Chapter-V

SIMILARITY ANALYSIS AND EXACT SOLUTION OF A VARIABLE COEFFICIENTS KORETEWEG-DE VRIES EQUATION

5.1. INTRODUCTION

One of the most important methods for developing exact solutions of PDEs is that of reducing the number of variables exploiting continuous symmetries of the system. The solutions obtained by this procedure are generally called similarity solutions. This method has been widely used in the past for developing solutions as well as for the test of PP of various systems [Shen and Ames (1974), and Lakshmanan and Kaliappan (1983)].

5.2. LIE GROUPS, LIE ALGEBRAS AND SIMILARITY SOLUTIONS

Sophus Lie has widely investigated systems of PDEs that are invariant under transformation groups called Lie groups. A Lie group is a topological group in which there exists some neighbourhood $N_0$ of the identity that can be mapped homeomorphically onto an open bounded subset of the real Euclidean space $\mathbb{R}_n$ for some $n$. Knowing the

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group of transformation the most general PDE invariant under the group can be constructed.

Given a Lie group $G$ it is possible to construct a corresponding Lie algebra $\mathcal{L}$ [Sudarshan and Mukunda (1974) and Olver (1986)] in some neighbourhood of the identity. A Lie algebra $\mathcal{L}$ is a finite $(n)$ dimensional real vector space in which a Lie bracket is defined which is linear, antisymmetric and satisfies Jacobi identity. For any $m$-parameter Lie group the infinitesimal operators form an $m$-dimensional Lie algebra.

A similarity solution is a solution obtained from group invariance. This integration procedure is based on the invariance of the differential equation under a continuous group of symmetries. The invariance of a first order differential equation under a group leads to the construction of an integrating factor and a reduction to quadrature. When a PDE is invariant under a transformation group, it is possible to find similarity solutions of the equation and its independent variables can be reduced by one. Knowing a symmetry group of a system of differential equations, we can construct new solutions of the system from known ones. Also new nonlinear PDEs reducible to the Painlevé equations
can be derived through special transformations constructed by similarity variables of well-known one-dimensional soliton equations [Kawamoto (1983)]. Group invariant solutions have been used to describe the asymptotic behaviour of much more general solutions to systems of PDEs.

In chapter IV we have analysed the existence of ABTs, LPs and the PP of the KdV equation with variable coefficients. In this chapter we are reporting some similarity solutions and in a particular case an exact solution of the equation using the standard similarity method.

5.3. SIMILARITY TRANSFORMATIONS OF A PDE

We shall give the essential details [Bluman and Cole (1974)] of the Lie continuous point group similarity transformation method to reduce the number of independent variables of a PDE,

\[ F(x, t, u, u_t, u_x, u_{xx}, \ldots) = 0 \quad (5.3.1) \]

under a family of one-parameter infinitesimal continuous point group transformations

\[ x = x + \varepsilon X(x,t,u) + O(\varepsilon^2), \quad (5.3.2) \]
\[ t = t + \varepsilon T(x,t,u) + O(\varepsilon^2), \quad (5.3.3) \]
\[ u = u + \varepsilon U(x,t,u) + O(\varepsilon^2). \quad (5.3.4) \]

Here X, T and U are the infinitesimals of the variables
\(x, t\) and \(u\), respectively, and \(\varepsilon\) is an infinitesimal parameter.

The derivatives of \(u\) are also transformed according to

\[
\begin{align*}
\frac{\partial u}{\partial x} &= u_x + \varepsilon [U_x] + O(\varepsilon^2), \\
\frac{\partial u}{\partial t} &= u_t + \varepsilon [U_t] + O(\varepsilon^2), \\
\frac{\partial^3 u}{\partial x^3} &= u_{xxx} + \varepsilon [U_{xxx}] + O(\varepsilon^2),
\end{align*}
\] (5.3.5)

(5.3.6)

(5.3.7)

where \([U_x]\), \([U_t]\) and \([U_{xxx}]\) are the infinitesimals of the transformations of derivatives \(u_x\), \(u_t\) and \(u_{xxx}\). These are called the first and third extensions depending on the order of the derivative term. These "extensions" [Bluman and Cole (1974)] are given by

\[
\begin{align*}
[U_x] &= U_x + (U_u - U_{xx})u_x - T_x u_t - X_u u_x^2 - T_u u_x u_t, \\
[U_t] &= U_t + (U_u - T_{xx}) u_t - X_t u_x - T_u u_t^2 - X_u u_x u_t,
\end{align*}
\] (5.3.8)

