# CHAPTER II

## THEORY OF ERRORS

The results of experimental observations will have always inaccuracies. It is necessary to know the magnitude of these inaccuracies in order to arrive at the inaccuracy in a computed result using the observations. A knowledge of the statistical behaviour of the inaccuracies or errors of observations will be of great help in reducing the effect of these uncertainties in the final result.

### 2.1. Errors of observations and measurements

All observations and measurements are subject to three kinds of errors: systematic errors, accidental or random errors, and mistakes.

Systematic errors are those which affect all measurements alike. They are mostly due to imperfections in the construction or adjustment of the instrument, the 'personal equation' of the observer, etc. Such errors can be remedied by applying proper corrections.

Accidental or random errors are those whose causes are unknown and indeterminate. They are usually small.

It is found experically that such random errors are frequently distributed according to a simple law, the law of chance. This makes it possible to use statistical methods to deal with random errors. The mathematical theory of errors deals with random errors only.

Mistakes are not, properly speaking, errors at all. They are blunders performed during reading of an instrument, recording of a result and in computations. They can be eliminated by careful work.

The word 'precision' is used in relation to random errors. A precise measurement will be free from random errors. An accurate measurement is one that is free from all kinds of errors - systematic, random and mistakes.

## 2.2. Errors in approximations

Numbers such as 2, 1/3, etc. are known as exact numbers because there is no uncertainty in them. On the other hand, numbers such as $\pi$, $\sqrt{3}$, etc. even though exact, cannot be expressed exactly by a finite number of digits. These numbers, when expressed in digital form, are known as approximate numbers.

Numbers of the above type and quotients of division which never terminate will have to be cut down to a manageable size to be used in practical computations. This process of cutting off superfluous digits and retaining as many as desired is known as 'rounding off'. This process, obviously, introduces an error into the number. The following rule for rounding off will cause the least possible error.

'To round off a number to n significant figures, discard all digits to the right of the $n^{th}$ place. If the discarded number is less than half a unit in the $n^{th}$ place, leave the $n^{th}$ digit unchanged; if the discarded number is greater than half a unit in the $n^{th}$ place, add one to the $n^{th}$ digit. If the discarded number is exactly half a unit in the $n^{th}$ place, round off so as to leave the $n^{th}$ digit an even number' (Scarborough, 1966). When a number is rounded off according to this rule, it will be correct to n significant figures. The error introduced into the number due to rounding off will not be greater than half a unit in the $n^{th}$ significant figure.

## 2.3. Errors in computations

When various quantities are used to calculate a result and when the different quantities have errors in them, then the result will also be in error by an amount which depends on the errors of the individual quantities.

2.3.1. The general formula for errors

Let

$$Q = f(a_1, a_2, a_3, \ldots\ldots, a_n) \qquad (2.1)$$

be a function of several independent quantities $a_1, a_2, \ldots\ldots a_n$ which are subject to errors $\delta a_1, \delta a_2, \ldots\ldots, \delta a_n$. These errors in a's will cause an error, $\delta Q$, in the function Q according to the relation

$$Q + \delta Q = f(a_1 + \delta a_1, a_2 + \delta a_2, \ldots\ldots, a_n + \delta a_n) \qquad (2.2)$$

To find an expression for $\delta Q$, we must expand the right hand side by Taylor's theorem.

$$f(a_1+\delta a_1, a_2+\delta a_2, \ldots\ldots\ldots\ldots, a_n + \delta a_n)$$
$$= f(a_1, a_2, \ldots\ldots, a_n) + \delta a_1 \frac{\partial f}{\partial a_1} + \delta a_2 \frac{\partial f}{\partial a_2} + \ldots\ldots$$
$$+ \delta a_n \frac{\partial f}{\partial a_n} + 1/2 \Big[ (\delta a_1)^2 \frac{\partial^2 f}{\partial a_1^2} +$$
$$(\delta a_2)^2 \frac{\partial^2 f}{\partial a_2^2} + \ldots\ldots + (\delta a_n)^2 \frac{\partial^2 f}{\partial a_n^2} +$$
$$2 \delta a_1 \delta a_2 \frac{\partial^2 f}{\partial a_1 \partial a_2} + \ldots\ldots \Big] + \ldots\ldots \qquad (2.3)$$

