3.1. INTRODUCTION

In this chapter we introduce a KdV equation with variable coefficients. First we give an account of different KdV-type equations which arise in the study of water waves.

3.2. KdV TYPE EQUATIONS WITH VARIABLE COEFFICIENTS

Equation (2.4.1) that we have investigated in the previous chapter is an example of a KdV equation with variable coefficients.

It is well-known that the waves reaching a shore can be considered as solitary waves since they are well separated. Thus the development of a solitary wave over a region of varying depth is of great practical importance. Notable contributions in this direction are due to Ippen and Kulin (1970) and the numerical studies of Peregrine (1967) and Madsen and Mei (1969). But it is rather surprising that except in the work of
Madsen and Mei no attempts were made to make use of the knowledge available concerning the KdV equation. Johnson's (1973a) work was perhaps the first serious attempt to fill this gap. It is to be mentioned that Grimshaw (1970) has considered the problem of waves on water of slowly varying depth, investigating the condition for solution to be a solitary wave with slowly varying coefficients.

Let us discuss Johnson's problem of a solitary wave moving onto a shelf. We consider a small amplitude motion defined by the amplitude parameter $\varepsilon$. The depth is allowed to vary slowly on the same scale $\varepsilon$. The far-field (distance $O(1/\varepsilon)$) approximation then incorporates the effects of changing depth and the near-field first approximation is unaltered since the depth approaches a constant as $\varepsilon \to 0$. If we assume that the nonlinear and dispersive effects are of the same order, the resulting equation has terms depending on the depth, nonlinearity and dispersion, all being of order unity and can be written in terms of the far-field distance co-ordinate

$$ X = \varepsilon x $$  \hspace{1cm} (2.2.1a)
and the appropriate characteristic co-ordinate

\[ \xi = \int_0^x d^{-\frac{1}{4}} (\varepsilon x) \, dx - t = 0(1), \tag{3.2.1b} \]

where 'x' and 't' are the original (non-dimensional) space and time variables respectively. When the attenuation factor \( d^{-1/4} \) is removed the final equation takes the form

\[ u_x + d^{-\frac{7}{4}} uu_{\xi} + d^{\frac{1}{2}} u_{\xi\xi\xi} = 0, \]

\[ d = d(x) \tag{3.2.2} \]

where \( u(\xi, x) \) is proportional to the elevation of the wave. It is to be noted that in (3.2.2) the region of changing depth (\( d=0(1), x = 0(1) \)) and the 'period' (\( \xi = 0(1) \)) of the wave are of the same order of magnitude. However, in the original non-dimensional variables the region of changing depth is extended (having length of \( 0(\varepsilon^{-1}), \varepsilon \to 0 \)) and the 'wave length' is still \( 0(1) \). The change in depth need not be sudden even as a function of the far-field co-ordinate \( x \). In fact, it may occur asymptotically rapidly or slowly. Also it was proved that if a solitary wave moves over the uniform depth (\( d=1 \)) without changing shape before reaching the shelf,
it breaks up into a finite number of solitons \( n \) on the shelf provided

\[ d_o = \left[ \frac{1}{2} n(n+1) \right]^{-\frac{4}{3}} \]  \hspace{1cm} (3.2.3)

where '\( d_o \)' is the depth of the shelf and '\( n \)' is an integer \( n \geq 1 \). In a subsequent paper [Johnson(1972)] this result was confirmed and some numerical solutions of (3.2.2) for various shelf depths were presented. But the problem of ultra-slowly varying depth \( (\varepsilon \to 0) \) was not examined. An approach suitable for dealing with such problems was developed by Johnson (1973b) and an asymptotic solution to (3.2.2) as \( \varepsilon \to 0 \), with a solitary-wave initial condition was constructed.

