CHAPTER V

ERRORS IN THE COMPUTATION OF DERIVED QUANTITIES

In chapter IV the magnitudes of errors that are caused in the computation of dependent oceanographic variables due to the random errors in the measurements of the independent oceanographic variables were discussed. These dependent oceanographic variables are used in the computations of the derived quantities, namely, the dynamic depth anomalies of isobaric surfaces and the relative currents. Since the dependent oceanographic variables are subject to errors, the derived quantities computed from them are also subject to errors, the magnitudes of which are discussed in this chapter.

5.1. Dynamic depth anomaly of isobaric surfaces

Level surfaces are surfaces that are everywhere normal to the force of gravity. Since potential energy of a mass remains constant on such a surface, they are surfaces of constant gravitational potential or constant geopotential. The ideal sea surface is a level surface with zero geopotential value assigned to it. The geopotential values of other equipotential surfaces are obtained by calculating the amount of work, W, required to move a unit mass from the ideal sea surface a geometrical distance h along a plumb line

$$\text{i.e.,} \qquad W = gh \qquad (5.1)$$

where g is the acceleration due to gravity.

The numerical value of the geopotential depends on the units used in gh. Bjerknes (1910) introduced, as a unit of the geopotential, the dynamic decimetre which represents the work that must be done to lift a unit mass along the plumb line about one geometric decimetre. The practical unit is the dynamic metre, D, and is defined by

$$D(\text{dyn.m.}) = \frac{gh}{10} \left(\frac{m}{sec}\right)^2 \qquad (5.2)$$

where h is expressed in metres and g in metre/sec^2. Since the numerical value of g, when expressed in m/sec^2, is less than 10, the dynamic metre is slightly ~~greater~~ smaller numerically than the geometric depth of one metre. A level surface in the ocean can then be defined as a surface of equal dynamic depth below the ideal sea surface.

The static pressure P at any depth h metres below the sea surface is given by the weight of the water column of unit cross-section between the sea surface and depth h.

If $\bar{\rho}$ is the mean density of the water column, then

$$P = \bar{\rho}\, gh \qquad (5.3)$$

Substituting for gh from equation (5.2),

$$P = 10\, \bar{\rho}\, D \text{ bar}$$
$$= \bar{\rho}\, D \text{ decibar} \qquad (5.4)$$

Hence dynamic depth expressed in dynamic metres is given by

$$D = \frac{P}{\bar{\rho}}$$
$$= P\bar{\alpha} \qquad (5.5)$$

where P is expressed in decibars and $\bar{\alpha}$ is the mean specific volume of the water column. Since the density and the specific volume of the waters in the ocean vary in both the horizontal and vertical directions, accurate computation of the dynamic depth requires the infenitesimal form of the equation (5.5).

$$dD = \bar{\alpha}\, dp \qquad (5.6)$$

The dynamic distance between two isobaric surfaces with pressures P_o and P is obtained from equation (5.6) as

$$D = \int_{P_o}^{P} \bar{\alpha}\, dP \qquad (5.7)$$

If P_o refers to the pressure at the sea surface, then $P_o = 0$ and D represents dynamic depth of an isobaric surface with pressure P.

Using equation (4.15), equation (5.7) may be written as

$$D = \int_{P_o}^{P} \alpha_{35,0,P}\, dp + \int_{P_o}^{P} \bar{\delta}\, dp$$

$$= D_{35,0,P} + \Delta D. \qquad (5.8)$$

The first term on the right hand side of the equation (5.8), namely,

$$D_{35,0,P} = \int_{P_o}^{P} \alpha_{35,0,P}\, dp \qquad (5.9)$$

contains the contribution of a standard ocean which is invariable. The second term, namely,

$$\Delta D = \int_{P_o}^{P} \bar{\delta}\, dp \qquad (5.10)$$

contains the departures of the real ocean from the standard ocean and is called the dynamic depth anomaly between the isobaric surfaces P_o and P. Since only ΔD is variable between two isobaric surfaces with pressures P_o and P, the relative geopotential intervals between the isobaric surfaces can be obtained from an evaluation of ΔD, if $\bar{\delta}$, is known as a function of depth.