Since the errors $\delta a_1, \delta a_2, \ldots\ldots$ are relatively small we may neglect their squares, products and higher powers and so

$$Q + \delta Q = f(a_1, a_2, \ldots\ldots\ldots, a_n) + \delta a_1 \frac{\partial f}{\partial a_1} + \delta a_2 \frac{\partial f}{\partial a_2} + \ldots\ldots\ldots + \delta a_n \frac{\partial f}{\partial a_n} \qquad (2.4)$$

Substracting equation (2.1) from equation (2.4),

$$\delta Q = \delta a_1 \frac{\partial f}{\partial a_1} + \delta a_2 \frac{\partial f}{\partial a_2} + \ldots\ldots\ldots + \delta a_n \frac{\partial f}{\partial a_n}$$

$$= \frac{\partial Q}{\partial a_1} \delta a_1 + \frac{\partial Q}{\partial a_2} \delta a_2 + \ldots\ldots + \frac{\partial Q}{\partial a_n} \delta a_n \ \ldots \qquad (2.5)$$

This is the general formula for computing the error of a function and gives the absolute error. The expression, obviously, is the total differential of the function Q. The relative error in the function Q is the quotient obtained when the absolute error is divided by the true value of the quantity.

i.e.

$$\frac{\delta Q}{Q} = \frac{\partial Q}{\partial a_1} \frac{\delta a_1}{Q} + \frac{\partial Q}{\partial a_2} \frac{\delta a_2}{Q} + \ldots\ldots\ldots\ldots + \frac{\partial Q}{\partial a_n} \frac{\delta a_n}{Q} \qquad (2.6)$$

## 2.3.2. Application of the general formula for errors in the fundamental operations of Arithmetic

(i) Addition

Let the function Q be of the form

$$Q = a_1 + a_2 + \dots + a_n \qquad (2.7)$$

Applying the general formula for errors to the function Q, the absolute error $E_a$ is,

$$E_a = dQ = da_1 + da_2 + \dots + da_n \qquad (2.8)$$

Here each of the $da_i$ are just as likely to be positive as negative. In order to be sure of the maximum error in the function Q, we must take all the terms with the positive sign.

Formula (2.8) indicates that when a few numbers of different accuracies are added it would be useless and absurd to retain all the decimal digits in all the numbers because the error in the result, any way, will be greater than half a unit in the last significant figure of the least accurate number. A safe rule for addition will be to retain one more decimal digit in the more accurate numbers than is contained in the least accurate number and round off the result to the decimal digit contained in the least accurate number. The result usually will be uncertain by one unit in the last figure. Retaining one more digit in the more accurate numbers than is contained in the least accurate number eliminates the possibility of the errors due to rounding off the more accurate numbers from affecting the error in the final result.

Since errors can be just as likely to be positive as negative, their algebraic sum will never be large when a large number of approximate numbers are added. And the mean of several approximate numbers can be more accurate than the numbers from which it was obtained because the computation involves addition as well as division by the total number (Scarborough, 1966).

(ii) Subtraction

Let the function Q be of the form

$$Q = a_1 - a_2 \tag{2.9}$$

Applying the general formula for errors to the function Q

$$\delta Q = \delta a_1 - \delta a_2 \tag{2.10}$$

Since the errors $\delta a_1$ and $\delta a_2$ are as likely to be positive as negative, we should take the sum of the errors to get the maximum error in the function Q. Hence, the absolute error $E_a$ is

$$E_a = dQ = da_1 + da_2 \tag{2.11}$$

A safe rule for the subtraction between two numbers of unequal accuracy is to round off the more accurate number to the same number of decimal places as the less accurate number and then subtract. The result usually will be in error by one unit in the last figure.

