos \), \( X(r,n,m,k) \), is given by
\[ f_{X(r,n,m,k)}(x) = \frac{C_r}{(r-1)!} \left[ F(x) \right]^{r-1} f(x) g_m^{-1}(F(x)), \] (5.2.4)

where
\[ g_m(z) = \begin{cases} \frac{1}{m+1}[1 - (1 - z)^{m+1}] & , m \neq -1 \\ -\ln(1-z) & , m = -1 \end{cases} \]

The \( pdf \) (5.2.4) can be written for the Gompertz distribution with \( pdf \) (5.2.1) and \( df \) (5.2.2) and making use of (5.2.3) in the following form
\[ f_{X(r,n,m,k)}(x) = \begin{cases} \frac{C_r}{(m+1)^{r-1}(r-1)!} \exp\left(-\frac{\beta r}{\alpha}(e^{\alpha x} - 1)\right) \beta e^{\alpha x} \\ \times \left[ 1 - \exp\left(-\frac{(m+1)\beta}{\alpha}(e^{\alpha x} - 1)\right) \right]^{r-1} , , m \neq -1 \\ \frac{k^r}{(r-1)!} \left( \frac{\beta}{\alpha} \right)^{r-1} (e^{\alpha x} - 1)^{r-1} \beta e^{\alpha x} \\ \times \exp\left(-\frac{\beta k}{\alpha}(e^{\alpha x} - 1)\right), , m = -1 \end{cases} \] (5.2.5)

By using binomial expansion, we can rewrite (5.2.5) as

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\[
f_X(r,n,m,k)(x) = \begin{cases} 
\frac{C_{r-1}}{(m+1)^{r-1}(r-1)!} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} (\beta e^{\alpha x}) \\
\times \exp\left(\frac{\beta r - u}{\alpha}\right) \exp\left(-\frac{\beta r - u}{\alpha} e^{\alpha x}\right), & m \neq -1 \\
\frac{k^r}{(r-1)!} \frac{\beta}{\alpha} \left(\beta \exp\left(\frac{\beta k}{\alpha}\right) \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} (\beta e^{\alpha x}) \\
\times (e^{\alpha x})^{r-u} \exp\left(-\frac{\beta k}{\alpha} e^{\alpha x}\right), & m = -1 
\end{cases}
\]

(5.2.6)

Let us denote the single moment generating function of \( X(r,n,m,k) \) by \( M_{X(r,n,m,k)}(t) \) and its \( j \) - th derivative by \( M_{X(r,n,m,k)}^{(j)}(t) \).

We shall first establish the explicit formula for \( M_{X(r,n,m,k)}(t) \). Using (5.2.6), we obtain when \( m \neq -1 \)

\[
M_{X(r,n,m,k)}(t) = A \int_0^\infty e^{(t+\alpha)x} \exp\left(-\frac{\beta r - u}{\alpha} e^{\alpha x}\right) dx,
\]

(5.2.7)

where

\[
A = \frac{\beta C_{r-1}}{(m+1)^{r-1}(r-1)!} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \exp\left(\frac{\beta r - u}{\alpha}\right)
\]

then by using the transformation \( z = \frac{\beta r - u}{\alpha} e^{\alpha x} \), we can show that

\[
\int_0^\infty e^{(t+\alpha)x} \exp\left(-\frac{\beta r - u}{\alpha} e^{\alpha x}\right) dx
\]

\[
= \frac{1}{\beta} \left(\frac{1}{\gamma_{r-u}}\right)^{t+\alpha} \left(\frac{\alpha}{\beta}\right)^{t/\alpha} \Gamma\left(\frac{t+\alpha}{\alpha}\right) IG\left(\frac{t+\alpha}{\alpha}, \frac{\beta r - u}{\alpha}\right),
\]

where \( IG(.,.) \) is the incomplete gamma function defined by

\[
IG(l,z) = \frac{1}{\Gamma(l)} \int_z^\infty u^{l-1} e^{-u} du
\]
and hence (5.2.7) reduces to

\[ M_{X(r,n,m,k)}(t) = \frac{C_{r-1}}{(m+1)^{r-1}} \left( \frac{\alpha}{\beta} \right)^{t/\alpha} \frac{1}{\gamma_{r-u}} \sum_{u=0}^{r-1} (-1)^u \left( \frac{1}{\gamma_{r-u}} \right)^{\alpha} \exp \left( \frac{\beta y_{r-u}}{\alpha} \right) \]

\[ \times \frac{1}{u!(r-u-1)!} \Gamma \left( \frac{t+\alpha(r-u)}{\alpha} \right) IG \left( \frac{t+\alpha(r-u)}{\alpha}, \frac{\beta y_{r-u}}{\alpha} \right). \]  

(5.2.8)

and when \( m = -1 \) that

\[ M_{X(r,n,-1,k)}(t) = M_{Y_r(k)}(t) \]

\[ = \exp \left( \frac{\beta k}{\alpha} \right) \sum_{u=0}^{r-1} (-1)^u \left( \frac{\beta k}{\alpha} \right)^{au-t} \frac{1}{u!(r-u-1)!} \]

\[ \times \Gamma \left( \frac{t+\alpha(r-u)}{\alpha} \right) IG \left( \frac{t+\alpha(r-u)}{\alpha}, \frac{\beta k}{\alpha} \right). \]  

(5.2.9)