Evolution of a wave should be determined according to the relative importance of non-linearity, dispersion and nonhomogeneity. Problems involving small and slowly varying nonhomogeneities, in a sense that a perturbation method in terms of a small parameter is applicable, lead to equations with slowly varying coefficients [Kakutani (1971) and Jeffrey and Kawahara (1982)]. Kakutani has shown that a modification of the KdV equation can describe shallow-water wave propagation
over gently sloping bottoms. The generalized KdV equation derived by Jeffrey and Kawahara

\[
\frac{\partial B}{\partial t_3} + c \frac{\partial B}{\partial x_3} + \mu \frac{\partial^2 B}{\partial x_3^2} + \gamma B \frac{\partial B}{\partial \phi} + \delta B = 0, \quad (3.2.4)
\]

where \( \phi = k(x_2, t_2, \ldots) \) and \( \delta \) is a phase variable, \( \mu, \gamma \) and \( \delta \) are functions of slow variables \( x_3 \) and \( t_3 \), \( B \) is a real function covering the result obtained by Kakutani. Another modification of the KdV equation was given by Grimshaw (1978).

\[
\eta_x + \delta \gamma^{-1} \eta \eta_x - c_1 \eta^{\circ 4} \eta^{\circ 4} = 0, \quad (3.2.5)
\]

where \( X = \varepsilon^{\frac{1}{3}} X \) (\( \varepsilon \) is a measure of weak dispersion),

\[
\mathcal{I} = \varepsilon^{-2} \int_0^X c_0(X')^{-1} dX' - \varepsilon t, \quad \eta = \gamma \eta^{(o)} \quad (\eta^{(o)} \) is the height of the interface), \( \gamma \) is an appropriate 'Green's law' factor, and \( \delta, c_0, c_1 \) are functions of \( X \).

The amplitude of a solitary wave in a channel of gradually varying depth would vary inversely as depth [Hiles (1980)]. A balance between geometry of depth and geometry of waves can be thought to exist
Approximate solutions of variable coefficients KdV equation for one-dimensional waves over a bottom of variable depth show how the wave shape changes as it moves into shallower water [Cramer et al. (1985)]. Some other works related to this are due to Peregrine (1968), Clements and Rogers (1975), Kawahara (1976), Miles (1979) and Watanabe and Yajima (1984).

3.3. INTEGRABLE AND NON-INTEGRABLE SYSTEMS

One of the important developments in mathematical physics was the discovery of inverse scattering transform (IST) method whereby the initial-value problem for a nonlinear wave system can be solved exactly through a succession of linear calculations. This method can be viewed as a generalization of Fourier analysis in the sense that it provides the exact solution to certain nonlinear evolution equations, just as the Fourier transform does for certain linear evolution equations. For any dynamical system, there exist true connections between solvability and integrability conditions. Nonlinear evolution equations which are exactly solvable by IST are said to satisfy the integrability condition. The term "integrable" is more
commonly referred to as "completely integrable" but this latter term has very different connotations in the study of Hamiltonian systems, which motivates our choice of the former. The existence theorem on the solution of ODEs indicate that the integrability of the dynamical system cannot influence the local character of the solution, as long as the analytic region is concerned. Thus the integrability is usually discussed in connection with the global or long time behaviour of the solution.

The existence of infinitely many conservation laws, the existence of multi-soliton solutions and solvability by inverse scattering are closely related. Miura, Gardner and Kruskal (1968) have discovered the existence of infinite sequence of explicit conservation laws for the KdV equation. The existence of infinite number of conserved quantities clearly added confidence that explicit solutions would be found.

Let us first consider the integrability of a system of ODEs. The solutions of a system of ODEs are regarded as (analytic) functions of a complex (time) variable 't'. The "movable" singularities [Ince(1956),
Hille (1976)] of the solution are the singularities of the solution (as a function of complex t) whose location depends on the initial conditions, and are hence movable (fixed singularities occur at points where the coefficients of the equation are singular). The system is said to possess the PP when all the movable singularities are single-valued (simple poles). When the system possesses PP it is integrable [Tabor and Weiss (1981)].