(iii) Multiplication

Let the function Q be of the form

$$Q = a_1 \times a_2 \times \ldots \times a_n \tag{2.12}$$

Taking the total differential and dividing by Q, the relative error $E_r$ is,

$$E_r = \frac{dQ}{Q} = \frac{da_1}{a_1} + \frac{da_2}{a_2} + \ldots\ldots + \frac{da_n}{a_n} \tag{2.13}$$

Here again, each of the $da_i$ can be as likely to be positive as negative and in order to be sure of the maximum relative error in the function Q, we must take the sum of the relative errors.

The accuracy of a product should be investigated by means of relative error. The absolute error, can be found from the relation

$$E_a = E_r \times Q \tag{2.14}$$

While finding the product of two or more approximate numbers, the safe rule is to retain one more significant figure in the more accurate factors than that contained in the least accurate factor and to round off the result to as many significant figures as in the least accurate factor (Scarborough, 1966).

(iv) Division

Let the function be of the form

$$Q = \frac{a_1}{a_2} \tag{2.15}$$

Taking the total differential of the function and dividing by Q

$$\frac{\delta Q}{Q} = \frac{\delta a_1}{a_1} - \frac{\delta a_2}{a_2} \tag{2.16}$$

The errors $\delta a_1$ and $\delta a_2$ are as likely to be positive as negative and to get the maximum relative error in the function, we must take the sum of the relative errors. Hence, the relative error $E_r$ is

$$E_r = \frac{\delta Q}{Q} = \frac{\delta a_1}{a_1} + \frac{\delta a_2}{a_2} \qquad (2.17)$$

The accuracy of quotients also should be investigated using relative error. The safe rule for division is to retain one more significant figure in the more accurate factor than the number of significant figures contained in the less accurate factor and to round off the result to the number of significant figures contained in the less accurate factor (Scarborough, 1966).

2.3.3. The normal law of errors

Since an error can be just as likely to be positive as negative, their algebraic sum will never be large in a computation involving a large number of approximate numbers. In such situations, the error in the result should be obtained using the normal law of errors, since it is generally found that the normal distribution or Gauss distribution describes the distribution of random errors (Young, 1962).

A random error may be assumed to be the result of a large number $\ell$ of elementary errors, all of equal magnitude $\epsilon$, and each equally likely to be positive or negative. Now we assume the following facts to be true for the distribution of random errors.

i) Small errors are more frequent than large errors.

ii) All errors are equally likely to be positive or negative.

iii) Very large errors do not occur.

Then the probability of occurrence of an error x in the range $(-\ell\epsilon)$ to $(+\ell\epsilon)$ is given by the probability equation

$$y = P(x) = \frac{h}{\sqrt{\pi}} \quad e^{-h^2 x^2} \qquad (2.18)$$

where h is called the measure or index of precision of the distribution represented by the equation and known as the normal error distribution. The measure of precision h and the standard deviation $\sigma$ of the distribution is related as

$$h = \frac{1}{\sqrt{2}\,\sigma} \qquad (2.19)$$

Hence the probability equation can be written in terms of $\sigma$ as:

$$Y = P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \; e^{-\frac{x^2}{2\sigma^2}} \qquad (2.20)$$

(Scarborough, 1966)

Equation (2.18) shows that zero error has maximum probability of occurrence and the probability decreases as the magnitude of the error increases. The quantity h is called the 'index of precision' because, for larger values of h the probability of occurrence of zero error is larger and also the probability decreases faster as the magnitude of the error increases. This means that small errors are more frequent than large errors which indicates precision of the data.

For the normal error distribution, the probability of an error to fall within 1,2 and 3 standard deviations from the mean, which is zero error, is given in Table I (Young, 1962).