Special cases

i) For \( m = 0, k = 1 \), the explicit moment generating function of \( r \) th order statistics of the Gompertz distribution can be obtained as

\[ M_{X_{r,n}}(t) = \frac{n!}{(n-r)!} \left( \frac{\alpha}{\beta} \right)^{t/\alpha} \frac{1}{\gamma_{n+u+1-r}} \exp \left( \frac{\beta (n+u+1-r)}{\alpha} \right) \]

\[ \times \left( \frac{1}{n+u+1-r} \right)^{t+\alpha} \frac{1}{u!(r-u-1)!} \]

\[ \times \Gamma \left( \frac{t+\alpha(r-u)}{\alpha} \right) IG \left( \frac{t+\alpha(r-u)}{\alpha}, \frac{\beta (n+u+1-r)}{\alpha} \right). \]

ii) Putting \( k = 1 \) in (5.2.9), explicit moment generating function of the upper record values of the Gompertz distribution can be obtained as

\[ M_{X_{U(n)}}(t) = \exp \left( \frac{\beta}{\alpha} \right)^{r-1} \sum_{u=0}^{r-1} (-1)^u \left( \frac{\beta}{\alpha} \right)^{au-t} \frac{1}{u!(r-u-1)!} \]

\[ \times \Gamma \left( \frac{t+\alpha(r-u)}{\alpha} \right) IG \left( \frac{t+\alpha(r-u)}{\alpha}, \frac{\beta}{\alpha} \right). \]
A recurrence relation for single moment generating function of $gos$ from $df$ (5.2.2) can be derived in the following theorem:

**Theorem 2.1** For a positive integer $k$. For $n \in N, \ m \in \mathbb{R}, \ 1 \leq r \leq n$ and $j = 1, 2, \ldots,$

$$M_{X(r,n,m,k)}^{(j)}(t) = M_{X(r-1,n,m,k)}^{(j)}(t) + \frac{t}{\gamma_r \beta} M_{X(r,n,m,k)}^{(j)}(t - \alpha)$$

$$+ \frac{j}{\gamma_r \beta} M_{X(r,n,m,k)}^{(j-1)}(t - \alpha). \quad (5.2.10)$$

**Proof** For $n \in N, \ m \in \mathbb{R}, \ 1 \leq r \leq n$, it follows from (5.2.4) that

$$M_{X(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} e^{tx} [\mathcal{F}(x)]^{\gamma_r-1} f(x) g_m^{-1}(F(x)) dx.$$ \quad (5.2.11)

Integrating by parts taking $[\mathcal{F}(x)]^{\gamma_r-1} f(x)$ as the part to be integrated, we get

$$M_{X(r,n,m,k)}(t) = M_{X(r-1,n,m,k)}(t)$$

$$+ \frac{tC_{r-1}}{\gamma_r (r-1)!} \int_{-\infty}^{\infty} e^{tx} [\mathcal{F}(x)]^{\gamma_r} g_m^{-1}(F(x)) dx,$$

the constant of integration vanishes since the integral considered in (5.2.11) is a definite integral. On using (5.2.3), we obtain

$$M_{X(r,n,m,k)}(t) = M_{X(r-1,n,m,k)}(t) + \frac{tC_{r-1}}{\gamma_r (r-1)!}$$

$$\times \int_{0}^{\infty} e^{tx} [\mathcal{F}(x)]^{\gamma_r-1} \left\{ \frac{1}{\beta} e^{-\alpha x} f(x) \right\} g_m^{-1}(F(x)) dx$$

$$= M_{X(r-1,n,m,k)}(t) + \frac{t}{\gamma_r \beta} M_{X(r,n,m,k)}(t - \alpha). \quad (5.2.12)$$

Differentiating both the sides of (5.2.12) $j$ times with respect to $t$, we get (5.2.10).
Remark 2.1 Putting \( m = 0, \ k = 1 \) in (5.2.10), we obtain the recurrence relation for single moment generating function of order statistics of the Gompertz distribution in the form

\[
M_{X_{r,n}}^{(j)}(t) = M_{X_{r-1,n}}^{(j)}(t) + \frac{1}{\beta(n-r+1)} t M_{X_{r,n}}^{(j)}(t-\alpha) + j M_{X_{r,n}}^{(j-1)}(t-\alpha). 
\]

Remark 2.2 Setting \( m = -1 \) in Theorem 2.1, we get the recurrence relation for single moment generating function of the upper \( k \) record values from Gompertz distribution.

2.2 Relations for joint moment generating function

The joint pdf of \( X(r,n,m,k) \) and \( X(s,n,m,k), 1 \leq r \leq s \leq n \), is given as

\[
f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \times [F(x)]^m g_m^{r-1} (F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1} \times [F(y)]^{s-1} f(x)f(y). \tag{5.2.13}
\]

Making use of (5.2.3), we can derive recurrence relations for joint moment generating function.

Theorem 2.2 For \( 1 \leq r \leq s \leq n, \ n \in \mathbb{N}, \ m \in \mathbb{R}, \ i, j = 0,1,2,\ldots \) and a fixed positive integer \( k \geq 1 \),

\[
M_{X(r,n,m,k),X(s,n,m,k)}^{(i,j)}(t_1,t_2) = M_{X(r,n,m,k),X(s-1,n,m,k)}^{(i,j)}(t_1,t_2) \\
+ \frac{t_2}{\gamma_s \beta} M_{X(r,n,m,k),X(s,n,m,k)}^{(i,j)}(t_1,t_2-\alpha) \\
+ \frac{j}{\gamma_s \beta} M_{X(r,n,m,k),X(s,n,m,k)}^{(i,j-1)}(t_1,t_2-\alpha). \tag{5.2.14}
\]

Proof Using (5.2.13), the joint moment generating function of \( X(r,n,m,k) \) and \( X(s,n,m,k) \) is given by
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\[ M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) \]

\[ = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_0^\infty \int_x^\infty \int_0^\infty \int_x^\infty e^{t_1 x + t_2 y} [F(x)]^m f(x) g_m^{r-1}(F(x)) \]

\[ \times [h_m(F(y)) - h_m(F(x)))]^{s-r-1}[\bar{F}(y)]^{\gamma s-1} f(y) dy dx \]

\[ = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_0^\infty \int_x^\infty [F(x)]^m f(x) g_m^{r-1}(F(x)) I(x) dx, \quad (5.2.15) \]

where

\[ I(x) = \int_x^\infty e^{t_1 x + t_2 y} [h_m(F(y)) - h_m(F(x)))]^{s-r-1}[\bar{F}(y)]^{\gamma s-1} f(y) dy . \]

Solving the integral in \( I(x) \) by parts and substituting the resulting expression in (5.2.15), we get

\[ M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) \]

\[ = M_{X(r,n,m,k),X(s-1,n,m,k)}(t_1,t_2) + \frac{t_2 C_{s-1}}{\gamma_s (r-1)!(s-r-1)!} \int_0^\infty \int_x^\infty e^{t_1 x + t_2 y} [F(x)]^m f(x) g_m^{r-1}(F(x)) \]

\[ \times [h_m(F(y)) - h_m(F(x)))]^{s-r-1}[\bar{F}(y)]^{\gamma s} dy dx . \]

On using the relation (5.2.3), we obtain

\[ M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) = M_{X(r,n,m,k),X(s-1,n,m,k)}(t_1,t_2) \]

\[ + \frac{t_2}{\gamma_s \beta} M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2 - \alpha). \quad (5.2.16) \]

Differentiating both the sides of (5.2.16) \( i \) times with respect to \( t_1 \) and then \( j \) times with respect to \( t_2 \), we get the result of Theorem 2.2.

