Chapter 6

Fixed point theorems in ordered metric spaces using $w$-distances

6.1 Introduction

As mentioned earlier time and again, Banach contraction principle has been extended, improved and generalized in several ways (e.g. [117, 159]). One variety of such generalizations is the contractive fixed point theorem contained in Khan et al. [117] wherein the authors utilized altering distance functions to alter the distance between two points in a metric space. Such altering distance functions are also sometimes referred as control functions.

The following altering distance function is instrumental in our presentation:

**Definition 6.1.1.** ([117]) A function $\phi : [0, \infty) \to [0, \infty)$ is said to be altering (distance) function if it satisfies in following:

(i) $\phi$ is continuous and non-decreasing.

(ii) $\phi(t) = 0$ if and only if $t = 0$.

Concretely, Khan et al. [117] proved the following result:

**Theorem 6.1.1.** ([117]) Let $T$ be a self-mapping defined on a complete metric space $(X, d)$ satisfying the condition

$$\phi(d(Tx, Ty)) \leq c \phi(d(x, y))$$

for $x, y \in X$ and $0 < c < 1$ where $\phi$ is earlier described altering distance function. Then $T$ has a unique fixed point.

In the recent past, the idea of altering distance function has been utilized by many researchers in metric fixed point theory (e.g. [24, 174]). On the other hand,

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A research paper based on the contents of this chapter have been submitted for publication in Fixed Point Theory and Applications.
Alber and Guerre-Delabriere [8] initiated the study of weakly contractive mappings which were primarily confined to Hilbert spaces. Rhoades [165] utilized this idea in the context of complete metric spaces and proved the following interesting theorem.

**Theorem 6.1.2.** ([165]) Let $\mathcal{T}$ be a self-mapping defined on a complete metric space $(\mathcal{X}, d)$ satisfying the condition

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$$

for $x, y \in \mathcal{X}$ where $\phi$ is earlier described altering distance function. Then $\mathcal{T}$ has a unique fixed point.

In fact, Alber and Guerre-Delabriere [8] assumed an additional assumption on $\phi$ namely: \(\lim_{t \to \infty} \phi(t) = \infty\). But Rhoades [165] proved theorem without this requirement on $\phi$.

In [67], Dutta and Choudhury presented a generalization of Theorem 6.1.2 by proving the following result:

**Theorem 6.1.3.** ([67]) Let $\mathcal{T}$ be a self-mapping defined on a complete metric space $(\mathcal{X}, d)$ satisfying the condition

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)), \text{ for all } x, y \in \mathcal{X},$$

where $\psi$ and $\phi$ are altering distance functions. Then $\mathcal{T}$ has a unique fixed point.

The purpose of this chapter is to prove some fixed point theorems in ordered metric spaces involving $w$-distance as well as altering distance function. Proving new fixed point theorems to improve earlier stated theorems is a subject of vigorous research interest and for the literature of this kind one can be referred to [67]. Our results, in this chapter, not only generalize the analogous fixed point theorems but are relatively simpler and natural over the related ones. Our improvements in this chapter are four fold:

(i) Generalized distance is used instead of metric.

(ii) We use a relatively more general contraction condition.

(iii) The mapping under consideration is weakened to orbital continuity as opposed to continuity.

(iv) The comparability condition used by earlier authors is also sharpened.

### 6.2 Preliminaries

Before presenting our results, we record the remaining background material which will be needed in the proofs of our main results.
The following example illustrate the relationship between metric and \( w \)-distance as every metric is a \( w \)-distance but not conversely.

**Example 6.2.1.** Let \((\mathcal{X}, d)\) be a metric space. A function \( p : \mathcal{X} \times \mathcal{X} \to [0, \infty) \) defined by \( p(x, y) = k \) for every \( x, y \in \mathcal{X} \) is a \( w \)-distance on \( \mathcal{X} \), where \( k \) is a positive real number. But \( p \) is not a metric since \( p(x, x) = k \neq 0 \) for any \( x \in \mathcal{X} \).

**Definition 6.2.1.** Let \( \mathcal{T} : \mathcal{X} \to \mathcal{X} \) be a function and \( F_{\mathcal{T}} = \{ u \in \mathcal{X} | u = \mathcal{T}(u) \} \) (set of fixed points of \( \mathcal{T} \)). Then function \( \mathcal{T} \) is called Picard Operator (briefly, PO) if there exists \( u \in \mathcal{X} \) such that \( F_{\mathcal{T}} = \{ u \} \) and \( \{ \mathcal{T}^n(x) \} \) converges to \( u \), for all \( x \in \mathcal{X} \).

The function \( \mathcal{T} \) is called orbitally \( U \)-continuous if for any \( U \subset \mathcal{X} \times \mathcal{X} \) the following condition holds:

For any \( x \in \mathcal{X} \), \( \lim_{i \to \infty} \mathcal{T}^{n_i}(x) = z \in \mathcal{X} \) and \( (\mathcal{T}^{n_i}(x), z) \in U \) for any \( i \in \mathbb{N} \), imply that \( \lim_{i \to \infty} \mathcal{T}^{n_i+1}(x) = \mathcal{T}z \).

Let \((\mathcal{X}, \preceq)\) be a partially ordered set. Let us denote by \( \mathcal{X}_{\preceq} \) the subset of \( \mathcal{X} \times \mathcal{X} \) defined by:

\[ \mathcal{X}_{\preceq} = \{ (x, y) \in \mathcal{X} \times \mathcal{X} | x \preceq y \text{ or } y \preceq x \}. \]

The following lemma is crucial in the proofs of our main results.