It was Kowalevskaya (1889) who first used PP to completely integrate a dynamical system of physical significance. It was shown that when the system is integrable there exists a converging power series expansion of solution in the neighbourhood of the singularities. With respect to the value of the Kowalevskaya exponents one can prove the existence of a number of first integrals that make the system integrable or non integrable. With a widely growing interest in dynamical systems and non-linear evolution equations in the 1970's these classical results were revived in a somewhat unexpected way.

The connection between ODEs of the Painlevé
type and the integrable PDEs has been pointed out by Ablowitz et al. (1977). Ablowitz, Ramani and Segur (1978, 1980 a, b) have conjectured that PDEs solvable by IST are closely connected with the six types of Painlevé equations (PI-P VI) [Ince (1956)]. Also, if the similarity reduced equation is any one of the six Painlevé equations then the given PDE is integrable and the respective dynamical system is fully deterministic, otherwise chaotic [Bountis (1985)]. The works of Jimbo, Kruskal and Miwa (1982), Weiss, Tabor and Carnevale (1983), Weiss (1983, 1984), Chudnovsky and Chudnovsky, and Tabor (1983); Ramani, Dorizzi and Grammaticos (1983); and Steeb et al. (1983) led to a conjecture that the integrability is related to the PP for PDEs also. A given PDE is said to be integrable if it possesses the PP or can be transformed to a PDE of Painlevé type. We shall note here that the last statement assumes a definition of PP in the case of PDEs. This will be discussed in the following chapters. Steeb and Grauel (1984) in their "Singular Point Analysis" for PDEs demonstrated that the Kadomstev-Petviashvili (K-P) equation has the PP.
3.4. A KdV EQUATION WITH VARIABLE COEFFICIENTS

We introduce a variable coefficients KdV equation

\[ u_{,t} + \alpha t^n u_{,x} + \beta t^m u_{,xxx} = 0, \tag{3.4.1} \]

where \( \alpha \) and \( \beta \) are constant parameters and \( n \) and \( m \) are real numbers. The celebrated KdV equation is obtained when \( n = m = 0 \). For \( \alpha = 3/2, \beta = 1/6 \) and \( m = 0, n = 1/2 \), we can transform (3.4.1) to the well-known purely concentric KdV equation.

\[ 2 v_{,t} + v/t + 3 v v_{,x} + \frac{1}{3} v_{,xxx} = 0 \tag{3.4.2} \]

through a nonlinear transformation

\[ u = v \sqrt{t}. \tag{3.4.3} \]

Equation (3.4.2) is studied by several authors [Calogero and Degasperis (1978 a,b), Nakumara (1980), Steeb et al. (1983), and Knickerbocker and Newell (1985)]. Some soliton like solutions of (3.4.2) in terms of Airy functions have also been developed.
Such equations like (3.4.1) is particularly significant in the study of the development of a steady solitary wave as it enters a region where the bottom is no longer level [Maxon and Viecelli (1974), Miles (1978) and Johnson and Thompson (1978)].

In terms of the transformation
\[ u = \frac{128}{a} t^{m-n} (\log F)_{xx}, \quad (3.4.4) \]
equation (3.4.1) can be rewritten into the bilinear form
\[
\frac{m-n}{t} F_{,x} F_{,x} - F(F_{,t} + \beta t^m F_{,xxx})_x - F_{,x}(F_{,t} + \beta t^m F_{,xxx}) + 3\beta t^m (F_{,xx}^2 - F_{,x} F_{,xxx}) = 0
\]
\[
(3.4.5)
\]
we shall note that for \( m = n \), equation (3.4.5) reduces to a bilinear equation which can be exactly solved by using a kind of perturbation method [Whitham (1974)] as in the case of KdV equation with constant coefficients. But the method fails when \( m \neq n \).

The following chapters are devoted to study the integrability of equation (3.4.1).