One can also note that Theorem 2.1 can be deduced from Theorem 2.2 by letting \( t_1 \) tending to zero.

**Remark 2.3** Putting \( m = 0, \ k = 1 \) in (5.2.14), we obtain a recurrence relation for joint moment generating function of order statistics of the Gompertz distribution.
Remark 2.4 Setting $m = -1$ in Theorem 2.2, we get the recurrence relation for joint moment generating function of the upper $k$ record values from Gompertz distribution.

2.3 Characterization

Let $X(r, n, m, k), r = 1, 2, \ldots, n$ be gos, then the conditional pdf of $X(s, n, m, k)$ given $X(r, n, m, k) = x$, $1 \leq r < s \leq n$, in view of (5.2.4) and (5.2.13), is

\[
f_{X(s, n, m, k) | X(r, n, m, k)}(y | x) = \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \times \left[ (\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1} \right]^{s-r-1} [\bar{F}(y)]^{y_{s-1}}
\]

\[
\times \frac{(m+1)^{s-r-1} \bar{F}(x)^{y_{r+1}}} \int_y^{\infty} e^{\alpha y} [\bar{F}(y)]^{y_{s-t}-1} f(y) dy.
\]

(5.2.17)

Theorem 2.3 Suppose $F(x) < 1$, for all $x \in (0, \infty)$ be a distribution function of the random variable $X$ and $F(0) = 0$, $F(\infty) = 1$, then

\[
\bar{F}(x) = \exp \left( -\frac{\beta}{\alpha} (e^{\alpha x} - 1) \right), \quad x \geq 0, \quad \alpha, \beta > 0
\]

(5.2.18)

if and only if, for $1 \leq r < s \leq n$

\[
E[e^{\alpha X(s, n, m, k)} | X(r, n, m, k) = x] = \frac{C_{s-1}}{(s-r-1)! C_{r-1}(m+1)^{s-r-1}}
\]

\[
\times \sum_{t=0}^{s-r-1} (-1)^t \left( \frac{s-r-1}{t} \right) \frac{1}{y_{s-t}} \left( e^{\alpha x} + \frac{\alpha}{\beta y_{s-t}} \right).
\]

(5.2.19)

Proof We have from (5.2.17)

\[
E[e^{\alpha X(s, n, m, k)} | X(r, n, m, k) = x] = \frac{C_{s-1}}{(s-r-1)! C_{r-1}(m+1)^{s-r-1}} \sum_{t=0}^{s-r-1} (-1)^t \left( \frac{s-r-1}{t} \right)
\]

\[
\times \frac{1}{[\bar{F}(x)]^{y_{s-t}}} \int_y^{\infty} e^{\alpha y} [\bar{F}(y)]^{y_{s-t}-1} f(y) dy.
\]

(5.2.20)
Integrating by parts and noting the relation (5.2.3) it is easy to prove the necessary part.

To prove sufficient part, we have from (5.2.20)

\[
\sum_{t=0}^{s-r-1} (-1)^t \left( \frac{s-r-1}{t} \right) \int_x^\infty e^{\alpha y} [F(y)]^{\gamma_{s-t}} f(y) dy
\]

\[
= \sum_{t=0}^{s-r-1} (-1)^t \left( \frac{s-r-1}{t} \right) \left( e^{\alpha x} + \frac{\alpha}{\beta \gamma_{s-t}} [F(x)]^{\gamma_{s-t}} \right).
\]

Differentiating both the sides with respect to \( x \) and rearranging, we get

\[
\frac{f(x)}{F(x)} = \beta e^{\alpha x}
\]

which gives

\[
F(x) = \exp \left( -\frac{\beta}{\alpha} (e^{\alpha x} - 1) \right), \quad x \geq 0, \quad \alpha, \beta > 0.
\]

3. Generalized exponential distribution

A random variable \( X \) is said to have generalized exponential distribution Gupta and Kundu (1999), if its pdf is of the form

\[
f(x) = \alpha (1 - e^{-x})^{\alpha-1} e^{-x}, \quad x > 0, \quad \alpha > 0
\]

and the corresponding df is

\[
F(x) = (1 - e^{-x})^\alpha, \quad x > 0, \quad \alpha > 0.
\]

Note that for generalized exponential distribution defined in (5.3.1)

\[
\alpha F(x) = (e^x - 1) f(x).
\]

Here \( \alpha \) is the shape parameter. The location and scale parameters can be added to this model. Without loss of generality, the location and scale parameters are taken to the zero and unity, respectively.

3.1 Relations for single moment generating function

The pdf of \( r \)-th lgos, \( X^* (r,n,m,k) \), is given by
We shall first establish some results which may be helpful in proving the main result.