Equation (5.10) may be used in the construction of dynamic topographies of given isobaric surfaces to provide maps showing lines of equal dynamic depth anomaly of an isobaric surface below the sea surface. The dynamic topographies thus obtained are known as relative topographies since they are obtained with reference to the sea surface. Since the actual topography of the sea surface more or less deviates from a level surface, to get the absolute topography of the different isobaric surfaces we should select, as a reference surface, an isobaric surface which coincides with a level surface. A reference surface of this kind is called a reference level and can usually be found only deep below the ocean surface. Since the isobaric surface and the level surface coincide at a reference level, no water motion is possible at that level, and hence, may be called a level of no motion. When such a reference level is selected, the absolute topographies of the different isobaric surfaces are obtained by computing the dynamic height anomalies of the different isobaric surfaces taking this reference level as the zero reference surface. For this computation, equation (5.10) may be modified by changing the origin of the co-ordinate system from the sea surface to the reference level and assuming the Z axis to be directed vertically upwards, as

$$\Delta D = -\int_{P_o}^{P} \bar{\delta}\, dp = -\int_{o}^{P-P_o} \bar{\delta}\, dp$$

$$= \int_{o}^{P_o-P} \bar{\delta}\, dp \qquad (5.11)$$

where P_o is the pressure at the reference level and P is the pressure at any other isobaric surface.

In practice, metres of depth are substituted for decibars of pressure in equation (5.11). This is permissible because the error introduced due to this substitution will be negligibly small, particularly in the lower latitudes and in the upper layers of the ocean, as we have already seen in section 3.2.

$$\therefore \Delta D = \int_{o}^{H_o-h} \bar{\delta}\, dz$$

$$= \int_{o}^{Z} \bar{\delta}\, dz \qquad (5.12)$$

where H_o and h are depths from the sea surface to the reference level and any other isobaric surface respectively and Z is the height of any isobaric surface above the reference level. Equation (5.12), in finite difference form, useful for computations with discrete data, is

$$\Delta D = \sum_{1}^{n} \bar{\delta}_i \, \Delta z_i \qquad (5.13)$$

where n is the number of intervals into which the distance between the level surface and the given isobaric surface is divided.

The error in the computation of dynamic height anomaly due to the errors inherent in the determination of specific volume anomaly and depth may be obtained by taking the total differential of equation (5.13).

$$d\Delta D = \sum_{1}^{n} \Delta z_i \; d\bar{\delta}_i + \sum_{1}^{n} \bar{\delta}_i \; d\Delta z_i \qquad (5.14)$$

The contribution of the second term on the right hand side of the expression (5.14) to the total error in the determination of dynamic height anomaly may be estimated as follows. If we assume that Δz_i are error free depth increments, then an error in the determination of depth will appear as a change in the value of $\bar{\delta}_i$. We have earlier, in section 3.2 estimated the magnitude of error in the determination of depth as ± 2 m upto a depth of 200 m and ± 4 m below 200 m. The change in value of $\bar{\delta}_i$, for such depth differences, is negligible except where the variation of specific volume anomaly with depth is large. Sufficiently large variation which can introduce an appreciable change in $\bar{\delta}_i$ for depth differences equal to the magnitude of error in the determination of depth is found only in the thermocline region, that is roughly in the depth range from 50 m to 350 m. If the selected reference level is sufficiently deep, the error contributed by this term to the total error in the computation of dynamic height anomaly will be small compared to that contributed by the first term and so, this may be neglected. Hence expression (5.14) becomes

$$d\Delta D = \sum_{1}^{n} \Delta z_i \; d\bar{\delta}_i$$

$$= \sum_{1}^{n} \Delta z_i \; d\delta_i \qquad (5.15)$$

$\bar{\delta}_i = \delta_i$ because $\bar{\delta}_i$ is obtained, in practical computations, from just two values of specific volume anomalies, found at the two extremities of the depth interval Δz_i.

Expression (5.15) shows that the error in the determination of dynamic height anomaly is the result of the sum of a large number of individual errors. Since the errors are just as likely to be positive as negative, their sum will never be large and the result should be obtained using the normal law of errors. Since we know only the maximum value of each individual error, we should use the formula (2.24) derived in section 2.3, which is given as

$$\sigma = \frac{\sqrt{s}\,\lambda\epsilon}{\sqrt{3}}$$

When the depth intervals Δz_i are constant and since we have assumed that $d\delta_i$ is constant throughout the water column

$$\ell\epsilon = \Delta z \; d\delta \qquad (5.16)$$

Hence expression (~~5.15~~ 2.24) may be written as

$$\sigma = \frac{\sqrt{S}\,\Delta z \; d\delta}{\sqrt{3}} \qquad (5.17)$$

Normally the total depth is divided into several depth ranges, each of which is divided into equal depth intervals. In such situations we should calculate σ for each depth range using formula (5.17) and then get the standard deviation of the combined error distribution which represents the distribution of error in the determination of dynamic height anomaly. This can be obtained using formula (2.22) of section 2.3, given as

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots\cdots}$$

where σ_1, σ_2, etc. are standard deviations for the different depth ranges. Twice the value of the standard deviation of the combined distribution may be taken as the maximum error in the determination of dynamic height anomaly, since more than 95%. of the time the error will be within the range of $\pm 2\,\sigma$ (Section 2.3).