**Lemma 6.2.1.**([108]) Let \( p \) be a \( w \)-distance on metric space \((\mathcal{X}, d)\) and \( \{ x_n \} \) be a sequence in \( \mathcal{X} \) such that for each \( \epsilon > 0 \) there exists \( N_\epsilon \in \mathbb{N} \) such that \( m > n > N_\epsilon \) implies \( p(x_n, x_m) < \epsilon \) (or \( \lim_{m,n} p(x_n, x_m) = 0 \)). Then \( \{ x_n \} \) is a Cauchy sequence.

### 6.3 Main results

Now, we present our main result as follows:

**Theorem 6.3.1.** Let \((\mathcal{X}, d, \preceq)\) be a complete partially ordered metric space equipped with a \( w \)-distance \( p \) and \( \mathcal{T} : \mathcal{X} \to \mathcal{X} \) be non-decreasing mapping. Suppose that:

(i) there exists \( x_0 \in \mathcal{X} \) such that \( (x_0, \mathcal{T}x_0) \in \mathcal{X}_{\preceq}, \)

(ii) there exists two altering distance functions \( \psi, \phi \) such that,

\[ \psi(p(\mathcal{T}x, \mathcal{T}y)) \leq \psi(M_{x,y}) - \phi(M_{x,y}) \]

for all \((x, y) \in \mathcal{X}_{\preceq}\), where

\[ M_{x,y} = \max\{p(x, y), \min\{p(x, \mathcal{T}x), p(y, \mathcal{T}y), p(\mathcal{T}x, x), p(\mathcal{T}y, y)\}\}, \]

(iii) either \( \mathcal{T} \) is orbitally continuous at \( x_0 \) or

(iv) \( \mathcal{T} \) is orbitally \( \mathcal{X}_{\preceq} \)-continuous and there exists a subsequence \( \{ \mathcal{T}^{n_k}x_0 \} \) of \( \{ \mathcal{T}^nx_0 \} \) converging to \( u \) such that \( (\mathcal{T}^{n_k}x_0, u) \in \mathcal{X}_{\preceq} \) for any \( k \in \mathbb{N} \).
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Then $\mathcal{F}_T \neq \emptyset$.

**Proof.** If $x_0 = T x_0$ for some $x_0 \in X$, then there is nothing to prove. Otherwise, let there be $x_0 \in X$ such that $x_0 \neq T x_0$, and $(x_0, T x_0) \in X_\leq$. Owing to monotonicity of $T$, we can write $(T^n x_0, T^{n+1} x_0) \in X_\leq$. Continuing this process inductively, we obtain

$$(T^n x_0, T^m x_0) \in X_\leq,$$

for any $n, m \in \mathbb{N}$. Now, we proceed to show that

$$\lim_{n \to \infty} p(T^n x_0, T^{n+1} x_0) = 0. \quad (6.3.1.1)$$

Write $p_0 = p(x_0, T x_0)$ and $p_n = p(T^n x_0, T^{n+1} x_0)$ for any $n \in \mathbb{N}$. On using condition (ii), we have

$$\psi(p_n) = \psi(p(T^n x_0, T^{n+1} x_0)) \leq \psi\left( \max\{p(T^{n-1} x_0, T^n x_0), \min\{p(T^{n-1} x_0, T^n x_0), p(T^n x_0, T^{n+1} x_0), p(T^{n+1} x_0, T^n x_0)\}\} \right) \leq \phi\left( \max\{p(T^{n-1} x_0, T^n x_0), \min\{p(T^{n-1} x_0, T^n x_0), p(T^n x_0, T^{n+1} x_0), p(T^{n+1} x_0, T^n x_0)\}\} \right),$$

so that

$$\psi(p_n) \leq \psi(p_{n-1}) - \phi(p_{n-1}),$$

for any $n \in \mathbb{N}$.

Also,

$$\psi(p_n) \leq \psi(p_{n-1}) - \phi(p_{n-1}) \leq \psi(p_{n-1}). \quad (6.3.1.2)$$

Therefore $p_n \leq p_{n-1}$ for every $n \in \mathbb{N}$ (by monotonicity of $\psi$), i.e. the sequence $\{p_n\}$ is decreasing, so that in respect of non-negative decreasing sequence $\{p_n\}$, there exists some $r > 0$ such that

$$\lim_{n \to \infty} p_n = \lim_{n \to \infty} p(T^n x_0, T^{n+1} x_0) = r,$$

which on letting $n \to \infty$, in (6.3.1.2), we get

$$\psi(r) \leq \psi(r) - \phi(r) \leq \psi(r),$$

which amounts to say that $\phi(r) = 0$. As $\phi$ is altering (distance) function, therefore $r = 0$, which is a contradiction to nonzeroness of $r$ yielding thereby

$$\lim_{n \to \infty} p_n = \lim_{n \to \infty} p(T^n x_0, T^{n+1} x_0) = 0,$$

which establishes (6.3.1.1).

Proceeding on earlier lines, we can also show that

$$\lim_{n \to \infty} p(T^{n+1} x_0, T^n x_0) = 0. \quad (6.3.1.3)$$
Write \( p_0 = p(Tx_0, x_0) \) and \( p_n = p(T^{n+1}x_0, T^nx_0) \) for any \( n \in \mathbb{N} \).

