Chapter-3

CHARACTERIZATIONS OF THE GUMBEL'S
BIVARIATE EXPONENTIAL DISTRIBUTION

3.1 Modified Gumbel's form

Instead of utilising the original Gumbel distribution as given in (2.1), in the present study, a more flexible model obtained by introducing two additional parameters is investigated. The motivation for considering this modified version arises from the observation on the original model made in Section 2.2. Accordingly the distribution of the random vector \((X_1, X_2)\) considered throughout the present study is specified by the probability density function

\[
f(x_1, x_2) = [(a_2 + \Theta x_1)(a_1 + \Theta x_2) - \Theta] \exp(-a_1 x_1 - a_2 x_2 - \Theta x_1 x_2) \quad (3.1)
\]

\(x_1, x_2 > 0; \ a_1, a_2 > 0; \ \Theta > 0.\)

The corresponding distribution function and survival functions are respectively

\[
F(x_1, x_2) = 1 - \exp(-a_1 x_1) - \exp(-a_2 x_2) + \exp(-a_1 x_1 - a_2 x_2 - \Theta x_1 x_2) \quad (3.2)
\]

and

\[
R(x_1, x_2) = \exp(-a_1 x_1 - a_2 x_2 - \Theta x_1 x_2) \quad (3.3)
\]
Although the distribution in the above form has appeared in Galambos and Kotz (1978) in connection with bivariate failure rates, a detailed investigation of its properties does not appear to have been undertaken so far. Therefore as a prelude to the focal theme of the present investigation, namely characteristic properties of the model, we examine some of its basic properties that are of relevance in the sequel. Most of the characterization theorems we present subsequently are motivated by these properties.

3.1.1 Marginal and conditional distributions

When the random vector \((X_1, X_2)\) has distribution (3.1), the marginal distributions are of the usual exponential form namely

\[
f_i(x_i) = \alpha_i^{-1}e^{-\alpha_i x_i}, \quad x_i > 0, \quad i = 1, 2.
\] (3.4)

with means \(\alpha_i^{-1}\).

The conditional distributions of \(X_i\) given \(X_j = t_j\) have density

\[
f(x_i|X_j = t_j) = [(\alpha_i + \Theta t_j)(\alpha_j + \Theta x_i) - \Theta] \alpha_j^{-1}\exp[-(\alpha_i + \Theta t_j)x_i] \\
x_i > 0; \quad i, j = 1, 2; \quad i \neq j
\] (3.5)
so that the means and variances are

\[ E(X_1|X_j=t_j) = (\alpha_i + \Theta t_j)^{-1} + \Theta \alpha_j^{-1}(\alpha_i + \Theta t_j)^{-2} \quad (3.6) \]

\[ V(X_1|X_j=t_j) = (\alpha_i + \Theta t_j)^{-2} + 2\Theta \alpha_j^{-1}(\alpha_i + \Theta t_j)^{-3} \]

\[ - \Theta^2 \alpha_j^{-2}(\alpha_i + \Theta t_j)^{-4} \quad (3.7) \]

It is to be noted that throughout the present investigation the suffixes i and j will be used in the manner explained in equation (3.5).

It is of considerable interest in our future investigation to introduce conditional distributions of a different kind than (3.5) in which the exceedences \( X_j > t_j \) is taken as the conditioning event. The conditional survival function of \( X_i \) given \( X_j > t_j \) is

\[ R(x_1|X_j > t_j) = P(X_i > t_1 | X_j > t_j) \]

\[ = \frac{R(t_1, t_2)}{R_j(t_j)} \]

\[ = \exp[-(\alpha_i + \Theta t_j)x_1] \quad (3.8) \]

From (3.8) the corresponding density is stated as

\[ f(x_1|X_j > t_j) = (\alpha_i + \Theta t_j) \exp[-(\alpha_i + \Theta t_j)x_1] \quad (3.9) \]

which is in the univariate exponential form.
Thus the marginal and conditional distributions (in the above sense) of the bivariate exponential distribution are exponential. The result (3.9) plays an important role in the characterizations of (3.1) by giving scope to extend theorems in the univariate case to higher dimensions. Further from the application side also the same condition remains quite meaningful. When \((X_1, X_2)\) represents the life time of a two component system, the exceedences \(X_j > t_j\) denote the survival of the component after time \(t_j\). Accordingly the condition enables to look at the life distribution of one of the components in a two-component system when the other is known to be performing adequately its intended function. In chapter 5 we examine in detail the implications of these observations, in connection with reliability analysis.