Formula (5.17) shows that σ increasea as S increases keeping Δz constant. This means that deeper we select the reference level, the more will be the error in the computed dynamic height anomaly. Also the formula shows that the larger the number of depth intervals into which a specific range of depth is divided, the smaller will be the computational error.

To illustrate computation of error in the dynamic height anomaly of an isobaric surface, let us compute the same for the zero decibar surface with reference to 1500 decibar surface. If we assume that data are available for the standard depths accepted by the National Oceanographic Data Centre, then there are three depth intervals of 10 m each in the depth range from 0 m to 30 m, one depth interval of 20 m in the depth range from from 30 m to 50 m, four depth intervals of 25 m each in the depth range from 50 m to 150 m, three depth intervals of 50 m each in the depth range from 150 m to 300 m and twelve depth intervals of 100 m each in the depth range from 300 to 1500 m. The value of the maximum error, $\ell\epsilon$, for the 10 m depth interval is obtained using formula (5.16) as

$$\ell\epsilon = \Delta z \, d\delta$$

$$= 10 \text{ decibar } \times 2 \text{ cl/ton (Section 4.2)}$$

$$= 10 \times 10^5 \frac{gm\ cm}{sec^2 \times cm^2} \times \frac{2 \times 10 \times cm^3}{10^5\ gm}$$

$$= 20 \frac{cm^2}{sec^2} = \frac{20}{10^4} \frac{m^2}{sec^2}$$

$$= \frac{20}{10^5} \text{ dyn. m} = 0.2 \text{ dyn. mm.}$$

Now the standard deviation σ for the depth range from 0 m to 30 m is obtained using formula (5.17) as

$$\sigma = \frac{\sqrt{S}\, \Delta z\, d\delta}{\sqrt{3}} = \frac{\sqrt{3} \times 0.2}{\sqrt{3}}$$

$$= 0.2 \text{ dyn. mm.}$$

Similarly standard deviation for the other depth ranges are computed after calculating the corresponding maximum errors and the results are tabulated in Table V. Using the values of the standard deviation for different depth ranges given in Table V, the standard deviation of the combined error distribution is obtained using formula (2.22) of section 2.3 as

$$\sigma = \sqrt{\sigma_1^2 + \sigma_1^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2}$$

$$= \sqrt{(0.2)^2 + (0.2)^2 + (0.6)^2 + (1.0)^2 + (4.0)^2}$$

$$= \sqrt{0.04 + 0.04 + 0.36 + 1 + 16}$$

$$= \sqrt{17.44} = 4.2 \text{ dyn. mm.}$$

Table V

Standard deviation of maximum error for different depth ranges

Depth range (m)	Depth interval (m)	Maximum error (dyn. mm)	σ for the depth range (dyn. mm)
0- 30	10	0.2	0.2
30- 50	20	0.4	0.2
50-150	25	0.5	0.6
150-300	50	1.0	1.0
300-1500	100	2.0	4.0

Table VI

Values of dΔD for different isobaric surfaces with reference to 1500 d bar surface

Depth (m)	0	10	20	30	50	75	100	125	150	200	250	300
dΔD (dyn.mm)	8	8	8	8	8	8	8	8	8	8	8	8

Depth (m)	400	500	600	700	800	900	1000	1100	1200	1300	1400	15
dΔD (dyn.mm)	8	7	7	7	6	6	5	5	4	3	2	

The maximum error in the computation of dynamic height anomaly of the zero isobaric surface with reference to 1500 d bar surface is twice the above value of σ and so

$$d\Delta D = 2 \times 4.2 = 8.4 \simeq 8 \text{ dyn. mm},$$

corrected to the nearest whole number. In a similar manner the maximum errors in the computation of dynamic height anomalies of the other isobaric surfaces with reference to 1500 d bar surface are computed and are presented in Table VI.