Now, on using (ii), we get

\[
\psi(p_n) = \psi(p(T^{n+1}x_0, T^nx_0)) \\
\leq \psi(\max\{p(T^nx_0, T^{n-1}x_0), \min\{p(T^n x_0, T^{n+1}x_0), \}
\p(T^{n-1}x_0, T^nx_0), p(T^{n+1}x_0, T^n x_0), p(T^n x_0, T^{n-1}x_0)\}\}) \\
- \phi(\max\{p(T^nx_0, T^{n-1}x_0), \min\{p(T^n x_0, T^{n+1}x_0), \}
\p(T^{n-1}x_0, T^nx_0), p(T^{n+1}x_0, T^n x_0), p(T^n x_0, T^{n-1}x_0)\}\}) \\
= \psi(p_{n-1}) - \phi(p_{n-1}),
\]

therefore, we get

\[
\psi(p_n) \leq \psi(p_{n-1}) - \phi(p_{n-1}) \leq \psi(p_{n-1}),
\]

which amounts to say that \( p_n \leq p_{n-1} \), i.e. non-negative sequence \( \{p_n\} \) is decreasing.

As earlier, we get

\[
\lim_{n \to \infty} p(T^{n+1}x_0, T^nx_0) = 0,
\]

which proves (6.3.1.3). Now, we proceed to show

\[
\lim_{n,m \to \infty} p(T^nx_0, T^mx_0) = 0,
\]

suppose is not true. Then we can find a \( \delta > 0 \) with sequences \( \{m_k\}_{k=1}^\infty, \{n_k\}_{k=1}^\infty \)
such that

\[
p(T^{m_k}x_0, T^{m_k}x_0) \geq \delta, \quad \text{for all } k \in \{1, 2, 3, \cdots \},
\]

wherein \( m_k > n_k \). By (6.3.1.1) there exists \( k_0 \in \mathbb{N} \), such that \( n_k > k_0 \) implies

\[
p(T^{n_k}x_0, T^{n_k+1}x_0) < \delta.
\]

Notice that in view of two last inequalities, \( m_k \neq n_{k+1} \). We can assume that \( m_k \) is a minimum index, so that

\[
p(T^{m_k}x_0, T^rx_0) < \delta, \quad \text{for } r \in \{n_{k+1}, n_{k+2}, \cdots, m_k - 1\},
\]

so that, we have

\[
0 < \delta \leq p(T^{n_k}x_0, T^{m_k}x_0) \\
\leq p(T^{n_k}x_0, T^{m_k-1}x_0) + p(T^{m_k-1}x_0, T^{m_k}x_0) \\
< \delta + p(T^{m_k-1}x_0, T^{m_k}x_0),
\]

thus

\[
\lim_{k \to \infty} p(T^{n_k}x_0, T^{m_k}x_0) = \delta.
\]

Now, we show that

\[
\lim_{k} \sup p(T^{n_k+1}x_0, T^{m_k+1}x_0) = \epsilon < \delta.
\]
If \( \limsup_k p(T^{n_k+1}x_0, T^{m_k+1}x_0) = \epsilon \geq \delta \), then there exists \( \{k_r\}_{r=1}^\infty \) such that
\[
\lim_{r \to \infty} p(T^{n_{k_r}+1}x_0, T^{m_{k_r}+1}x_0) = \epsilon \geq \delta.
\]
Since \( \psi \) is continuous and non-decreasing and also \((T^{n_{k_r}x_0}, T^{m_{k_r}x_0}) \in X_\preceq \), by using condition (ii), one gets
\[
\psi(p(T^{n_{k_r}+1}x_0, T^{m_{k_r}+1}x_0)) \leq \psi(M_{T^{n_{k_r}x_0}, T^{m_{k_r}x_0}}) - \phi(M_{T^{n_{k_r}x_0}, T^{m_{k_r}x_0}})
\]
with
\[
M_{T^{n_{k_r}x_0}, T^{m_{k_r}x_0}} = \max\{p(T^{n_{k_r}x_0}, T^{m_{k_r}x_0}), \min\{p(T^{n_{k_r}x_0}, T^{m_{k_r}+1}x_0), p(T^{m_{k_r}x_0}, T^{m_{k_r}+1}x_0), p(T^{m_{k_r}+1}x_0, T^{m_{k_r}x_0})\}\},
\]
implying thereby
\[
\lim_{r \to \infty} M_{T^{n_{k_r}x_0}, T^{m_{k_r}x_0}} = \max\{0, \delta\} = \delta,
\]
therefore, by letting \( k \to \infty \), we get
\[
\psi(\delta) \leq \psi(\epsilon) \leq \psi(\delta) - \phi(\delta) \leq \psi(\delta),
\]
so that \( \phi(\delta) = 0 \) implying thereby \( \delta = 0 \) which is a contradiction. Hence
\[
\limsup_k p(T^{n_{k+1}x_0, T^{m_{k+1}x_0}} < \delta,
\]
so we have
\[
0 < \delta \leq p(T^{n_kx_0}, T^{m_kx_0}) \leq p(T^{n_kx_0}, T^{n_{k+1}x_0}) + p(T^{n_{k+1}x_0}, T^{m_{k+1}x_0}) + p(T^{m_{k+1}x_0}, T^{m_kx_0}).
\]
Therefore, owing to (6.3.1.1) and (6.3.1.3), we get
\[
o < \delta \leq \lim_{k \to \infty} p(T^{n_kx_0}, T^{n_{k+1}x_0}) + \limsup_{k \to \infty} p(T^{n_{k+1}x_0}, T^{m_{k+1}x_0}) + \lim_{k \to \infty} p(T^{m_{k+1}x_0}, T^{m_kx_0})
\]
\[
= \limsup_{k \to \infty} p(T^{n_{k+1}x_0}, T^{m_{k+1}x_0}) < \delta,
\]
which is a contradiction. Hence \( \lim_{m,n \to \infty} p(T^nx_0, T^mx_0) = 0 \). Owing to Lemma 6.2.1, \( \{T^nx_0\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete metric space, there exists \( u \) such that \( \lim_{n \to \infty} T^nx_0 = u \).