Table I

| $\sigma$ | P(x) | % P(x) |
|---|---|---|
| 1 | 0.683 | 68.3 |
| 2 | 0.954 | 95.4 |
| 3 | 0.997 | 99.7 |

If $M_1$, $M_2$, $M_3$, ......, $M_n$ are n independent normal error distributions having standard deviations $\sigma_1$, $\sigma_2$, $\sigma_3$ ... $\sigma_n$ respectively, then their sum

$$F = M_1+M_2+M_3 \ldots\ldots + M_n \tag{2.21}$$

will also be a normal error distribution whose standard deviation is given by (Scarborough 1966)

$$\sigma_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \ldots\ldots\ldots + \sigma_n^2} \qquad (2.22)$$

The standard deviation of the normal error distribution which results from the summation of S values, each of which may, with equal probability, contain an error equal to one of the values of the finite sequence

$$-l\epsilon,\ -(l-1)\epsilon,\ -(l-2)\epsilon,\ \ldots\ldots\ldots,\ -2\epsilon,\ -\epsilon,\ 0,$$
$$\epsilon,\ 2\epsilon,\ \ldots\ldots\ldots\ldots,\ (l-2)\epsilon,\ (l-1)\epsilon,\ l\epsilon$$

may be obtained as follows. The standard deviation of the above sequence is given by

$$\sigma = \frac{\sqrt{S\sum_{-l}^{l}(n\epsilon)^2}}{\sqrt{\sum_{-l}^{l} n}} = \frac{\epsilon\sqrt{S.2\sum_{0}^{l} n^2}}{\sqrt{2l+1}}$$

$$= \frac{\epsilon\sqrt{S\cdot 2l(l+1)(2l+1)}}{\sqrt{6(2l+1)}} = \frac{\epsilon\sqrt{Sl(l+1)}}{\sqrt{3}}$$

Assuming $\ell$ to be large, the above may be approximated to

$$\sigma = \epsilon \frac{\sqrt{S\ell^2}}{\sqrt{3}}$$

$$= \frac{\sqrt{S}\ell\epsilon}{\sqrt{3}} \; * \qquad (2.24)$$

The formula for the standard deviation in this form will be useful when only the maximum value of the individual error, $\ell\epsilon$ , is known

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*Formula (2.24) is derived in the same way as was done by Fomin (1964) except for his assumption that each of the S values of the summation contains an error, with equal probability equal to one of the intigers in the range $-\ell$ to $\ell$. With this assumption he got the formula for the index of precision as

$$h = \frac{\sqrt{6}}{2\sqrt{\ell(\ell+1)S}} \qquad (2.25)$$

The formula for the standard deviation can easily be obtained from the above as

$$\sigma = \frac{\sqrt{S\ell(\ell+1)}}{\sqrt{3}} \qquad (2.26)$$

There is a conceptual error in the above assumption that the error can assume only values of the intigers. Formula (2.26) will also result in different magnitudes for the standard deviation for different units used for the error $\ell$.

## 2.4. The law of propagation of errors

Consider a quantity Q which is calculated from several observed quantities $a_1$, $a_2$, $a_3$ ......

$$Q = f(a_1, a_2, a_3 \, .....) \qquad (2.27)$$

Suppose that $a_1, a_2, a_3$, ........ are all measured N times. Then the law of propagation of errors is expressed by the equation (Young, 1962)

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial a_1}\right)^2 {\sigma_{a_1}}^2 + \left(\frac{\partial Q}{\partial a_2}\right)^2 {\sigma_{a_2}}^2 + \left(\frac{\partial Q}{\partial a_3}\right)^2 {\sigma_{a_3}}^2 + \, .... \qquad (2.28)$$

where $\sigma_{a_1}$, $\sigma_{a_2}$, $\sigma_{a_3}$, ......... are the standard deviations of the quantities $a_1, a_2, a_3$, ........... Where the observations are not repeated, as in the case of field sciences like oceanography, meteorology, etc., the equation expressing the law of propagation of errors reduces to the general formula for errors expressed by the equation (2.5).