Lemma 3.1  For generalized exponential distribution as given in (5.3.2) and any non-negative and finite integers \( a \) and \( b \) with \( m \neq -1 \),

\[
I_j(a,0) = \alpha \sum_{p=0}^{\infty} \frac{(t)_p}{p! [\alpha(a+1)+p]},
\]

where

\[
(t)_p = \begin{cases} \frac{t(t+1)\ldots(t+p-1)}{1}, & p = 0,1,2,\ldots \\ 1, & p = 1,2,\ldots \end{cases}
\]

and

\[
I_j(a,b) = \int_0^\infty e^{tx} [F(x)]^a f(x) g_m^b (F(x)) dx.
\]

Proof  From (5.3.6), we have

\[
I_j(a,0) = \int_0^\infty e^{tx} [F(x)]^a f(x) dx.
\]

Making the substitution \( z = [F(x)]^{1/\alpha} \) in (5.3.7), we find that

\[
I_j(a,0) = \alpha \int_0^1 (1-z)^{-t} z^{\alpha(a+1)-1} dz.
\]

On using Maclaurine series expansion

\[
(1-z)^{-t} = \sum_{p=0}^{\infty} \frac{(t)_p}{p!} z^p
\]

and integrating the resulting expression, we derive the result given in (5.3.5).

Lemma 3.2  For the generalized exponential distribution as given in (5.3.2) and any non-negative and finite integers \( a \) and \( b \),
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\[ I_j(a,b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} I_j(a+u(m+1),0) \] (5.3.8)

\[ = \frac{\alpha}{(m+1)^b} \sum_{p=0}^{\infty} \sum_{u=0}^{b} (-1)^{u} \binom{b}{u} (t)^p p! [\alpha(a+u(m+1)+1+p)]^{b+1}, \quad m \neq -1 \] (5.3.9)

\[ = b! \alpha^{b+1} \sum_{p=0}^{\infty} \frac{(t)^p}{p! [\alpha(a+1)+p]^{b+1}}, \quad m = -1 \] (5.3.10)

where \( I_j(a,b) \) is as given in (5.3.6).

**Proof** On expanding \( g_m^b(F(x)) = \left[ \frac{1}{m+1} \{1 - (F(x))^{m+1} \} \right]^{b} \) binomially in (5.3.6), we get when \( m \neq -1 \)

\[ I_j(a,b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \int_{0}^{\infty} e^{tx} [F(x)]^{a+u(m+1)} f(x) dx \]

and hence the result given in (5.3.8).

Making use of Lemma 3.1, we establish the result given in (5.3.9) and when \( m = -1 \) that

\[ I_j(a,b) = \frac{0}{0} \quad \text{as} \quad \sum_{u=0}^{b} (-1)^u \binom{b}{u} = 0. \]

Since (5.3.9) is of the form \( \frac{0}{0} \) at \( m = -1 \), so after applying L’ Hospital rule on the lines of (4.2.8) [see Section 2, Chapter IV], we get

\[ \lim_{m \to -1} I_j(a,b) = \frac{A \alpha^b}{[\alpha(a+1)+p]^{b+1}} \sum_{u=0}^{b} (-1)^u \binom{b}{u} u^b. \] (5.3.11)

Making use of the result that (Ruiz, 1996)

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x+i)^n = n!. \] (5.3.12)

for all integers \( n \geq 0 \) and for all real numbers \( x \).
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Therefore,

\[
\sum_{u=0}^{b} (-1)^{u+b} \binom{b}{u} u^b = b!.
\]  

(5.3.13)

Now on substituting (5.3.13) in (5.3.11), we have the result given in (5.3.9).

**Theorem 3.1** For the generalized exponential distribution as given in (5.3.2) and \(1 \leq r \leq n\), \(k = 1, 2, \ldots\), and \(m \neq -1\),

\[
M_{X^*_{(r,n,m,k)}}(t) = \frac{C_{r-1}}{(r-1)!} I_j(\gamma_r - 1, r - 1)
\]  

(5.3.14)

\[
= \frac{\alpha C_{r-1}}{(r-1)!} \left( \sum_{p=0}^{\infty} \sum_{u=0}^{r-1} (-1)^{u}\left( \frac{r-1}{u} \right) p \right) \frac{(t_p)}{p!\left[\alpha \gamma_{r-u} + p\right]},
\]  

(5.3.15)

where

\(I_j(\gamma_r - 1, r - 1)\) is as defined in (5.3.6).

**Proof** From (5.3.4), we have

\[
M_{X^*_{(r,n,m,k)}}(t) = \frac{C_{r-1}}{(r-1)!} \left[ \int_0^{x} e^{t x} [F(x)]^{r-1} f(x) g_m^{-1}(F(x)) dx \right]
\]

\[
= \frac{C_{r-1}}{(r-1)!} I_j(\gamma_r - 1, r - 1).
\]

Making use of Lemma 3.2, we establish the relation given in (5.3.15).

**Special cases**

i) Putting \(m = 0\), \(k = 1\) in (5.3.15), the explicit formula for single moment generating functions of order statistics of the generalized exponential distribution can be obtained as

\[
M_{X_{n-r+1:n}}(t) = \alpha C_{r:n} \sum_{p=0}^{\infty} \sum_{u=0}^{r-1} (-1)^{u}\left( \frac{r-1}{u} \right) \frac{(t_p)}{p!\left[\alpha (n-r+1+u) + p\right]}.
\]

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That is

\[ M_{X_{rn}}(t) = \alpha C_{r,n} \sum_{p=0}^{\infty} \sum_{u=0}^{n-r} (-1)^u \binom{n-r}{u} \frac{(t)_p}{p!(\alpha(r+u)+p)^r}, \]

where

\[ C_{r,n} = \frac{n!}{(r-1)!(n-r)!}. \]

ii) Setting \( m = -1 \) in (5.3.15), we get the explicit expression for single moment generating function of lower \( k \) record values from generalized exponential distribution in view of (5.3.14) and (5.3.10)

\[ M_{X^*(r,n,-1,k)}(t) = M_{(Z^{(k)}_r)}(t) = (\alpha k)^r \sum_{p=0}^{\infty} \frac{(t)_p}{p!(\alpha k + p)^r} \]

and hence for lower records, at \( k = 1 \)

\[ M_{X_{L(r)}}(t) = \alpha^r \sum_{p=0}^{\infty} \frac{(t)_p}{p!(\alpha + p)^r}, \]

as obtained by Raqab (2002).