(5.3.9)

and

\[
\begin{align*}
[U_{xxx}] &= U_{xxx} + (3U_{xxx} - X_{xxx}) u_x - T_{xxx} u_t \\
&\quad + 3(U_{xxx} - X_{xxx}) u_x^2 - 3T_{xxx} u_x u_t \\
&\quad + (U_{xxx} - 3X_{xxx}) u_x^3 + 3(U_{xxx} - X_{xxx}) u_x
\end{align*}
\]
The invariance requirement of (5.3.1) under the set of transformations (5.3.2)-(5.3.10) leads to the invariant surface condition

\[ T \frac{\partial F}{\partial t} + X \frac{\partial F}{\partial x} + U \frac{\partial F}{\partial u} + [U_x] \frac{\partial F}{\partial u_x} + [U_t] \frac{\partial F}{\partial u_t} + [U_{xxx}] \frac{\partial F}{\partial u_{xxx}} = 0 \]  

(5.3.11)

On solving (5.3.11), the infinitesimals \(X, T\) and \(U\) can be uniquely determined, which give the similarity group under which the system (5.3.1) is invariant. By the infinitesimal
transformations (5.3.2)-(5.3.4) we have

\[ u(x + \epsilon x + O(\epsilon^2), t + \epsilon t + O(\epsilon^2)) \]

\[ = u + \epsilon U + O(\epsilon^2) \quad (5.3.12) \]

On expanding and equating the \( O(\epsilon) \) terms on either side of (5.3.12) we get

\[ T \frac{du}{dt} + X \frac{du}{dx} - U = 0 \quad (5.3.13) \]

The solutions of (5.3.13) are obtained by Lagrange's condition

\[ \frac{dt}{T} = \frac{dx}{X} = \frac{du}{U} \quad (5.3.14) \]

Equations (5.3.14) give the solution

\[ x = x(t, c_1, c_2), \quad (5.3.15) \]

\[ u = u(t, c_1, c_2), \quad (5.3.16) \]

where \( c_1, c_2 \) are arbitrary integration constants. The constant \( c_1 \) plays the role of an independent variable called the similarity variable \( x \) and \( c_2 \) that of a dependent variable called the similarity solution \( f(x) \) such that
\[ u(x,t) = f(\sigma) \] (5.3.17)

Substituting (5.3.17) in the original equation (5.3.1) the resultant equation is an ODE involving only the derivatives with respect to the similarity variable \( \sigma \).

5.4. SIMILARITY TRANSFORMATION AND LIE ALGEBRA OF VARIABLE COEFFICIENTS KdV EQUATION

Under the family of infinitesimal transformations (5.3.2)-(5.3.4) the variable coefficients KdV equation (3.4.1) yields

\[
[U_t] + at^n (u_x U + u[U_x]) + ant^{n-1} u_{xx} T
\]
\[
+ \beta t^m [U_{xxx}] + \beta m t^{m-1} u_{xxx} T = 0 \] (5.4.1)

On substituting the expressions for the extensions from (5.3.8)-(5.3.10) and solving for the infinitesimals \( \xi, T \) and \( U \) we get the constraint equations

\[- x_t + at^n (U_u (U_x - x_x)) + nat^{n-1} u_T = 0, \] (5.4.2)

\[ u_t + at^n u U_x + \beta t^m U_{xxx} = 0, \] (5.4.3)
The constraints (5.4.2)-(5.4.6) can be uniquely solved. Then we get the following solutions for $X, T$ and $U$.

(i) when $m$ and $n$ are arbitrary,

$$T = 0,$$  \hfill (5.4.7)

$$X = a\frac{x^{n+1}}{(n+1)} + b,$$  \hfill (5.4.8)

$$U = a$$  \hfill (5.4.9)

For the Lie algebra,

$$g_1 = \frac{at^{n+1}}{n+1} \frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$  \hfill (5.4.10)

$$g_2 = \frac{\partial}{\partial x},$$  \hfill (5.4.11)

$$[g_1, g_2] = 0$$  \hfill (5.4.12)

(ii) when $m = 3n + 5$,

$$T = t,$$  \hfill (5.4.13)
\[ X = (2+n)x + a\left[t^{n+1}/(n+1)\right] + b, \]  
\[ U = u + a. \]

The Lie algebra is the same as in the last case [(5.4.10)-(5.4.12)].

(iii) when \( m = -2 \) and \( n = -\frac{3}{2} \),

\[ T = t^{\frac{1}{2}}, \]

\[ X = -(xt^{-\frac{1}{4}}/2) - 2at^{-\frac{1}{2}} + b, \]

\[ U = (ut^{-\frac{1}{2}}/2) + (x/4a) + a. \]

The Lie algebra is same as in (5.4.10)-(5.4.12) with \( n = -\frac{3}{2} \).

In all the above cases [(5.4.7)-(5.4.18)], \( a \) and \( b \) are arbitrary integration constants.