Now, we show that \( u \) is fixed point of \( T \). If (iii) holds, then \( \lim_{n \to \infty} T^{n+1}x_0 = Tu \). By lower semi-continuity of \( p(T^nx_0, \cdot) \), we have
\[
p(T^nx_0, u) \leq \liminf_{m \to \infty} p(T^nx_0, T^mx_0) = \alpha_n \quad \text{(say)},
\]
\[
p(T^nx_0, Tu) \leq \liminf_{m \to \infty} p(T^nx_0, T^{n+1}x_0) = \beta_n \quad \text{(say)}.
\]
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By using $\lim_{m,n \to \infty} p(T_n x_0, T_m x_0) = 0$, we have $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0$. Now, in view of Lemma 1.6.1, we conclude that

$$Tu = u.$$ 

Next, suppose that (iv) holds. Since $\{T_n x_0\}$ converges to $u$, $(T_n x_0, u) \in X_\prec$ and $T$ is $X_\prec$-continuous, it follows that $\{T_n x_0, u\}$ converges to $Tu$. As earlier, by lower semi-continuity of $p(T_n x_0, .)$, we conclude that $Tu = u$. This completes the proof.

In Theorem 6.3.1, setting $\psi = I$ the identity mapping, we deduce the following:

**Corollary 6.3.1.** Let $(X, d, \preceq)$ be a complete partially ordered metric space equipped with a $w$-distance $p$ and $T : X \to X$ be non-decreasing mapping. Suppose that

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_\prec$,

(ii) there exists an altering distance function $\phi$, such that

$$p(Tx, Ty) \leq M_{x,y} - \phi(M_{x,y})$$

for all $(x, y) \in X_\prec$, where

$$M_{x,y} = \max\{p(x, y), \min\{p(x, Tx), p(y, Ty), p(Tx, x), p(Ty, y)\}\},$$

(iii) either $T$ is orbitally continuous at $x_0$ or

(iv) $T$ is orbitally $X_\prec$-continuous and there exists a subsequence $\{T^{nk} x_0\}$ of $\{T^n x_0\}$ converges to $u$ such that $(T^{nk} x_0, u) \in X_\prec$ for any $k \in \mathbb{N}$.

Then $F_T \neq \emptyset$.

Choosing $\psi = I$ the identity mapping and $\phi(t) = (1 - \alpha)t$, (for all $t \in [0, \infty)$) in Theorem 6.3.1, we deduce the following:

**Corollary 6.3.2.** Let $(X, d, \preceq)$ be a complete partially ordered metric space equipped with a $w$-distance $p$ and $T : X \to X$ be non-decreasing mapping. Suppose that

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_\prec$,

(ii) if for all $(x, y) \in X_\prec$ and $\alpha \in [0, 1)$

$$p(Tx, Ty) \leq \alpha M_{x,y}$$

where

$$M_{x,y} = \max\{p(x, y), \min\{p(x, Tx), p(y, Ty), p(Tx, x), p(Ty, y)\}\},$$

(iii) either $T$ is orbitally continuous at $x_0$ or
(iv) $\mathcal{T}$ is orbitally $\mathcal{X}_\preceq$-continuous and there exists a subsequence $\{\mathcal{T}^{n_k}x_0\}$ of $\{\mathcal{T}^n x_0\}$ converges to $u$ such that $(\mathcal{T}^{n_k}x_0, u) \in \mathcal{X}_\preceq$ for any $k \in \mathbb{N}$.

Then $\mathcal{F}_T \neq \emptyset$.

As an application of Corollary 6.3.2, we can also prove the following related result:

**Theorem 6.3.2.** Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a $w$-distance $p$ and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be non-decreasing mapping. Suppose that

(i) there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{T}x_0) \in \mathcal{X}_\preceq$,

(ii) for all $(x, y) \in \mathcal{X}_\preceq$,

\[ p(\mathcal{T}x, \mathcal{T}y) \leq \alpha p(x, y) + \beta (\min \{ p(x, \mathcal{T}x), p(y, \mathcal{T}y), p(\mathcal{T}x, x), p(\mathcal{T}y, y) \}) \]

where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$,

(iii) either $\mathcal{T}$ is orbitally continuous at $x_0$ or

(iv) $\mathcal{T}$ is orbitally $\mathcal{X}_\preceq$-continuous and there exists a subsequence $\{\mathcal{T}^{n_k}x_0\}$ of $\{\mathcal{T}^n x_0\}$ converges to $u$ such that $(\mathcal{T}^{n_k}x_0, u) \in \mathcal{X}_\preceq$ for any $k \in \mathbb{N}$.

Then $\mathcal{F}_T \neq \emptyset$.

**Proof.** On using condition (ii), we can write

\[ p(\mathcal{T}x, \mathcal{T}y) \leq \alpha p(x, y) + \beta (\min \{ p(x, \mathcal{T}x), p(y, \mathcal{T}y), p(\mathcal{T}x, x), p(\mathcal{T}y, y) \}) \]

\[ \leq (\alpha + \beta) \max \{ p(x, y), \min \{ p(x, \mathcal{T}x), p(y, \mathcal{T}y), p(\mathcal{T}x, x), p(\mathcal{T}y, y) \} \}, \]

where $k = \alpha + \beta \in [0, 1)$. Therefore, all the conditions of Corollary 6.3.2 are satisfied which ensure the conclusion.