3.1.2 Local lack of memory

One of the most well studied property of the exponential law that forms the basis of many theoretical and applied researches is the lack of memory. It is therefore, important to investigate how our model accommodates this property in the bivariate set up.
We presently establish that for the model (3.1), the extended version of memorylessness in the form

\[
P[X_i > t_i + s_i | X_i > s_i, X_j > t_j] = P[X_i > t_i | X_j > t_j]
\]

holds.

To verify this, we take \(i=1\) and note that the above statement is equivalent to

\[
R_2(t_2) R(t_1 + s_1, t_2) = R(s_1, t_2) R(t_1, t_2)
\]

where \(R(x_1, x_2)\) is as defined in (3.3) and

\[
R_2(t_2) = P[X_2 > t_2]
= \exp(-\alpha_2 t_2)
\]

Substituting the relevant expressions from equation (3.3) the result is seen to hold for \(i=1\). The proof for \(i=2\) is similar.

The above result indicates that each of the components \(X_i\) lacks memory and depends only on the other component or in other words the residual life of each component depends on the life time of the other. This property will be referred to as the local lack of memory.
of bivariate distribution and it will be shown in a subsequent section that the only absolutely continuous bivariate model that exhibits this property is (3.1).

3.1.3 Moments

When the random vector \((X_1, X_2)\) has distribution (3.1), the \((r,s)\)th order raw moment

\[
\mu_{rs} = \text{E}(X_1^r X_2^s)
\]
simplifies to

\[
\mu_{rs} = s! \left[ a_1 J(r,s) + \Theta s J(r,s+1) \right]
\]

where

\[
J(r,s) = \int_0^\infty x_1^r (\alpha_2 + \Theta x_1)^{-s} e^{-\alpha_1 x_1} \, dx_1
\]

In particular

\[
\mu_{r0} = \text{E}(X_1^r) = r! \, \alpha_1^{-r} \quad \text{and}
\]

\[
\mu_{0s} = \text{E}(X_2^s) = s! \, \alpha_2^{-s}
\]

The characteristic function of the distribution is
\[ \phi(t_1, t_2) = \left( 1 - \frac{it_1}{\alpha_1} \right) + \left( 1 - \frac{it_2}{\alpha_2} \right) - 1 \]

\[ - t_1 t_2 \theta^{-1} \exp[(\alpha_1 - it_1)(\alpha_2 - it_2)\theta^{-1}] \]

\[ E_1[(\alpha_1 - it_1)(\alpha_2 - it_2)\theta^{-1}] \]  

(3.11)

where

\[ E_1(x) = \int_x^\infty e^{-z} z^{-1} dz \]  

(3.12)

Bivariate distributions are primarily intended to provide models when there is some kind of dependency between the underlying variables. It is therefore important to look at the correlation structure associated with (3.1) to be able to know the type of random phenomena it can represent reasonably well. We first notice that the regression equation of \( x_1 \) given \( x_j \) is

\[ E(x_1|x_j) = \frac{\alpha_j (\alpha_i + \theta x_j \alpha_i) + \Theta}{\alpha_j (\alpha_i + \theta x_j \alpha_i)^2} \]  

(3.13)

which are non-linear. The above function is ever decreasing and crosses the \( x_1 \) axis at \( \alpha_i \alpha_j + \Theta/\alpha_j \alpha_i^2 \).

The \( x_j \) axis becomes an asymptote when \( x_j \) increases indefinitely. The regression curves does not intersect at the means of the variables as in the normal case, except when \( \Theta = 0 \) in which case the variables are
independent. On the other hand, the correlation coefficient is

$$\text{Cor}(X_1, X_2) = \alpha_1 \alpha_2 \theta^{-1} \exp(\alpha_1 \alpha_2 \theta^{-1}) E_{\theta}(\alpha_1 \alpha_2 \theta^{-1}) - 1 \ (3.14)$$

