CHAPTER IV

ERRORS IN THE COMPUTATION OF DEPENDENT OCEANOGRAPHIC VARIABLES

In the preceding chapter we have discussed the random errors that occur in the measurements of the independent oceanographic variables, namely, temperature, depth and salinity. The values of these variables are used to calculate the dependent oceanographic variables, namely, density in situ, sigma-t, specific volume and specific volume anomaly. Since the measurements of the independent variables are subject to random errors, the resulting dependent variables will also be in error, the magnitude of which may be obtained using the law of propagation of errors.

4.1. Density in situ and sigma-t

The density of any substance is defined as the mass per unit volume and in the cgs system it is stated in grammes per cubic centimetre. The specific gravity of any substance is defined as the ratio of its density to that of distilled water at $4^{\circ}C$ and under atmospheric pressure. Since in the cgs system the density of distilled water at $4^{\circ}C$ is unity, the specific gravities are numerically

identical with densities. In oceanography, the term density is generally used eventhough specific gravity is the one that is meant.

The density of sea water depends upon the three independent variables, namely, temperature, salinity and pressure. The density of a sea water sample at the temperature and pressure at which it was collected, known as the density _in situ_, is denoted by the symbol $\rho_{s,t,p}$. Knudsen (1901) introduced, for the purpose of convenient handling of numerical values, a symbol $\sigma_{s,t,p}$ as a measure of density _in situ_ defined as

$$\sigma_{s,t,p} = (\rho_{s,t,p}-1)\ 10^3 \qquad (4.1)$$

The corresponding quantity at atmospheric pressure is written as σ_t and that at atmospheric pressure and $0^{\circ}C$ as σ_0. σ_t may be obtained from the relation (Knudsen, 1901)

$$\sigma_t = -\Sigma_t + (\sigma_0+0.1324)\ [1-A_t+3_t(\sigma_0-0.1324)] \qquad (4.2)$$

where

$$\Sigma_t = \frac{(T-3.98)^2}{503.570} \times \frac{(T+283)}{(T+67.26)}, \qquad (4.3)$$

$$\sigma_0 = -0.069+1.4708\ Cl-0.001570\ Cl^2$$

$$+0.0000398\ Cl^3, \qquad (4.4)$$

$$A_t = [4.7867 \ T - 0.098185 \ T^2 + 0.0010843 \ T^3] \ 10^{-3};$$

$$\cdot (4.5)$$

and

$$B_t = [18.030T - 0.8164 \ T^2 + 0.01667 \ T^3] \ 10^{-6}$$

$$(4.6)$$

$\sigma_{s,t,p}$ as a measure of density _in situ_ may be obtained by the addition of the correction terms computed from formula established by Ekman (1908).

The error in the result of computation for σ_t due to the errors in the temperature and chlorinity measurements may be obtained by taking the total differential of the expression given above and the result may be written as follows.

$$d\sigma_t = \alpha \ dCl + \beta dT \qquad (4.7)$$

where

$$\alpha = [1.4708 - 0.003140 \ Cl + 0.0001194 \ Cl^2] \ x$$
$$[1-A_t + 2 \ \sigma_o \ B_t] \qquad (4.8)$$

$$\beta = (\sigma_o + 0.1324) \, [(\sigma_o - 0.1324) \, (18.030 -$$
$$1.6328 \, T + 0.05001 \, T^2) \, 10^{-6} -$$
$$(4.7867 - 0.196370 \, T + 0.0032529 \, T^2) \, 10^{-3}] -$$
$$\Sigma_t (\frac{2}{T-3.98} + \frac{1}{T+283} - \frac{1}{T+67.26})^* \qquad (4.9)$$

The coefficients α and β calculated for different values of temperature and chlorinity are shown as Table III(A) and Table III(B) respectively. Table III(B) shows that β Can be either positive or negative. But, since the error dT is just as likely to be positive as negative, we should take the term β dT with a positive sign to be sure of the maximum error in σ_t.