**Corollary 6.3.3.** Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a $w$-distance $p$ and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be non-decreasing mapping. Suppose that

(i) there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{T}x_0) \in \mathcal{X}_\preceq$,

(ii) for all $(x, y) \in \mathcal{X}_\preceq$,

\[ \int_0^{p(\mathcal{T}x, \mathcal{T}y)} \theta(\xi) d\xi \leq \alpha \int_0^{\max \{ p(x, y), \min \{ p(x, \mathcal{T}x), p(y, \mathcal{T}y), p(\mathcal{T}x, x), p(\mathcal{T}y, y) \} \}} \theta(\xi) d\xi \]

where $0 \leq \alpha < 1$ and $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is Lebesgue integrable mapping which is summable and $\int_0^\epsilon \theta(\xi) d\xi > 0$, (for each $\epsilon > 0$)

(iii) either $\mathcal{T}$ is orbitally continuous at $x_0$ or
(iv) $\mathcal{T}$ is orbitally $\mathcal{X}_\preceq$-continuous and there exists a subsequence $\{\mathcal{T}^{n_k}x_0\}$ of $\{\mathcal{T}^nx_0\}$ converges to $u$ such that $(\mathcal{T}^{n_k}x_0, u) \in \mathcal{X}_\preceq$ for any $k \in \mathbb{N}$.

Then $\mathcal{F}_\mathcal{T} \neq \emptyset$.

**Proof.** We choose $\psi(t) = \int_0^t \theta(\xi)d\xi$ and $\phi(t) = (1-\alpha)\int_0^t \theta(\xi)d\xi$, (for all $t \in [0, \infty)$). Clearly, $\psi$ and $\phi$ are altering distance functions. Then by making use of Theorem 6.3.1 the proof is completed.

**Remark 6.3.1.** In Theorem 6.3.1, let $p = d$, $\psi = \mathcal{I}$ (identity) and $\phi = (1-\alpha)t(0 \leq \alpha < 1)$. Then Theorem 6.3.1 is the classical Banach fixed point theorem.

**Lemma 6.3.1.** Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space and $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a map wherein $p$ is a $w$-distance on $(\mathcal{X}, d)$. If

(i) $u \in \mathcal{F}_\mathcal{T}$,

(ii) there exists two altering distance functions $\psi, \phi$ such that

$$\psi(p(\mathcal{T}x, \mathcal{T}y)) \leq \psi(M_{x,y}) - \phi(M_{x,y})$$

for all $(x, y) \in \mathcal{X}_\preceq$, with

$$M_{x,y} = \max \{p(x, y), \min\{p(x, \mathcal{T}x), p(y, \mathcal{T}y), p(\mathcal{T}x, x), p(\mathcal{T}y, y)\}\}.$$

Then $p(u, u) = 0$.

**Proof.** Suppose $p(u, u) \neq 0$. As $(u, u) \in \mathcal{X}_\preceq$, and

$$M_{u,u} = \max \{p(u, u), \min\{p(u, \mathcal{T}u), p(u, \mathcal{T}u), p(\mathcal{T}u, u), p(\mathcal{T}u, u)\}\} = p(u, u)$$

therefore

$$\psi(p(\mathcal{T}u, \mathcal{T}u)) = \psi(p(u, u)) \leq \psi(p(u, u)) - \phi(p(u, u)) \leq \psi(p(u, u))$$

which amounts to say that $\phi(p(u, u)) = 0$. As $\phi$ is an altering distance function, we infer that $p(u, u) = 0$. This completes the proof.

### 6.4 Uniqueness of fixed point

In what follows, we give a sufficient condition for the uniqueness of fixed point in Theorem 6.3.1 which runs as follows:

(A): for every $x, y \in \mathcal{X}$, there exists a lower bound or an upper bound.

In [139], it is proved that condition (A) is equivalent to the following one:

(B): for every $x, y \in \mathcal{X}$, there exists $z \in \mathcal{X}$ for which $(x, z) \in \mathcal{X}_\preceq$ and $(y, z) \in \mathcal{X}_\preceq$. 
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Theorem 6.4.1. With the addition of condition (B) to the hypotheses of Theorem 6.3.1, the fixed point of \( T \) turns out to be unique. Moreover,

\[
\lim_{n \to \infty} T^n(x) = u,
\]

for every \( x \in X \), provided \( u \in \mathcal{F}_T \), i.e. the map \( T : X \to X \) is a Picard operator.

**Proof.** Following the proof of Theorem 6.3.1, \( \mathcal{F}_T \neq \emptyset \). Suppose, there exist two fixed points \( u_1 \) and \( u_2 \) of \( T \) in \( X \). We distinguish two cases:

Case 1: If \((u_2, u_1) \in X_\preceq \), owing to condition (ii) and Lemma 6.3.1, we have

\[
\psi(p(Tu_2, Tu_1)) \leq \psi(M_{u_2, u_1}) - \phi(M_{u_2, u_1}).
\]

As

\[
M_{u_2, u_1} = \max \{p(u_2, u_1), \min\{p(u_2, Tu_2), p(u_1, Tu_1), p(Tu_2, u_2), p(Tu_1, u_1)\}\}
\]

therefore

\[
\psi(p(Tu_2, Tu_1)) \leq \psi(p(u_2, u_1)) - \phi(p(u_2, u_1)) \leq \psi(p(u_2, u_1))
\]

which amounts to say that \( \phi(p(u_2, u_1)) = 0 \). As \( \phi \) is altering distance function, therefore for every \( n \in \mathbb{N} \),

\[
p(u_2, u_1) = 0,
\]

also in view of Lemma 6.3.1, we get \( p(u_2, u_2) = 0 \) which by using Lemma 1.6.1, we have \( u_2 = u_1 \), i.e. the fixed point of \( T \) is unique.