A recurrence relation for single moment generating function for \( lgos \)
from \( df \) (5.3.2) can be obtained in the following theorem:

**Theorem 3.2** For the distribution given in (5.3.2) and for \( 2 \leq r \leq n, n \geq 2 \) and \( k = 1,2, \ldots, \)

\[ \left( 1 - \frac{t}{\alpha r} \right) M_{X^*(r,n,m,k)}^{(j)}(t) = M_{X^*(r-1,n,m,k)}^{(j)}(t) + \frac{j}{\alpha r} M_{X^*(r,n,m,k)}^{(j-1)}(t) \]

\[ - \frac{1}{\alpha r} \left\{ t M_{X^*(r,n,m,k)}^{(j)}(t+1) + j M_{X^*(r,n,m,k)}^{(j-1)}(t+1) \right\}. \quad (5.3.16) \]

**Proof** From (5.3.4), we have

\[ M_{X^*(r,n,m,k)}(t) = \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{r-1} f(x) g_m^{r-1} (F(x)) dx. \]

\[ (5.3.17) \]
Integrating by parts treating \([F(x)]^{r-1}f(x)\) for integration and rest of the integrand for differentiation, we get

\[
M_{X^*(r,n,m,k)}(t) = M_{X^*(r-1,n,m,k)}(t) - \frac{t}{\gamma_r} \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{r-1} \left\{ \frac{(e^x - 1)}{\alpha} f(x) \right\} g^{-1}_m(F(x))dx,
\]

the constant of integration vanishes since the integral considered in (5.3.17) is a definite integral. On using (5.3.3), we obtain

\[
M_{X^*(r,n,m,k)}(t) = M_{X^*(r-1,n,m,k)}(t) - \frac{t}{\alpha \gamma_r} \{M_{X^*(r,n,m,k)}(t+1) - M_{X^*(r,n,m,k)}(t)\}.
\]

(5.3.18)

Differentiating both the sides of (5.3.18) \(j\) times with respect to \(t\), we get

\[
M^{(j)}_{X^*(r,n,m,k)}(t) = M^{(j)}_{X^*(r-1,n,m,k)}(t) - \frac{t}{\alpha \gamma_r} M^{(j)}_{X^*(r,n,m,k)}(t+1) - \frac{j}{\alpha \gamma_r} M^{(j-1)}_{X^*(r,n,m,k)}(t+1) + \frac{t}{\alpha \gamma_r} M^{(j)}_{X^*(r,n,m,k)}(t) + \frac{j}{\alpha \gamma_r} M^{(j-1)}_{X^*(r,n,m,k)}(t).
\]

The recurrence relation in equation (5.3.16) is derived simply by rewriting the above equation.

By differentiating both sides of equation (5.3.16) with respect to \(t\) and then setting \(t = 0\), we obtain the recurrence relations for moments of \(g_{gos}\) from generalized exponential distribution in the form
\[ E[X^*j(r,n,m,k)] = E[X^*(r-1,n,m,k)] + \frac{j}{\alpha r} \{E[X^*j-1(r,n,m,k)] - E[\phi(X^*(r,n,m,k))]\}, \]

where
\[ \phi(x) = x^{j-1}e^x. \]

**Remark 3.1**  Putting \( m = 0, \ k = 1 \) in (5.3.16), we obtain the recurrence relation for single moment generating function of order statistics for generalized exponential distribution in the form
\[
\left(1 - \frac{t}{\alpha(n-r+1)}\right)M^{(j)}_{X_{n-r+\ln}}(t) = M^{(j)}_{X_{n-r+2\ln}}(t) + \frac{j}{\alpha(n-r+1)}M^{(j-1)}_{X_{n-r+\ln}}(t)
\]
\[- \frac{1}{\alpha(n-r+1)}\{tM^{(j)}_{X_{n-r+\ln}}(t+1) + jM^{(j-1)}_{X_{n-r+\ln}}(t+1)\}.\]

Replacing \((n-r+1)\) by \((r-1)\), we have
\[
M^{(j)}_{X_{r-\ln}}(t) = \left(1 - \frac{t}{\alpha(r-1)}\right)M^{(j)}_{X_{r-\ln}}(t) - \frac{j}{\alpha(r-1)}M^{(j-1)}_{X_{r-\ln}}(t)
\]
\[+ \frac{1}{\alpha(n-1)}\{tM^{(j)}_{X_{r-\ln}}(t+1) + jM^{(j-1)}_{X_{r-\ln}}(t+1)\}.\]

For \( r = r + 1 \), the result was obtained by Raqab (2004).

**Remark 3.2**  Setting \( m = -1 \) and \( k \geq 1 \), in Theorem 3.2, we get a recurrence relation for single moment generating function of lower \( k \) record values for generalized exponential distribution in the form
\[
\left(1 - \frac{t}{\alpha k}\right)M^{(j)}_{Z_{r}}(t) = M^{(j)}_{Z_{r-1}}(t) + \frac{j}{\alpha k}M^{(j-1)}_{Z_{r}}(t)
\]
\[- \frac{1}{\alpha k}\{tM^{(j)}_{Z_{r}}(t+1) + jM^{(j-1)}_{Z_{r}}(t+1)\}.\]

### 3.2 Relations for joint moment generating function

The joint pdf of \( X^*(r,n,m,k) \) and \( X^*(s,n,m,k) \), \( 1 \leq r < s \leq n \), is
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\[ f_{X^* (r,n,m,k),X^* (s,n,m,k)} (x,y) = \frac{C_{s-1}}{(r-1)! (s-r-1)!} [F(x)]^m f(x) \times g_m^{-1} (F(x))[h_m (F(y)) - h_m (F(x))]^{s-r-1} [F(y)]^{y_s-1} f(y), \]

\[ y < x. \quad (5.3.19) \]

Before coming to main results we shall prove the following Lemmas.