In chapter 3 we have discussed the magnitudes of the random errors in the measurements of temperature and

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*Fomin (1964) gives the expression for β as

$$\beta = (\sigma_o + 0.1324) [\![(\sigma_o - 0.1324) \, (18.030 - 1.6328 \, T +$$
$$0.05001 \, T^2) \, 10^{-6} - (4.7867 - 0.196370 \, T +$$
$$0.0032529 \, T^2) \, 10^{-3} - \Sigma_t (\frac{2}{T-3.98} + \frac{1}{T+283} - \frac{1}{T+67.26})]\!]$$
$$(4.10)$$

Here the position of the double bracket is at the end of the expression which is wrong. This is not a printing mistake because Fomin actually used the wrong expression for the computation of values of β.

chlorinity. It was established that the error in the
measurement of temperature in a tropical sea should be
considered as $\pm 0.01^{\circ}C$ upto a depth of 200 m and as $\pm 0.02^{\circ}C$
below 200 m. The temperature observed at a depth of 200 m
in a tropical sea is around $13^{\circ}C$ and so we can estimate
the error committed in the measurement of temperature above
$13^{\circ}C$ to be $\pm 0.01^{\circ}C$ and that below $13^{\circ}C$ to be $\pm 0.02^{\circ}C$.
We have also established that the error in the chlorinity
determinations by the method of chlorinity titration is
$\pm 0.02\%$. and that by the method of conductivity ratio
measurement is $\pm 0.01\%$.. Assuming as above the errors in
the determination of temperature and chlorinity, the
errors in the computed values of σ_t for different values
of temperature and chlorinity may be obtained using
expression (4.7) and Tables III(A) and III(B). It can
easily be observed that for almost all values of
temperature and chlorinity, the error in σ_t is greater
than 0.015 when chlorinity is determined by the method of
conductivity ratio measurement and that the error is around
± 0.03 when the chlorinity is estimated by the method of
chlorinity titration. Hence in the discussion that
follows we will assume that the error in the computation
of σ_t is ± 0.02 when the chlorinity is determined by the
method of conductivity ratio measurement and is ± 0.03

Table III(A)

Variation of coefficient α with temperature and chlorinity

Cl ‰	T°C											
	-2	1	4	7	10	13	16	19	22	25	28	31
10	1.46	1.45	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.34	1.34
12	1.46	1.44	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.34	1.34
14	1.46	1.44	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.35	1.34
16	1.45	1.45	1.43	1.42	1.40	1.39	1.38	1.37	1.36	1.36	1.35	1.34
18	1.46	1.45	1.43	1.42	1.40	1.39	1.38	1.37	1.37	1.36	1.35	1.35
20	1.47	1.45	1.44	1.42	1.41	1.40	1.39	1.38	1.37	1.36	1.36	1.35
22	1.47	1.45	1.44	1.43	1.41	1.40	1.39	1.38	1.38	1.37	1.36	1.36

Table III(B)

Variation of coefficient β with temperature and chlorinity

Cl‰	T°C											
	-2	1	4	7	10	13	16	19	22	25	28	31
10	+0.03	-0.01	-0.06	-0.10	-0.13	-0.17	-0.20	-0.23	-0.26	-0.28	-0.31	-0.33
12	+0.02	-0.03	-0.07	-0.11	-0.14	-0.17	-0.20	-0.23	-0.26	-0.29	-0.31	-0.34
14	+0.01	-0.04	-0.08	-0.12	-0.15	-0.18	-0.21	-0.24	-0.27	-0.29	-0.32	-0.34
16	0.00	-0.05	-0.09	-0.12	-0.16	-0.19	-0.22	-0.25	-0.27	-0.30	-0.32	-0.34
18	-0.02	-0.06	-0.10	-0.15	-0.17	-0.20	-0.22	-0.25	-0.28	-0.30	-0.32	-0.35
20	-0.03	-0.07	-0.11	-0.14	-0.17	-0.20	-0.23	-0.26	-0.28	-0.31	-0.33	-0.35
22	-0.04	-0.08	-0.12	-0.15	-0.18	-0.21	-0.24	-0.26	-0.29	-0.31	-0.33	-0.35

when the chlorinity is determined by the method of chlorinity titration.