Case 2: If \((u_1, u_2) \notin X_\preceq \), then owing to condition (B), there exists \( z \in X \) such that \((u_1, z) \in X_\preceq \) and \((u_2, z) \in X_\preceq \). As \((z, u_1) \in X_\preceq \), due to monotonicity of \( T \), we get \((T^{n-1}z, u_1) \in X_\preceq \) for any \( n \in \mathbb{N} \) and henceforth

\[
\psi(p(T^n z, u_1)) = \psi(p(T^n z, Tu_1)) \leq \psi(M_{T^{n-1}z, u_1}) - \phi(M_{T^{n-1}z, u_1}),
\]

with

\[
M_{T^{n-1}z, u_1} = \max \{p(T^{n-1}z, u_1), \min\{p(T^{n-1}z, T^n z), p(u_1, Tu_1), p(T^n z, T^{n-1}z), p(u_1, u_1)\}\}
\]

therefore

\[
\psi(p(T^n z, u_1)) \leq \psi(p(T^n z, Tu_1)) \leq \psi(p(T^{n-1} z, u_1)) - \phi(p(T^{n-1} z, u_1)) \leq \psi(p(T^{n-1} z, u_1)).
\]

Since \( \psi \) is non-decreasing function, therefore \( p(T^n z, u_1) \leq p(T^{n-1} z, u_1) \) i.e. the non-negative sequence \( \{p(T^n z, u_1)\} \) is decreasing. As earlier, we have

\[
\lim_{n \to \infty} p(T^n z, u_1) = 0.
\]
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Also, since \((z, u_2) \in X_{\leq}\), therefore proceeding as earlier, we can prove that

\[
\lim_{n \to \infty} p(T^n z, u_2) = 0,
\]

which by using Lemma 1.6.1, we infer that \( u_2 = u_1 \) i.e. the fixed point of \( T \) is unique.

Now, we proceed to show

\[
\lim_{n \to \infty} T^n(x) = u,
\]

for every \( x \in X \), provided \( u \in F_T \). We distinguish two cases:

Case 1: Let \( x \in X \) and \((u, x) \in X_{\leq}\). As earlier, we have

\[
\lim_{n \to \infty} p(T^n u, T^n x) = 0.
\]

Also, in view of Lemma 6.3.1, we have

\[
\lim_{n \to \infty} p(T^n u, u) = 0,
\]

which by using Lemma 1.6.1, we get

\[
\lim_{n \to \infty} T^n(x) = u.
\]

Case (2): Let \( x \in X \) and \((u, x) \notin X_{\leq}\). Owing to condition \((B)\), there exists some \( z \) in \( X \) such that \((u, z) \in X_{\leq}\) and \((x, z) \in X_{\leq}\). As earlier, we can prove \( \lim_{n \to \infty} p(T^n z, u) = 0 \) and \( \lim_{n \to \infty} p(u, T^n z) = 0 \), which by triangular inequality,

\[
p(T^{n-1} z, T^n z) \leq p(T^{n-1} z, u) + p(u, T^n z),
\]

one get

\[
\lim_{n \to \infty} p(T^{n-1} z, T^n z) = 0.
\]

Since \((x, z) \in X_{\leq}\), due to monotonicity of \( T \), we can write \((T x, T z) \in X_{\leq}\). Continuing this process inductively, we obtain

\[
(T^n x, T^n z) \in X_{\leq}.
\]

Now, we proceed to show that

\[
\lim_{n} \inf_{n} p(T^n z, T^n x) = 0.
\]

Suppose, \( \lim_{n} \inf_{n} p(T^n z, T^n x) = \delta > 0 \). Since \( \lim_{n \to \infty} p(T^{n-1} z, T^n z) = 0 \), then for arbitrary \( \epsilon \) \((0 < \epsilon < \delta)\), there exists \( N_1 \in \mathbb{N} \) such that for every \( n > N_1 \), we have \( p(T^{n-1} z, T^n z) < \epsilon \). Also, since \( \lim_{n} \inf_{n} p(T^n z, T^n x) = \delta > \epsilon > 0 \), then there exists
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\( N_2 \in \mathbb{N} \) such that for every \( n > N_2 \), we have \( p(T^n z, T^n x) > \epsilon \). Therefore, for every \( n > N = \max\{N_1, N_2\} \), we have

\[
\max\{p(T^{n-1} z, T^{n-1} x), \min\{p(T^{n-1} z, T^n z), p(T^{n-1} x, T^n x), p(T^n z, T^{n-1} z), \}
\]

\[
p(T^n x, T^{n-1} x)\} = p(T^{n-1} z, T^n x).
\]

Now, on using (ii), for every \( n > N \) we get

\[
\psi(p(T^n z, T^n x)) \leq \psi(p(T^{n-1} z, T^{n-1} x)) - \phi(p(T^{n-1} z, T^{n-1} x)) \leq \psi(p(T^{n-1} z, T^n x)).
\]