**Lemma 3.3**  For generalized exponential distribution as given in (5.3.1) and non-negative integers \( a, b, c \) with \( m \neq -1 \),

\[ I_{i j} (a,0,c) = \alpha^2 \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t_1)^q (t_2)^p}{p! q!} [\alpha(c+1) + p][\alpha(a + c + 2) + p + q], \]

where

\[ I_{i j} (a,b,c) = \int_0^\infty \int_0^x e^{t_1 x + t_2 y} [F(x)]^a f(x) \times [h_m (F(y)) - h_m (F(x))]^b [F(y)]^c f(y) dy dx. \]

**Proof**  From (5.3.19), we have

\[ I_{i j} (a,0,c) = \int_0^\infty \int_0^x e^{t_1 x + t_2 y} [F(x)]^a f(x) [F(y)]^c f(y) dy dx \]

\[ = \int_0^\infty e^{t_1 x} [F(x)]^a f(x) G(x) dx, \quad (5.3.22) \]

where

\[ G(x) = \int_0^x e^{t_2 y} [F(y)]^c f(y) dy. \]

(5.3.23)

By setting \( z = [F(y)]^{1/\alpha} \) in (5.3.23), we get

\[ G(x) = \alpha \int_0^{[F(x)]^{1/\alpha}} (1 - z)^{-t_2} z^{\alpha(c+1) - 1} dz \]

\[ = \alpha \sum_{p=0}^{\infty} \frac{(t_2)^p}{p!} \int_0^{[F(x)]^{1/\alpha}} z^{\alpha(c+1) + p - 1} dz \]

\[ = \alpha \sum_{p=0}^{\infty} \frac{(t_2)^p}{p! [\alpha(c+1) + p]} [F(x)]^{c+1+p/\alpha}. \]
On substituting the above expression of $G(x)$ in (5.3.22), we find that

$$I_{ij}(a,0,c) = \alpha \sum_{p=0}^{\infty} \left( \frac{t_2}{p!} \right)^{p} e^{t_1 x} F(x)^{a+c+1+p/\alpha} f(x) dx.$$  

(5.3.24)

Again by setting $w = [F(x)]^{1/\alpha}$ in (5.3.24) and simplifying the resulting expression, we derive the relation given in (5.3.20).

**Lemma 3.4** For the distribution as given in (5.3.1) and any non-negative integers $a, b, c$,

$$I_{ij}(a,b,c) = \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^b \binom{b}{v} I_{ij}(a + (b-v)(m+1), 0, c + v(m+1))$$

(5.3.25)

$$= \frac{\alpha^2}{(m+1)^b} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{v=0}^{b} (-1)^b \binom{b}{v} \frac{(t_1)_q}{p! q! [\alpha (c + v(m+1) + 1) + p]}$$

$$\times \frac{(t_2)_p}{[\alpha (a + c + b(m+1) + 2) + p + q]}, \quad m \neq -1$$

(5.3.26)

$$= b! \alpha^{b+2} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t_1)_q (t_2)_p}{p! q! [\alpha (c+1) + p]^{b+1} [\alpha (a + c + 2) + p + q]}, \quad m = -1,$$

(5.3.27)

where

$I_{ij}(a,b,c)$ is as given in (5.3.21).

**Proof** When $m \neq -1$, we have

$$[h_m(F(y)) - h_m(F(x))]^b = \frac{1}{(m+1)^b} [(F(x))^{m+1} - (F(y))^{m+1}]^b$$

$$= \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^v \binom{b}{v} [F(x)]^{(b-v)(m+1)} [F(y)]^{v(m+1)}.$$

Now substituting for $[h_m(F(y)) - h_m(F(x))]^b$ in (5.3.21), we get
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\[ I_{i,j}(a,b,c) = \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^v \binom{b}{v} \]

\[ \times \int_0^\infty \int_0^\infty e^{t_1 x + t_2 y} [F(x)]^{a+(b-v)(m+1)} f(x)[F(y)]^{c+v(m+1)} f(y) dy dx \]

and hence the result given in (5.3.25).

Making use of Lemma 3.3, we derive the relation given in (5.3.26).

When \( m = -1 \), we have

\[ I_{i,j}(a,b,c) = \frac{0}{0} \quad \text{as} \quad \sum_{v=0}^{b} (-1)^v \binom{b}{v} = 0. \]

Since at \( m = -1 \) (5.3.26) is of the form \( \frac{0}{0} \), so after applying L’ Hospital rule and (5.3.13), (5.3.27) can be proved.

**Theorem 3.3** For the distribution as given in (5.3.2) and \( 1 \leq r < s \leq n \), \( k = 1,2, \ldots \) and \( m \neq 1 \),

\[ M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}} \]

\[ \times \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} I_{i,j}(m+u(m+1),s-r-1,\gamma_s-1) \quad (5.3.28) \]

\[ = \frac{\alpha^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^u \binom{r-1}{u} \binom{s-r-1}{v} \frac{(t_1)_q (t_2)_p}{p!q![\alpha \gamma_{s-v} + p][\alpha \gamma_{r-u} + p + q]} \quad (5.3.29) \]

**Proof** From (5.3.19), we have

\[ M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \]

\[ \times \int_0^\infty \int_0^\infty e^{t_1 x + t_2 y} [F(x)]^{m} f(x) g_m^{r-1}(F(x)) \]

\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1} f(y) dy dx. \quad (5.3.30) \]
On expanding $g_{m}^{r-1}(F(x))$ binomially in (5.3.30), we get

$$M_{X^{*}(r,n,m,k),X^{*}(s,n,m,k)}(t_1,t_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}}$$

$$\times \sum_{u=0}^{r-1} (-1)^{u} \binom{r-1}{u} I_{i_j} \left(m+u(m+1),s-r-1,\gamma_{s}-1\right).$$

Making use of the Lemma 3.4, we derive the relation in (5.3.29).