Cox et al. (1968) have pointed out that the absolute density of distilled water at $4^\circ C$ and at atmospheric pressure is about 3 parts in 10^5 below unity and hence the absolute density of sea water is not quite equal to its specific gravity. They also pointed out that the distilled water used by Forch et al. (1902) in their specific gravity determinations was of unknown isotopic composition. Since the distillation procedure used by them is not given, it is now not possible to reproduce the standard used by Forch et al. (1902). This situation warranted a new investigation into the relationship between salinity, temperature and σ_t. Cox et al. (1970) have described an apparatus designed for this purpose. They found that Knudsen's tables gave slightly lower values of specific gravity over the salinity range of normal sea water and the deviation became larger for lower salinities. Investigations of Kremling (1971,1972) using a more accurate density comparison instrument, developed by Kratky et al. (1969), confirmed the above conclusions of Cox et al. (1970). Kremling's work showed that Knudsen's values were low, on an average, by 0.013 in σ_t in the salinity range from

9‰. to 39‰. and the deviation was about 0.025 in σ_t at a salinity of 5‰..

Using the data obtained from his experiments Cox et al. (1970) have developed an emperical relationship between temperature, salinity and σ_t, valid in the salinity range from 9‰. to 41‰., as:

$$\sigma_t = 8.00969062 \times 10^{-2} + 5.88194023 \times 10^{-2} \, T +$$
$$7.97018644 \times 10^{-1} S - 8.11465413 \times 10^{-3} \, T^2 -$$
$$3.25310441 \times 10^{-3} \, ST + 1.31710842 \times 10^{-4} \, S^2 +$$
$$4.76600414 \times 10^{-5} \, T^3 + 3.89187483 \times 10^{-5} \, ST^2 +$$
$$2.8797153C \times 10^{-6} \, S^2 T - 6.11831499 \times 10^{-8} \, S^3$$

$$(4.11)$$

As before, the error in the result of computation for σ_t due to the errors in the temperature and salinity measurements may be obtained by taking the total differential of the above expression and the result may be written as:

$$d\sigma_t = a \, ds + b \, dt \qquad (4.12)$$

where

$$a = 7.9702 \times 10^{-1} - 3.2531 \times 10^{-3} \, T +$$
$$2.6342 \times 10^{-4} \, S + 3.8919 \times 10^{-5} \, T^2 +$$
$$5.7594 \times 10^{-6} \, ST - 1.8355 \times 10^{-7} \, S^2 \qquad (4.13)$$

$$b = 5.8319 \times 10^{-2} - 1.6629 \times 10^{-2} \, T -$$
$$3.2531 \times 10^{-3} \, S + 1.4298 \times 10^{-4} \, T^2 +$$
$$7.7838 \times 10^{-5} \, ST + 2.8797 \times 10^{-6} \, S^2 \qquad (4.14)$$

The coefficients a and b are calculated for different values of temperature and salinity and are shown as Table IV(A) and Table IV(B) respectively. Table IV(B) shows that b can be either positive or negative. But, since the error dT is just as likely to be positive as negative, we should take the term bdT with a positive sign to be sure of the maximum error in σ_t.

Considering a tropical sea and following the same argument discussed in connection with the computational error in σ_t when Knudsen's relation is used for the purpose, we can estimate that the error committed in the measurement of temperature above $13^{\circ}C$ is $\pm 0.01^{\circ}C$ and that below $13^{\circ}C$ is $\pm 0.02^{\circ}C$. As established in section 3.3, the error in the determination of salinity by the chlorinity titration method is $\pm 0.04\%$. and that by the method of conductivity ratio measurement is $\pm 0.02\%$.. Quantifying as above the errors in the determination of temperature and salinity, the error in the computed value of σ_t for different values of temperature and salinity may be obtained using the relation (4.12) and Tables IV(A) and IV(B). Here also it may be observed

Table IV(A)