Since \( \psi \) is an altering distance function, we get non-negative sequence \( \{p(T^n z, T^n x)\} \) is decreasing, as earlier, we can prove \( \lim_{n \to \infty} p(T^n z, T^n x) = 0 \), which is indeed a contradiction to nonzeroness of \( \delta \), implying thereby

\[
\liminf_{n} p(T^n z, T^n x) = 0,
\]

also since \( (u, z) \in \mathcal{X}_\prec \), therefore using the arguments of the earlier case, we can prove

\[
\lim_{n \to \infty} p(T^n z, T^n u) = 0,
\]

which by lower semi-continuity of \( p(T^n z, \cdot) \), we have

\[
p(T^n z, \lim_{m \to \infty} T^m x) \leq \liminf_{m \to \infty} p(T^n z, T^m x) = \alpha_n \quad (\text{say}),
\]

\[
p(T^n z, u) \leq \liminf_{m \to \infty} p(T^n z, T^m u) = \beta_n \quad (\text{say}).
\]

As \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0 \), thus, in view of Lemma 1.6.1, we conclude that

\[
\lim_{n \to \infty} T^n(x) = u.
\]

This completes the proof.

**Corollary 6.4.1.** With the addition of condition (B) to the hypotheses of Corollary 6.3.1 (or Corollary 6.3.2, Corollary 6.3.3) the fixed point of \( T \) turns out to be unique. Moreover,

\[
\lim_{n \to \infty} T^n(x) = u,
\]

for every \( x \in \mathcal{X} \), provided \( u \in \mathcal{F}_T \), i.e. the map \( T : \mathcal{X} \to \mathcal{X} \) is a Picard operator.

**Corollary 6.4.2.** With the addition of condition (B) to the hypotheses of Theorem 6.3.2 the fixed point of \( T \) turns out to be unique. Moreover,

\[
\lim_{n \to \infty} T^n(x) = u,
\]

for every \( x \in \mathcal{X} \), provided \( u \in \mathcal{F}_T \), i.e. the map \( T : \mathcal{X} \to \mathcal{X} \) is a Picard operator.
6.5 Illustrative examples

Finally, we conclude this chapter with the following two examples:

Example 6.5.1. Consider $X = [0, 1]$, equipped with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$ and $p = d$ be a $w$-distance on $(X, d)$. We consider $X_{\leq}$ as follows,

$$X_{\leq} = \{(x, y) \in X \times X : x = y \text{ or } x, y \in \{0\} \cup \{\frac{1}{n} : n = 2, 3, \ldots\}\}$$

where $\leq$ is the usual ordering.

Let $T : X \to X$ be given by

$$T(x) = \begin{cases} 
0, & \text{if } x = 0 \\
\frac{1}{n+1}, & \text{if } x = \frac{1}{n} \\
\sqrt{2}/2, & \text{otherwise.}
\end{cases}$$

Obviously, $(X, d, \leq)$ is a complete partially ordered metric space. It is easy to see that $T$ is non-decreasing. Also, there is $x_0 = 0$ in $X$ such that $x_0 = 0 \leq 0 = Tx_0$, i.e. $(x_0, Tx_0) \in X_{\leq}$ and $T$ satisfies (iv).

We now show that $T$ satisfies (ii) with $\psi, \phi : [0, \infty) \to [0, \infty)$ which are defined as $\psi(t) = t^2$ and $\phi(t) = t^4$ ($t \in [0, \infty)$).

If $(x, y) \in X_{\leq}$ and $x = y$, then $p(Tx, Ty) = p(x, y) = 0$. Otherwise, If $(x, y) \in X_{\leq}$ with $x \neq y$, then either $x = \frac{1}{n}, y = 0$ or $x = \frac{1}{n}, y = \frac{1}{m}$, $(m > n \geq 2)$, which evolve two cases as follows:

Case 1. If $x = \frac{1}{n}(n \geq 2)$ and $y = 0$, then

$$M_{\frac{1}{n}, 0} = \max \left\{ \frac{1}{n}, \min \left\{ \frac{1}{n}, T\frac{1}{n} \right\}, p(0, 0), p(T\frac{1}{n}, \frac{1}{n}), p(0, 0) \right\} = \frac{1}{n}$$

and

$$\psi(M_{\frac{1}{n}, 0}) - \phi(M_{\frac{1}{n}, 0}) = \left(\frac{1}{n}\right)^2 - \left(\frac{1}{n}\right)^4 > \frac{1}{n^2} - \frac{1}{n^2(n+1)} = \frac{n}{n^2(n+1)}$$

$$= \frac{1}{n(n+1)} > \frac{1}{(n+1)^2} = \psi(p(Tx, Ty)).$$

Case 2. Next, if $x = \frac{1}{n}$ and $y = \frac{1}{m}$, $(m > n \geq 2)$, then

$$M_{\frac{1}{n}, \frac{1}{m}} = \max \left\{ \frac{1}{n}, \frac{1}{m}, \min \left\{ \frac{1}{n}, T\frac{1}{n}, p(\frac{1}{n}, T\frac{1}{n}), p(T\frac{1}{n}, \frac{1}{n}), p(T\frac{1}{n}, \frac{1}{m}) \right\} \right\}$$