3.1.4. Truncated moments

Another type of moments that are of interest in practical applications is the truncated version defined as follows. For a random vector $X=(X_1, X_2)$ admitting absolutely continuous distribution in the support of the first quadrant

$$Q = \{(x_1, x_2); x_1 \geq 0, x_2 \geq 0\}$$

of the two dimensional space $\mathbb{R}^2$, we define its $(r,s)^{th}$ bivariate truncated moment as

$$\phi^{r,s}(t_1, t_2) = E[(X_1 - t_1)^r (X_2 - t_2)^s \mid X > t]$$

or

$$R(t_1, t_2) \phi^{rs}(t_1, t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_1 - t_1)^r (x_2 - t_2)^s dF \ (3.15)$$

Taking $s = 0$, we get

$$\phi^{r,0}(t_1, t_2) = E[(X_1 - t_1)^r \mid X > t] \ (3.16)$$
Notice however that this is different from the \( r \)th truncated moment of the component variable \( X_1 \), which is in fact

\[
\phi^r_1(t_1) = E[(X_1 - t_1)^r \mid X_1 > t_1] \quad (3.17)
\]

\[
= \phi^{r,0}(t_1, 0)
\]

When \( X \) has the Gumbel distribution \( \phi^{r,0}(t_1, t_2) \), simplifies to

\[
\phi^{r,0}(t_1, t_2) = r!(\alpha_1 + \Theta t_2)^{-r} \quad (3.18)
\]

which is independent of \( t_1 \).

A symmetric expression is available corresponding to \( r = 0 \) in (3.15).

A detailed discussion of some properties associated with bivariate truncated moments in general and also some features peculiar to the bivariate exponential distribution will be taken up in Section 3.3.

3.1.5. Partial moments

The \((r, s)\)th partial moment of the random variable \( X \) defined in Section 3.1.4 is given by

\[
\psi^{r,s}(t_1, t_2) = E[(X_1 - t_1)^r \mid (X_2 - t_2)^+)^s] \quad (3.19)
\]
where

\[(X_i-t_i)^+ = \max(X_i-t_i, 0) \text{ for } i=1,2.\]

From (3.19)

\[
\varphi_{r, s}(t_1, t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_1-t_1)^r (x_2-t_2)^s \, dF \quad (3.20)
\]

and from (3.15)

\[
\varphi_{r, s}(t_1, t_2) = R(t_1, t_2) \varphi(t_1, t_2) \quad (3.21)
\]

In particular for \(s=0\) in (3.19)

\[
\varphi_{r, 0}(t_1, t_2) = E \left[ (X_1-t_1)^+ \right]^r \quad (3.22)
\]

\[
= \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_1-t_1)^r \, dF
\]

We distinguish this with the \(r^{th}\) partial moment of the random variable \(X_1\) which is

\[
\varphi_r(t_1) = \int_{t_1}^{\infty} (x_1-t_1)^r f_1(x_1) \, dx_1 \quad (3.23)
\]

\[
= \varphi_{r, 0}(t_1, 0)
\]
For the Gumbel model from (3.18) and (3.21)

\[ \Psi_{\mathcal{R}}(t_1, t_2) = \exp \left[ -\alpha_1 t_1 - \alpha_2 t_2 - \Theta t_1 t_2 \right] \left( \alpha_1 + \Theta t_2 \right)^{-\frac{1}{\alpha}} \quad (3.24) \]

3.1.6 Distributions of maxima and minima

In life length studies of two component systems, certain systems operate on the condition that it fails when one of the components fails. In such situations, the random variable of interest is \( Y = \min(X_1, X_2) \). The distribution of minimum is also useful in other contexts as well. From the formula

\[ F_Y(y) = F_{X_1}(y) + F_{X_2}(y) - F_{X_1, X_2}(y, y) \]

the distribution of \( Y \) for the Gumbel distribution is represented by

\[ f_Y(y) = (\alpha_1 + \alpha_2 + 2\Theta y) \exp(-\alpha_1 y - \alpha_2 y - \Theta y^2) \quad (3.25) \]

The distribution (3.25) reduces to the standard Rayleigh distribution under the transformation

\[ t = \alpha_1 y + \alpha_2 y + \Theta y^2 \]