Variation of coefficient 'a' with temperature and salinity

S‰	T°C											
	-2	1	4	7	10	13	16	19	22	25	28	31
15	0.81	0.80	0.79	0.78	0.77	0.77	0.76	0.75	0.75	0.75	0.74	0.74
20	0.81	0.80	0.79	0.78	0.77	0.77	0.76	0.76	0.75	0.75	0.74	0.74
25	0.81	0.80	0.79	0.78	0.78	0.77	0.76	0.76	0.75	0.75	0.75	0.74
30	0.81	0.80	0.79	0.78	0.78	0.77	0.77	0.76	0.76	0.75	0.75	0.75
35	0.81	0.80	0.79	0.78	0.78	0.77	0.77	0.76	0.76	0.75	0.75	0.75
40	0.81	0.80	0.80	0.79	0.78	0.77	0.77	0.76	0.76	0.76	0.75	0.75

Table IV(B)

Variation of coefficient 'b' with temperature and salinity

S‰	T°C											
	-2	1	4	7	10	13	16	19	22	25	28	31
15	+0.04	0.00	-0.05	-0.09	-0.13	-0.17	-0.20	-0.23	-0.26	-0.29	-0.31	-0.33
20	+0.03	-0.02	-0.06	-0.10	-0.14	-0.18	-0.21	-0.24	-0.27	-0.29	-0.31	-0.33
25	+0.01	-0.04	-0.08	-0.12	-0.15	-0.19	-0.22	-0.25	-0.27	-0.30	-0.32	-0.34
30	-0.01	-0.05	-0.09	-0.13	-0.16	-0.20	-0.23	-0.26	-0.28	-0.30	-0.32	-0.34
35	-0.02	-0.07	-0.10	-0.14	-0.18	-0.21	-0.24	-0.26	-0.29	-0.31	-0.33	-0.34
40	-0.04	-0.08	-0.12	-0.15	-0.19	-0.22	-0.25	-0.27	-0.29	-0.31	-0.33	-0.35

that the magnitude of the error in the computed value of σ_t is such that d σ_t may be taken as ± 0.02 when the salinity is determined by the method of conductivity ratio measurement and as ± 0.03 when the salinity is determined by the method of chlorinity titration.

This shows that the result can be in error by ± 0.02 in σ_t even when using the best available methods for the measurement of temperature and salinity. For more accurate results we may have to go for instrumental methods. One such instrument described by Cox et al. (1970), eventhough gives accurate results (the accuracy claimed is ± 0.008 in σ_t), is not suitable for routine work because of its complexity. Kratky et al. (1969) has developed a more accurate density comparison instrument suitable for routine work. Although this instrument appears to be rather slower in operation compared to modern salinometers, it gives better results compared to computational methods. The accuracy claimed for the instrument is $\pm 1.5 \times 10^{-6}$ in density for a variation of temperature by $\pm 0.01^{\circ}C$. Assuming the temperature variation in the instrument to be within $\pm 0.01^{\circ}C$, since the maximum error in the determination of temperature is $\pm 0.02^{\circ}C$, the maximum error in the determination of

density $\underline{in}$ $\underline{situ}$ will be 4.5×10^{-6} or less than half

a unit in the second decimal place of the σ_t value.

This means that, if the accuracy claimed for the

instrument can really be achieved in the routine work,

the result will be at least four times more accurate than

the result obtained from computational methods.

4.2. Specific volume and specific volume anomaly

Instead of density $\underline{in}$ $\underline{situ}$, its reciprocal,

$\alpha_{s,t,p}$, called the specific volume $\underline{in}$ $\underline{situ}$, is generally

used in Dynamic Oceanography. In order to avoid writing

a large number of decimals, the specific volume is

commonly expressed as an anomaly, δ defined as

$$\delta = \alpha_{s,t,p} - \alpha_{35,0,p} \qquad (4.15)$$

where $\alpha_{35,0,p}$ is the specific volume of a 'standard ocean'

of salinity 35‰ at $0^{\circ}C$ and at pressure p expressed in

decibars. The values of $\alpha_{35,0,p}$ for the 'standard ocean'

were first tabulated by Bjerknes and Sandstrom (1910).