5.2. Relative current

Relative current perpendicular to the vertical plane between two stations A and B is computed using Helland-Hansen formula (Sandstrom and Helland-Hansen, 1903), derived from Bjerkne's circulation theorem (Bjerknes 1898, 1900) with the assumptions that motion is non-accelerated and frictionless and that the observations are taken simultaneously. Though not absolutely justified, these assumptions are used for the computation of currents in the ocean, since the oceanic condition is generally quasi-permanent.

Helland-Hansen's formula is derived by equating the pressure gradient force acting down slope of a sloping isobaric surface with the Coriolis force acting upslope. Solving for V, we get

$$V = \frac{10\ (\Delta D_A - \Delta D_B)}{L.\ 2\ \omega \sin \phi} \tag{5.18}$$

where

$\Delta D_A - \Delta D_B$ = difference in anomalies of dynamic height at stations A and B in dynamic metres.

L = distance between stations in metres

ω = angular velocity of the earth equal to 0.729×10^{-4} radians/sec

ϕ = mean latitude between the stations

V = relative current velocity normal to the line joining the two stations, in metres per second.

The current is called relative current because the current computed using the formula is relative to any unknown current at the reference level. To get the absolute current, the reference level selected should be a level of no motion.

The error in the computation of relative current is due to the errors in the computed value of difference in dynamic height anomalies, in the measurements of distance between stations and in the latitude angle. This may be obtained by taking the total differential of the expression (5.18).

$$dV = \frac{10d\ (\Delta D_A - \Delta D_B)}{L.\ 2\ \omega \sin \phi} - \frac{10\ (\Delta D_A - \Delta D_B)}{L^2.\ 2\ \omega \sin \phi}\ dL$$

$$- \frac{10(\Delta D_A - \Delta D_B)}{L.2\ \omega \sin^2 \phi} \cos \phi\ d\phi \qquad (5.19)$$

The second and third terms on the right hand side of the equation (5.19) may be discarded being negligible compared to the first term assuming that the measurements of distance between stations, L and latitude angle, ϕ are done accurately. This is generally true, particularly with the use of modern navigational aids. Hence the error in the relative current may be written as

$$dV = \frac{10d\ (\Delta D_A - \Delta D_B)}{L.\ 2\ \omega \sin \phi} \qquad (5.20)$$

Expression (5.20) shows that the error in the computed current is directly proportional to the error in the difference in dynamic height anomaly and is inversely proportional to the distance between stations and the sine of the latitude angle*.

To illustrate the computation of error in the relative current relative to 1500 d bar surface, let us assume that the distance between the two stations A and B is 100 kilometres and the average latitude angle is 5°. The maximum error in the difference between the dynamic height anomalies at the two stations should be computed as twice the standard deviation of the combined distribution resulting from the combination of the

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equation (5.18) will lead to computation of infinite current at the equator. But the fact that halving the distance between stations will also lead to unrealistic results, by doubling the error in the computed current, is not recognised by many oceanographers. Montgomery and Stroup (1962) found that decrease of station spacing will not always result in more details of the current distribution and stated: 'In effect, the stations (9,11,......., 27) midway between whole degrees of latitude are neglected. It was thought that the details gained by halving the spacing of verticals would be largely unreal'. Again while discussing the representation of distribution of geostrophic flow through a vertical section they stated: 'It might be thought that a continuous distribution of geostrophic velocity component could be attained by sufficiently two error distributions of the dynamic height anomalies at the two stations: Hence

$$d\,(\Delta D_A - \Delta D_B) = 2\sqrt{4^2 + 4^2} = 2\sqrt{2} \times 4 = 11.3$$

$$\triangleq \quad 11 \text{ dyn. mm}$$

The error in the computed current, dV, is computed from

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reducing the station spacing. The result, however, would be an increasingly irregular pattern, ultimately bearing no resemblence to the actual current (because with decreasing spacing even small fluctuations of the observed specific volume would lead to increasingly swift and narrow current bands of alternating direction)'. One of the reasons for the above results obtained by Montgomery and Stroup (1962), even though they have taken a comparatively shallow reference surface at 300 m which should have resulted in less computational error, may be the increased error inherent in the method of obtaining values of thermosteric anomaly described by Montgomery (1954) and Montgomery and Wooster (1954) (Section 4.2). Ofcourse, smaller spacing of verticals will result in gained details for the vertical sections of the dependent and independent parameters, but not of the distribution of currents.

the formula (5.20) and using the LaFond's tables (LaFond, 1951) as

$$dV = 9 \text{ cm/sec}$$

Figure 2 shows the variation of the error in the computed current (relative to 1500 d bar surface) with latitude angle for an error of 10 dyn. mm in the difference in dynamic height anomaly and for a pair of stations separated by 100 km. The curve shows that the error is less than 1 cm/sec above 40^{o} latitude angle and increases towards the equator. Very near the equator the rate of increase is very large.