However, when the system fails only when both components fail, its life time is \( Z = \max(X_1, X_2) \).
For the particular distribution (3.1), $Z$ has density function

$$f_Z(z) = a_1 \exp(-a_1 z) + a_2 \exp(-a_2 z)$$

$$- (a_1 + a_2 + 2\theta z) \exp(-a_1 z - a_2 z - \theta z^2) \quad (3.26)$$

3.2 Characterization problems

It is evident from the review of literature, in the previous chapter, that most characterizations on bivariate exponential distributions revolve around suitable extension of the properties in the univariate case. Since such extensions can be achieved in a variety of ways our aim is to find meaningful definitions analogous to the concepts in one dimension that can characterize the Gumbel's form. The properties of the distribution discussed in the previous section form the basis of our investigation. In the following sections we identify those properties that are unique to the Gumbel's bivariate exponential distribution and which have meaningful physical interpretations related to real world phenomena.

The theorems that follow in the succeeding sections are broadly classified under three heads
(i) those based on properties of truncated moments,
(ii) by geometric compounding and (iii) by form of
conditional distributions.

3.3 Characterizations based on truncated moments *

Theorem 3.1

Let \( X = (X_1, X_2) \) be a vector of non-negative
random variables admitting probability density function
with respect to Lebesgue measure given by \( f(x_1, x_2) \) such
that \( E(X_1^k) < \infty \). Then \( X \) follows the Gumbel's bivariate
exponential distribution specified by (3.1) if and only
if for all positive integers \( k \)

\[
E[(X_1-t_1)^k \mid X_1>t_1, X_2>t_2] = a_k^{(i)}(t_3-i)
\]  
(3.27)

where

\[
a_k^{(i)}(t_3-i) = E[X_1^k \mid X_3-i > t_3-i]
\]  
(3.28)

are non-increasing, \( a_k^{(i)} \) is independent of \( t_1 \) for
all \( t_1 > 0 \) with

\[
a_1^{(i)}(o) = a_1^{-1}
\]  
(3.29)

* Some results in this section have appeared in the
p.267-271. (Reference 36)
Proof

When the conditions of Theorem 3.1 are true, equation (3.27) can be written as

\[ a_k^{(1)} (t_{3-i}) \int_{t_1}^{\infty} \int_{t_2}^{\infty} f(x_1, x_2) \, dx_1 \, dx_2 \]

\[ = \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_1 - t_1)^k f(x_1, x_2) \, dx_1 \, dx_2 \quad (3.30) \]

Taking \( i = 1 \)

\[ a_k^{(1)} (t_2) \, R(t_1, t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_1 - t_1)^k \frac{\partial^2 R}{\partial x_1 \partial x_2} \, dx_1 \, dx_2 \quad (3.31) \]

where \( R = R(t_1, t_2) \) is the survival function of \((X_1, X_2)\) given by

\[ R(t_1, t_2) = P[X_1 > t_1, X_2 > t_2] \quad (3.32) \]

Integrating the right side of (3.31) with respect to \( x_2 \)

\[ a_k^{(1)} (t_2) \, R(t_1, t_2) = \int_{t_1}^{\infty} (x_1 - t_1)^k \frac{\partial}{\partial x_1} [F_1(x_1) - F(x_1, t_2)] \, dx_1 \quad (3.33) \]

where \( F_1(.) \) and \( F(., .) \) are the distribution functions of
Differentiating (3.33) with respect to $t_1$

$$a_{k}^{(l)}(t_2) \frac{\partial R}{\partial t_1} = - \int_{t_1}^{\infty} k(x_1-t_1)^{k-1} \frac{\partial}{\partial x_1} [F_1(x_1)-F(x_1,t_2)] dx_1$$

and performing the same operation successively

$$a_{k}^{(l)}(t_2) \frac{\partial^{k} R}{\partial t_1^{k-1}} = (-1)^{k-1} k! \int_{t_1}^{\infty} (x_1-t_1)^{k-1} \frac{\partial}{\partial x_1} [F_1(x_1)-F(x_1,t_2)] dx_1$$

$$= (-1)^{k-1} k! \int_{t_1}^{\infty} [F_1(x_1)-F(x_1,t_2)] dx_1$$

or

$$a_{k}^{(l)}(t_2) \frac{\partial^{k} R}{\partial t_1^{k}} = (-1)^{k} k! R(t_1,t_2) \quad (3.34)$$

For $k=1$, equation (3.34) reduces to

$$\frac{\partial \log R}{\partial t_1} = \frac{-1}{a_{1}^{(l)}(t_2)}$$

The solution of this equation is

$$R(t_1,t_2) = c_1(t_2) \exp \left[ \frac{-t_1}{a_{1}^{(l)}(t_2)} \right] \quad (3.35)$$
When \( t_1 \) tends to zero in the last expression,

\[
1 - F_2(t_2) = c_1(t_2)
\]

where \( F_2(.) \) is the distribution function of \( X_2 \).