Differentiating $M_{X^{*}(r,n,m,k),X^{*}(s,n,m,k)}(t_1,t_2)$ and evaluating at $t_1=t_2=0$, we get the product moments of $\log$ when $m \neq -1$

$$E[X^{*}(r,n,m,k),X^{*}(s,n,m,k)] = \frac{\alpha^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v} \binom{r-1}{u} \binom{s-r-1}{v}$$

$$\times \frac{1}{pq(\alpha \gamma_{s-v} + p)(\alpha \gamma_{r-u} + p + q)}.$$

**Special cases**

i) Putting $m = 0, k = 1$ in (5.3.29), the explicit formula for joint moment generating functions of order statistics for generalized exponential distribution can be obtained as

$$M_{X_{n-r+1,n-s+1n}}(t_1,t_2) = \alpha^2 C_{r,s,n} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v}$$

$$\times \binom{r-1}{u} \binom{s-r-1}{v} \frac{(t_2)_p (t_1)_q}{p!q!\left[\alpha(n-s+1+v) + p\right]\left[\alpha(n-r+1+u) + p + q\right]}.$$

That is

$$M_{X_{r,s:n}}(t_1,t_2) = \alpha^2 C_{r,s:n} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{u=0}^{n-s} \sum_{v=0}^{s-r-1} (-1)^{u+v} \binom{n-s}{u} \binom{s-r-1}{v}$$

$$\times \frac{(t_2)_p (t_1)_q}{p!q!\left[\alpha(r+v) + p\right]\left[\alpha(s+u) + p + q\right]},$$
where

$$C_{r,s,n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$  

ii) Setting $m = -1$ in (5.3.29), we deduce the explicit expression for joint moment generating function of lower $k$ record values for generalized exponential distribution in view of (5.3.28) and (5.3.27) in the form

$$M_{(Z_r^{(k)}, Z_s^{(k)})}(t_1, t_2) = (\alpha k)^j \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t_2)^p (t_1)^q}{p!q!(\alpha k + p)^{s-r}(\alpha k + p + q)^r}$$

and hence for lower records

$$M_{X_{L(r)}, X_{L(s)}}(t_1, t_2) = \alpha^s \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t_2)^p (t_1)^q}{p!q!(\alpha + p)^{s-r}(\alpha + p + q)^r}.$$  

Making use of (5.3.3), we can derive the recurrence relations for joint moment generating function of $l gos$ from (5.3.2).

**Theorem 3.4** For the distribution given in (5.3.2) and for $1 \leq r < s \leq n$, $n \geq 2$ and $k = 1, 2, \ldots$,

$$\left(1 - \frac{t_2}{\alpha \gamma_s}\right)M^{(i,j)}_{X^*(r,n,m,k), X^*(s,n,m,k)}(t_1, t_2) = M^{(i,j)}_{X^*(r,n,m,k), X^*(s-1,n,m,k)}(t_1, t_2)$$

$$- \frac{1}{\alpha \gamma_s} \{t_2 M^{(i,j)}_{X^*(r,n,m,k), X^*(s,n,m,k)}(t_1, t_2 + 1)$$

$$+ j M^{(i,j-1)}_{X^*(r,n,m,k), X^*(s,n,m,k)}(t_1, t_2 + 1)\}$$

$$+ \frac{j}{\alpha \gamma_s} M^{(i,j-1)}_{X^*(r,n,m,k), X^*(s,n,m,k)}(t_1, t_2). \quad (5.3.31)$$

**Proof** Using (5.3.19), the joint moment generating function of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$ is given by
\[ M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1, t_2) \] 
\[ = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_0^\infty [F(x)]^m f(x)g_m^{r-1}(F(x))I(x)dx, \quad (5.3.32) \]

where
\[ I(x) = \int_0^x e^{t_1x+t_2y}[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s}f(y)dy. \]

Solving the integral in \( I(x) \) by parts and substituting the resulting expression in (5.3.32), we get
\[ M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1, t_2) \]
\[ = M_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1, t_2) - \frac{t_2C_{s-1}}{\gamma_s(r-1)!(s-r-1)!} \]
\[ \times \int_0^\infty \int_0^x e^{t_1x+t_2y}[F(x)]^m f(x)g_m^{r-1}(F(x)) \]
\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s} dydx \]

the constant of integration vanishes since the integral in \( I(x) \) is a definite integral. On using the relation (5.3.3), we obtain
\[ M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1, t_2) \]
\[ = M_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1, t_2) \]
\[ - \frac{t_2}{\alpha \gamma_s} M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1, t_2 + 1) \]
\[ + \frac{t_2}{\alpha \gamma_s} M_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1, t_2). \quad (5.3.33) \]

Differentiating both the sides of (5.3.33) \( i \) times with respect to \( t_1 \) and then \( j \) times with respect to \( t_2 \), we get

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\[
M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1,t_2) = M^{(i,j)}_{X^*(r,n,m,k),X^*(s-1,n,m,k)}(t_1,t_2)
- \frac{t_2}{\alpha \gamma_s} M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1, t_2 + 1)
- \frac{j}{\alpha \gamma_s} M^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1, t_2 + 1)
+ \frac{t_2}{\alpha \gamma_s} M^{(i,j)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1, t_2)
+ \frac{j}{\alpha \gamma_s} M^{(i,j-1)}_{X^*(r,n,m,k),X^*(s,n,m,k)}(t_1, t_2)
\]

which, when rewritten gives the recurrence relation in (5.3.31).

One can also note that Theorem 3.2 can be deduced from Theorem 3.4 by letting \( t_1 \) tends to zero.

**Remark 3.3** Putting \( m = 0, \ k = 1 \) in (5.3.31), we obtain the recurrence relations for joint moment generating function of order statistics for generalized exponential distribution in the form

\[
\left(1 - \frac{t_2}{\alpha(n-s+1)}\right) M^{(i,j)}_{X_{n-r+1,n-s+1:n}}(t_1, t_2) = M^{(i,j)}_{X_{n-r+1,n-s+2:n}}(t_1, t_2)
+ \frac{j}{\alpha(n-s+1)} M^{(i,j-1)}_{X_{n-r+1,n-s+1:n}}(t_1, t_2) - \frac{1}{\alpha(n-s+1)}
\times \{ t_2 M^{(i,j)}_{X_{n-r+1,n-s+1:n}}(t_1, t_2 + 1) + j M^{(i,j-1)}_{X_{n-r+1,n-s+1:n}}(t_1, t_2 + 1) \}. \]

That is

\[
M^{(i,j)}_{X_{r,s:n}}(t_1, t_2) = \left(1 - \frac{t_1}{\alpha(r-1)}\right) M^{(i,j)}_{X_{r-1,s:n}}(t_1, t_2) - \frac{i}{\alpha(r-1)} M^{(i-1,j)}_{X_{r-1,s:n}}(t_1, t_2)
+ \frac{1}{\alpha(r-1)} \{ t_1 M^{(i,j)}_{X_{r-1,s:n}}(t_1 + 1, t_2) + i M^{(i-1,j)}_{X_{r-1,s:n}}(t_1 + 1, t_2) \}
\]

as obtained by Raqab (2004) for \( r = r + 1 \).
Remark 3.4 Sustituting $m=-1$ and $k \geq 1$, in Theorem 3.4, we get recurrence relation for joint moment generating function of lower $k$ record values for generalized exponential distribution.