Any computed current which is less than or equal to the magnitude of the error itself should be considered as unreal. Since the error is given in absolute magnitude, its effect is serious where the current velocities are small, particularly very near the equator where the magnitude of error increases due to latitude effect.

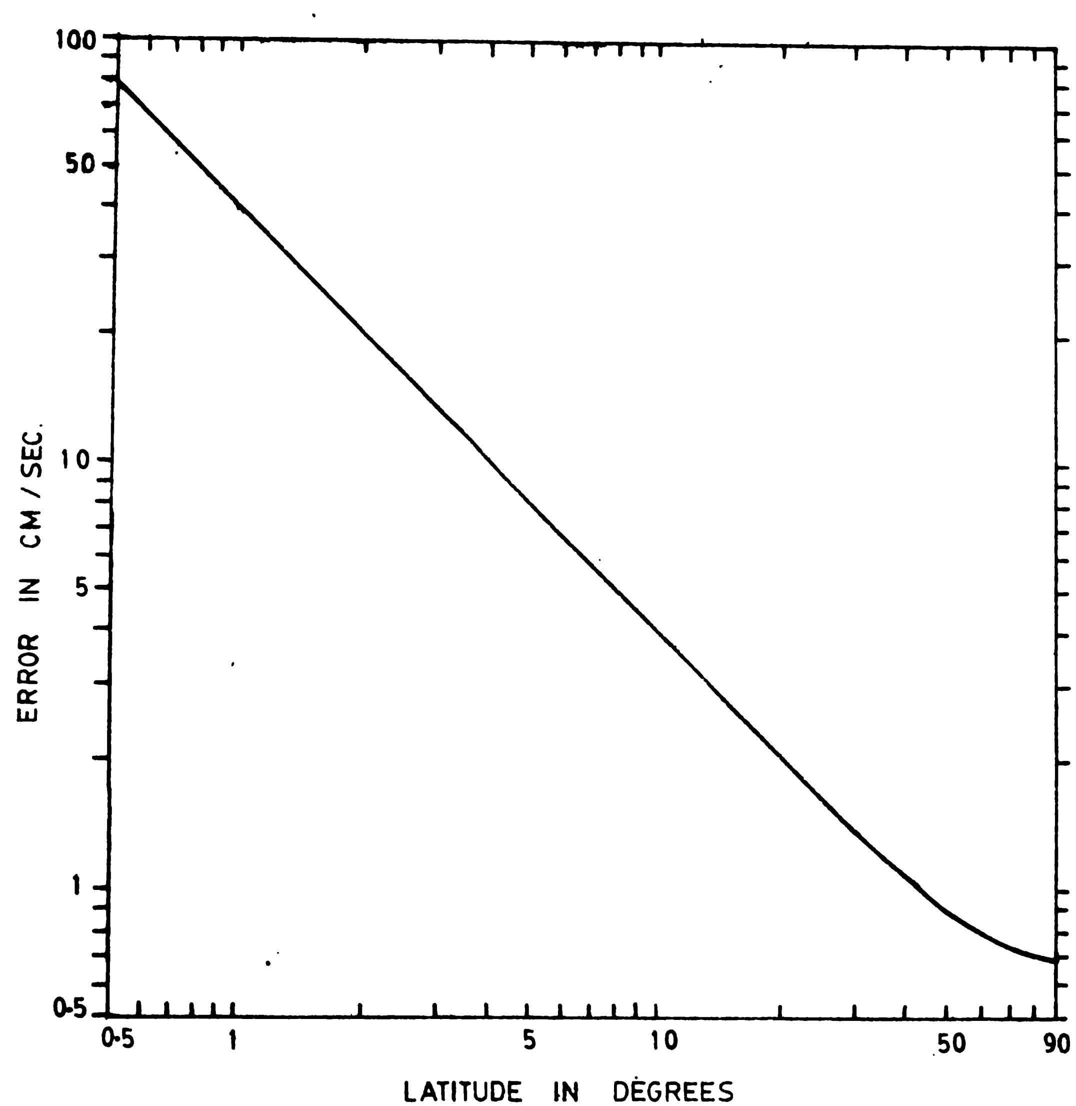

Fig. 2: Absolute error in the computation of geotrophic current for 10 dyn.mm. error in the difference in dynamic height anomaly and for 100 Km station spacing (relative to 1500 dbar)