Substituting in (3.35)

\[
R(t_1,t_2) = 1 - F_1(t_1) - F_2(t_2) + F(t_1,t_2)
\]

\[
= [1 - F_2(t_2)] \exp \left[ \frac{-t_1}{a_1^{(1)}(t_2)} \right]
\]  \hspace{1cm} (3.36)

To obtain a general solution of (3.34), notice that the equation may be rewritten in the form

\[
a_k^{(1)}(t_2) \frac{\delta^k R(t_1,t_2)}{\delta t_1^k} = R(t_1,t_2)
\]  \hspace{1cm} (3.37)

which is satisfied by the function

\[
R(t_1,t_2) = \sum_{j=1}^{k} c_j(t_2) e^{b_j(t_2)t_1}
\]  \hspace{1cm} (3.38)

where \( c_j(t_2) \) are arbitrary functions independent of \( t_1 \)
and \( b_j(t_2) \) are the \( k \) solutions of the auxiliary equation
\[
\frac{a_k(t_2)}{(-1)^k k!} m^k = 1 \quad (3.39)
\]

For \( k = 3 \), the equation (3.39) takes the form

\[(am^3 + 6) = 0\]

whose solutions are a negative root with value \(-\left(\frac{b}{a}\right)^{1/3}\)

and two complex roots

\[\frac{1}{2} - \frac{1}{6^{1/3}} \left(1 \pm \sqrt{3}i\right)\]

where \( a = a_3(t_2) \)

Since \( a > 0 \), the real parts are positive and accordingly

\[|e^{b_1(t_2)t_1}| \rightarrow \infty \text{ as } t_1 \rightarrow \infty\]

Since \( R(t_1, t_2) \) tends to zero as \( t_1 \) tends to infinity, we must have \( c_2(t_2) = c_3(t_2) = 0 \) in (3.38). Thus

\[R(t_1, t_2) = c_1(t_2) \exp[-b_1(t_2)t_1] \quad (3.40)\]

When \( k = 4 \), the equation to be considered is

\[(am^4 - 24) = 0\]
with real roots

\[ m = \pm \left( \frac{24}{a} \right)^{\frac{1}{4}} \]

and imaginary roots

\[ m = \pm i \left( \frac{24}{a} \right)^{\frac{1}{4}} \]

with real parts zero.

When \( m = \left( \frac{24}{a} \right)^{\frac{1}{4}} \), arguing as before \( c_4(t_2) = 0 \).

For the two imaginary roots, the expression

\[ c_2(t_2)e^{b_2(t_2)t_1} + c_3(t_2)e^{b_3(t_2)t_1} \]

has to decrease to zero for large \( t_1 \) and hence once again we have the form in (3.40) for \( R(t_1,t_2) \). For \( k = 5, 6, \ldots \) the argument is similar and therefore the general solution of (3.39) is

\[ R(t_1,t_2) = c_1(t_2) \exp[-b_1(t_2)]t_1 \]

As \( t_1 \to 0 \) in the above equation

\[ c_1(t_2) = 1 - F_2(t_2) \]
Hence from (3.36) and (3.40) we find

\[ b_1(t_2) = \frac{1}{a_1(1)(t_2)} \]

for all values of \( k \).