### 3.3 Characterization

Let $X^*(r,n,m,k), r=1,2,\ldots,n$ be lgos from a continuous population with df $F(x)$ and pdf $f(x)$, then the conditional pdf of $X^*(s,n,m,k)$ given $X^*(r,n,m,k) = x$, $1 \leq r < s \leq n$, in view of (5.3.4) and (5.3.19), is

$$f_{X^*(s,n,m,k)|X^*(r,n,m,k)}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}[F(x)]^{m-\gamma_r+1}$$

$$\times[(h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y). \quad (5.3.34)$$

**Theorem 3.5** Let $X$ be a non negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0 < F(x) < 1$ for all $x > 0$, then

$$E[e^{tX^*(s,n,m,k)} | X^*(l,n,m,k) = x]$$

$$= \sum_{p=0}^{\infty} \frac{(t)_p}{p!}(1-e^{-x})^p \prod_{j=1}^{s-1} \frac{\gamma_{l+j}}{\gamma_{l+j} + p/\alpha}, \quad l = r, r+1 \quad (5.3.35)$$

if and only if

$$F(x) = (1-e^{-x})^\alpha, \quad x > 0, \quad \alpha > 0.$$ 

**Proof** From (5.3.34), we have

$$E[e^{tX^*(s,n,m,k)} | X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times \int_0^\infty e^{ty} \left[1 - \left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1} \left(\frac{F(y)}{F(x)}\right)^{\gamma_s-1} \frac{f(y)}{F(x)} dy. \quad (5.3.36)$$
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By setting $u = \frac{F(y)}{F(x)} = \left(\frac{1-e^{-y}}{1-e^{-x}}\right)^{\alpha}$ from (5.3.2) in (5.3.36), we obtain

$$E[e^{tX^*(s,n,m,k)} | X^*(r,n,m,k) = x] = A \int_0^1 [1-(1-e^{-x})u^{1/\alpha}]^{-t} u^{\gamma_s-1}(1-u^{m+1})^{s-r-1} du$$

$$= A \sum_{p=0}^{\infty} \frac{(t)_p (1-e^{-x})^p}{p!} \int_0^1 u^{\gamma_s-1+p/\alpha} (1-u^{m+1})^{s-r-1} du, \quad (5.3.37)$$

where

$$A = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}. $$

Again by setting $v = u^{m+1}$ in (5.3.37), we get

$$E[e^{tX^*(s,n,m,k)} | X^*(r,n,m,k) = x] = A \sum_{p=0}^{\infty} \frac{(t)_p (1-e^{-x})^p}{p!} \int_0^1 \frac{\alpha k + p + n-s-1}{\alpha(m+1)+n-s} (1-v)^{s-r-1} dv$$

$$= \frac{A}{m+1} \sum_{p=0}^{\infty} \frac{(t)_p (1-e^{-x})^p}{p!} \frac{\Gamma\left(\frac{\alpha k + p + n-s-1}{\alpha(m+1)+n-s}\right)}{\Gamma\left(\frac{\alpha k + p + n-s-1}{\alpha(m+1)+n-s}\right)} \Gamma(s-r)$$

$$= \frac{C_{s-1}}{C_{r-1}} \sum_{p=0}^{\infty} \frac{(t)_p (1-e^{-x})^p}{p!} \prod_{j=1}^{s-r} (\gamma_r+j + p/\alpha)$$

and hence the relation given in (5.3.35).

To prove sufficient part, we have from (5.3.34) and (5.3.35)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_0^x e^{fy} [(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1}$$

$$\times [F(y)]^{\gamma_s-1} f(y) dy = [F(x)]^{\gamma_{r+1}} H_r(x), \quad (5.3.38)$$
where

\[ H_r(x) = \sum_{p=0}^{\infty} \frac{(t)(p)(1-e^{-x})^p}{p!} \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\gamma_{r+j} + p/\alpha} \right). \]

Differentiating (5.3.38) both the sides with respect to \( x \), we get

\[
\frac{C_{s-1}[F(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}} \int_0^x e^{ty} \left[ (F(x))^{m+1} - (F(y))^{m+1} \right]^{s-r-2} \\
\times [F(y)]^{s-1} f(y) dy = H'_r(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x)[F(x)]^{\gamma_{r+1}-1} f(x) \]

or

\[
\gamma_{r+1} H_{r+1}(x)[F(x)]^{\gamma_{r+1}+m} f(x) \\
= H'_r(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x)[F(x)]^{\gamma_{r+1}-1} f(x),
\]

where

\[
H'_r(x) = \sum_{p=0}^{\infty} \frac{(t)(p)(1-e^{-x})^{p-1} e^{-x}}{p!} \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\gamma_{r+j} + p/\alpha} \right)
\]

and

\[
H_{r+1}(x) - H_r(x) = \sum_{p=0}^{\infty} \frac{(t)(p)(1-e^{-x})^p}{p!} \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\gamma_{r+j} + p/\alpha} \right) \frac{p/\alpha}{\gamma_{r+1}}.
\]

Therefore,

\[
\frac{f(x)}{F(x)} = \alpha e^{-x} \frac{1}{1-e^{-x}}
\]

which proves that

\[ F(x) = (1-e^{-x})^\alpha, \quad x > 0, \quad \alpha > 0. \]