The value of the anomaly δ , which contains the departures

of the real ocean from the 'standard ocean' depends on

the temperature, salinity and pressure and hence can be

expressed as

$$\delta = \delta_s + \delta_t + \delta_{s,t} + \delta_{s,p} + \delta_{t,p} + \delta_{s,t,p} \qquad (4.16)$$

The last term of the above expression, $\delta_{s,t,p}$, is so small

that it can always be neglected. The first three terms, which are independent of pressure, are usually grouped together and the sum is known as the thermo-steric anomaly, δ_T. The value of δ_T may be obtained from the value of σ_t using the relation

$$\delta_T = 0.02736 - \frac{10^{-3}\sigma_t}{1+10^{-3}\sigma_t} \qquad (4.17)$$

Hence

$$\delta = \delta_T + \delta_{s,p} + \delta_{t,p} \qquad (4.18)$$

The values of the three terms on the right hand side of the above expression may be obtained from Oceanographic Tables and hence the specific volume anomaly in situ may be computed. Sverdrup (1933) was the first to compile these tables.

The error in the result of computation for δ due to the errors in the measurements of temperature, salinity and pressure may be obtained by taking the total differential of the above expression.

$$d\delta = d\delta_T + d\delta_{s,p} + d\delta_{t,p} \qquad (4.19)$$

The magnitudes of the errors $d\delta_{s,p}$ and $d\delta_{t,p}$ are small compared to the magnitude of error $d\delta_T$ so that the former two are neglected in the following discussions.

Hence

$$d\delta \simeq d\delta_T \qquad (4.20)$$

$d\delta_T$ may easily be obtained from the relation (4.17) as

$$d\delta_T = - \frac{10^{-3}d\ \sigma_t}{(1+10^{-3}\sigma_t)^2} \qquad (4.21)$$

Hence

$$d\delta \simeq d\delta_T = - \frac{10^{-3}\ \sigma_t}{(1+10^{-3}\sigma_t)^2} \qquad (4.22)$$

The negative sign in the above expression is immaterial since $d\sigma_t$ is just as likely to be positive as negative.

We have already discussed the magnitude of $d\sigma_t$ and found that it is ± 0.02 in σ_t when salinity is obtained using the method of conductivity ratio measurement and is ± 0.03 in σ_t, when the salinity is measured using the method of chlorinity titration. Using these values for the error in σ_t it can easily be estimated, using the expression (4.22), that the error in the specific volume anomaly is ± 2 c l/ton when the former method is used for the determination of salinity and is ± 3 c l/ton when the latter method is used for the purpose[*]. Since the error in the determination of σ_t by

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[*]There is a widely used graphical method described by Montgomery (1954) and Montgomery and Wooster (1954) for

instrumental method, more specifically, when the
instrument developed by Kratky et al. (1969) is used for
the purpose, is less than half a unit in its second
decimal place, the corresponding error in the value of

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the determination of the values of thermosteric anomaly at
different depths from known values of temperature and
salinity. The method involves, as a first step, drawing
of the temperature-depth curve for a station on a graph
sheet which has the temperature marked along the abscissa,
depth and salinity marked along the ordinate and super
imposed with a family of curves representing thermosteric
anomaly. Salinity is then plotted against temperature
(This is essentially a T-S curve). Using these two
curves, depths of chosen values of thermosteric anomalies
are obtained.

The smallest division on the above mentioned graph
sheet is equivalent to $0.05^{\circ}C$ in temperature and $0.05‰$.
in salinity and so plotting of a point on the T-S curve
introduces an additional error of $\pm 0.025^{\circ}C$ in temperature
and $\pm 0.025‰$. in salinity. Consequently the total error
in the values of temperature and salinity, to be used for
the purpose of computing the error in the value of
thermosteric anomaly obtained from the station curves,

specific volume anomaly will be less than half cl/ton.

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should be taken as $\pm0.045^{\circ}$C and $\pm0.045\%$ respectively,

assuming that salinity is obtained using the method of

conductivity ratio measurement. This results in an

error of approximately ±4 cl/ton in the value of

thermosteric anomaly obtained graphically.