5.5. SIMILARITY SOLUTIONS

Using (5.3.14) and (5.3.17) we can find the similarity variables, similarity reduced equations, and similarity solutions for the above three cases [(5.4.7)-(5.4.18)].
The set of infinitesimals (5.4.7)-(5.4.9) gives the similarity variable

$$\sigma_1 = t$$

(5.5.1)

and the similarity reduced equation

$$\frac{df_1}{d\sigma_1} + \frac{(n+1)a\sigma_1^n}{as^n + (n+1)b} f_1 = 0$$

(5.5.2)

The corresponding similarity solution is

$$u(x,t) = \left[(n+1)ax/as^n + (n+1)b\right]f_1$$

(5.5.3)

Equations (5.5.2) and (5.5.3) give an exact solution of the variable coefficients KdV equation (3.4.1)

$$u(x,t) = \left[a(n+1)x+c\right]\left[as^n + b(n+1)\right]$$

(5.5.4)

The solution (5.5.4) is not so useful as the third derivative with respect to the variable $x$ vanishes.

The set of infinitesimals (5.4.13)-(5.4.15) yields the similarity variable
The corresponding similarity reduced equation is

\[
\frac{\beta d^3 f_2}{d\sigma_2^2} + \sigma_2 f_2 \frac{df_2}{d\sigma_2} + f_2 - (n+2)\sigma_2 \frac{df_2}{d\sigma_2} = 0
\]

(5.5.6)

and the similarity solution is

\[
u(x,t) = t f_2(\sigma_2) - a.
\]

(5.5.7)

When \( n = -3 \), equation (5.5.6) can be reduced to a second-order equation by integration with respect to \( \sigma_2 \). This yields

\[
\beta \frac{d^3 f_2}{d\sigma_2^2} + \frac{\sigma_2}{2} f_2^2 + \sigma_2 f_2 = \text{const.}
\]

(5.5.8)

Equation (5.5.8) is not easily solvable. From equation (5.4.16)-(5.4.18) we get the similarity variable

\[
\sigma_3 = x t^\frac{1}{4} + 4sat^\frac{1}{4} - bt
\]

(5.5.9)

The corresponding similarity reduced equation is
\[ \beta \frac{d^3f_3}{d\sigma_3^3} + \alpha f_3 \frac{df_3}{d\sigma_3} + \frac{b}{2\alpha} = 0 \quad (5.5.10) \]

and the similarity solution is

\[ u(x,t) = -\frac{\sigma_3}{2\alpha} + \frac{bt}{2\alpha} + t^\frac{1}{2} f_3(\sigma_3) \quad (5.5.11) \]

Equation (5.5.10) can be exactly solved for the case \( b = 0 \). This gives the following solution of the variable coefficients KdV equation (3.4.1) for \( m = -2, n = -\frac{3}{2} \):

\[ u(x,t) = \frac{-(4\alpha+x)t^\frac{1}{2}}{2\alpha} + \frac{4t^\frac{1}{2}}{[(-\alpha/3b)(x+4\alpha t^\frac{1}{2}) + c]^2} \quad (5.5.12) \]

The exact solution (5.5.12) is real valued only when \( \alpha < 0 \) or \( \beta < 0 \) and not both simultaneously negative. The solution (5.5.12) has no characteristics of a stable configuration like "soliton" [Scott, Chu and Mc Laughlin (1973)].

5.6. SELF-SIMILAR SOLUTION

The self-similar solution can be developed for the variable coefficients KdV equation (3.4.1) using the dimensional analysis. The self-similar transformation is very
much identical to the similarity transformations; nevertheless self-similar solutions are not always obtainable by similarity procedure.

For the variable coefficients KdV equation (3.4.1) we get the self-similar transformation

\[ u(x,t) = t^{(m-3n-2)/3} P(\eta) \]  \hspace{1cm} (5.6.1)

where \( \eta(x,t) \) is the self-similar variable

\[ \eta(x,t) = xt^{-(m+1)/3} \]  \hspace{1cm} (5.6.2)

Equation (5.6.1) yields the following self-similarity reduced ODE, on substituting in (3.4.1):

\[ \beta \frac{d^3 P}{d\eta^3} + \alpha P \frac{dP}{d\eta} - \left( \frac{m+1}{3} \right) \eta \frac{dP}{d\eta} + \frac{m-3n-2}{3} P = 0 \]  \hspace{1cm} (5.6.3)

Unfortunately equation (5.6.3) cannot be easily solved for any values of \( m \) and \( n \).
5.7. DISCUSSION

Using the well-known Ablowitz-Ramani-Segur (ARS) conjecture [Ablowitz, Ramani and Segur (1978, 1980a, b), Ablowitz and Segur (1981)] one can study the PP of a PDE by reducing it to an ODE, using similarity or self-similar transformations. Equation (5.5.2) is linear and so it is clearly Painlevé-type. For \( n = -3 \), the equation (5.5.8) is not a Painlevé-type equation whereas (5.5.10) can be integrated once and reduced to Painlevé-type. This equation (5.5.15) can be reduced to a second-order equation for \( n = -1 \), but it is not Painlevé-type.

The exact solution (5.5.12) that we developed has no smooth property of a soliton solution, which indicates that the system has decaying solutions other than soliton solutions when coefficients of KdV equation are variables.