$$= \max \left\{ \frac{1}{n} - \frac{1}{m}, \min \left\{ \frac{1}{n} - \frac{1}{n+1}, \frac{1}{n} - \frac{1}{m+1} \right\} \right\}.$$
If for $m > n \geq 2$, we have
\[
\left| \frac{1}{n} - \frac{1}{n+1} \right| \leq \left| \frac{1}{n} - \frac{1}{m} \right|
\]
which is equivalent to
\[
\frac{1}{n(n+1)} \leq \frac{(m-n)}{mn}
\]
or
\[
\frac{m}{n+1} \leq (m-n).
\]
The preceding inequality holds as
\[
\frac{m}{n+1} \leq 1 \leq (m-n),
\]
so that
\[
\mathcal{M}_{\frac{1}{n},\frac{1}{m}} = \left| \frac{1}{n} - \frac{1}{m} \right|.
\]
Also (with $m > n \geq 2$),
\[
\psi(p(Tx, Ty)) \leq \psi(M_{x,y}) - \phi(M_{x,y})
\]
or
\[
\left| \frac{1}{n+1} - \frac{1}{m+1} \right|^2 \leq \left| \frac{1}{n} - \frac{1}{m} \right|^2 - \left| \frac{1}{n} - \frac{1}{m} \right|^4
\]
or
\[
\frac{(m-n)^2}{(n+1)^2(m+1)^2} \leq \frac{(m-n)^2}{(mn)^2} - \frac{(m-n)^4}{(mn)^4}
\]
or
\[
\frac{(m-n)^4}{(mn)^4} \leq \frac{(m-n)^2}{(mn)^2} - \frac{(m-n)^2}{(n+1)^2(m+1)^2}
\]
or
\[
\frac{(m-n)^2}{(mn)^2} \leq \frac{(n+1)(m+1) - mn}{(n+1)^2(m+1)}
\]
therefore,
\[
\left( \frac{1}{m} - \frac{1}{n} \right)^2 \leq \frac{(n+m+1)((n+1)(m+1) + mn)}{(n+1)^2(m+1)^2}. \tag{6.5.1.1}
\]
Also, we can write
\[
\left( \frac{1}{m} - \frac{1}{n} \right)^2 \leq \frac{1}{m^2} \leq \frac{m}{(n+1)m} = \frac{m}{(n+1)m^2} \leq \frac{m}{(n+1)(m+1)}
\]
\[
\leq \frac{m}{(n+1)(m+1)} + \frac{1}{m+1} = \frac{m + n + 1}{(n+1)(m+1)}
\]
\[
\leq \frac{(m + n + 1)(n+1)(m+1)}{(n+1)^2(m+1)^2} + \frac{(m + n + 1)(mn)}{(n+1)^2(m+1)^2}
\]
\[
= \frac{(n+m+1)((n+1)(m+1) + mn)}{(n+1)^2(m+1)^2},
\]
which amounts to say that the inequality (6.5.1.1) holds and hence so is the inequality (ii). Thus, all the conditions of Theorem 6.3.1 are satisfied implying thereby the existence of fixed point of the map $T$ which are indeed two in number namely: 0 and $\sqrt{2}/2$. Here, it is worth pointing out that the condition (B) does not hold in respect of this example.

We give another example that illustrates Theorem 6.4.1.

**Example 6.5.2.** Let $\mathcal{X} = \{0\} \cup \{\frac{1}{2^n} : n \geq 1\}$, where $(\mathcal{X}, d, \preceq)$ is a complete partially ordered metric space with $d$ and usual order $\preceq$. Clearly, condition (B) holds in $\mathcal{X}$. We define $p : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by $p(x, y) = y$. Let $\phi(t) = \frac{1}{4}t$ and $\psi(t) = \frac{1}{2}t$. Assume that $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by $\mathcal{T}x = \frac{x}{32}$ for any $x \in \mathcal{X}$. Obviously, $\phi$ and $\psi$ are altering distance functions, it is easy to see that $\mathcal{T}$ is non-decreasing and self-map. Also, there is $x_0 = 0$ in $\mathcal{X}$ such that $(x_0, \mathcal{T}x_0) \in \mathcal{X}_\preceq$, and $\mathcal{T}$ satisfies (iv). Now, we show that $\mathcal{T}$ satisfies (ii). If $y = 0$, then clearly condition (ii) is satisfied. Now, suppose that $y = \frac{1}{2^m}$, then we have

$$M_{x, \frac{1}{2^m}} = \max \{p(x, \frac{1}{2^m}), \min\{p(x, \mathcal{T}x), p(\frac{1}{2^m}, \mathcal{T} \frac{1}{2^m}), p(\mathcal{T}x, x), p(\mathcal{T} \frac{1}{2^m}, \frac{1}{2^m})\}\} = \frac{1}{2^m}.$$

By making use of condition (ii), one gets

$$\psi p(\mathcal{T}x, \mathcal{T}y) = \psi(\frac{1}{32 \cdot 2^m}) \leq \psi(\frac{1}{2^m}) - \phi(\frac{1}{2^m}),$$

so that

$$\frac{1}{2} \cdot \frac{1}{32 \cdot 2^m} \leq \frac{1}{2} \cdot \frac{1}{2^m} - \frac{1}{4} \cdot \frac{1}{2^m},$$

or

$$\frac{1}{2^{m+6}} \leq \frac{1}{2^{m+1}} - \frac{1}{2^{m+2}} = \frac{1}{2^{m+2}}.$$

The preceding inequality holds and hence so is the inequality (ii). Thus, all the conditions of Theorem 6.3.3 are satisfied. We note that $x = 0$ is unique fixed point for $\mathcal{T}$. Moreover $\lim_{n \to \infty} \mathcal{T}^n(x) = \lim_{n \to \infty} \frac{x}{32^n} = 0$. 