Equation (3.36) can also be written as

\[ 1-F_1(t_1)-F_2(t_2)+F(t_1, t_2) = [1-F_2(t_2)]\exp\left[\frac{-t_1}{a_1(t_2)}\right] \]

(3.41)

As \( t_2 \) tends to zero

\[ 1-F_1(t_1) = \exp\left[\frac{-t_1}{a_1(t_1)}\right] \]

\[ = \exp(-\alpha_1 t_1), \text{ using } (3.29) \]

or

\[ F_1(t_1) = 1-\exp(-\alpha_1 t_1) \]

Similarly

\[ F_2(t_2) = 1-\exp(-\alpha_2 t_2) \]

Thus from (3.41)

\[ R(t_1, t_2) = \exp\left[-\alpha_2 t_2 - \frac{t_1}{a_1(t_2)}\right] \]  (3.42)
On similar lines one can show by taking \( i=2 \) and \( k=1 \) in equation (3.30) that

\[
R(t_1, t_2) = \exp \left[ -\alpha_1 t_1 - \frac{t_2}{a_1^{(2)}(t_1)} \right]
\]  
(3.43)

Equating the expressions for \( R(t_1, t_2) \) in (3.42) and (3.43) results in the functional equation

\[
\alpha_2 t_2 + \frac{t_1}{a_1^{(1)}(t_2)} = \alpha_1 t_1 + \frac{t_2}{a_1^{(2)}(t_1)}
\]  
(3.44)

To solve (3.44) we write it in the form

\[
\frac{1-\alpha_2 a_1^{(1)}(t_2)}{t_2 a_1^{(1)}(t_2)} = \frac{1-\alpha_2 a_1^{(2)}(t_1)}{t_1 a_1^{(2)}(t_1)}
\]  
(3.45)

Since (3.45) has to be true for all \( t_1, t_2 > 0 \)

\[
\frac{1-\alpha_1 a_1^{(1)}(t_{3-i})}{t_{3-i} a_1^{(1)}(t_{3-i})} = 0
\]  
(3.46)

a constant, independent of \( t_1 \) and \( t_2 \) for \( i=1,2 \).
Hence

\[ a_1^{(i)}(t_{3-i}) = \frac{1}{\alpha_1 + \Theta t_{3-i}} \]

and therefore, the survival function of X is

\[ R(t_1, t_2) = \exp(-\alpha_1 t_1 - \alpha_2 t_2 - \Theta t_1 t_2) \tag{3.47} \]

and this completes the proof of the necessity of the condition. From the monotonicity of \( a_1(t_2) \) we have \( \Theta > 0 \). Further for the marginals of the bivariate exponential distribution to be proper densities we should have \( \alpha_1, \alpha_2 > 0 \).

The sufficiency part follows from the actual expression for truncated moments of the bivariate exponential distribution namely

\[ E[(X_i - t_i)^k | X > t] = k! (\alpha_i + \Theta t_{3-i})^{-k} \]

from where, it is easy to verify that the conditions of the theorem are true.

Corollary-1

Taking \( k=1 \) in equation (3.27) we get the characterizing property

\[ E[(X_i - t_i) | X_1 > t_1, X_2 > t_2] = E[X_i | X_j > t_j] \tag{3.48} \]
of (3.1) which is proved in Nair and Nair (1988).

Corollary-2

Setting \( i=1 \) and allowing \( t_2 \) to tend to zero the relationship

\[
E\left[(X_1-t_1)^k \mid X_1 > t_1\right] = E(X_1)^k
\]

(3.49)

for \( k=1,2,3,\ldots \) characterizes the univariate exponential distribution with survival function

\[
P[X_1 > x_1] = \exp(-\alpha_1 x_1)
\]

the result due to Sahobov and Geshev cited in Galambos and Kotz (1978).

Corollary-3

When \( k=1 \) in (3.49), we have

\[
E[X_1-t_1 \mid X_1 > t_1] = E(X_1)
\]

(3.50)

which is the well known constancy property of the mean residual life function of the exponential distribution proved in several investigations as Reinhardt (1968), Shanbhag(1970), Gupta (1975) etc.
In the following theorem we prove that the local lack of memory property explained in Section 3.1.2 is characteristic of the distribution (3.1) in the class of absolutely continuous bivariate models.

Theorem 3.2

The random vector $X$ in Theorem 3.1 has the Gumbel's bivariate exponential distribution if and only if for all $t_i, s_i \geq 0$, there holds the relations

$$P[X_i > t_i + s_i | X_1 > t_1, X_2 > t_2] = P[X_i > s_i | X_j > t_j]$$  \hspace{1cm} (3.51)  

$i=1, 2, i \neq j$

Proof

It is enough to establish the equivalence of (3.51) and (3.48). To prove this we note that when $i=1$ (3.51) is equivalent to

$$R(t_2) R(t_1 + s_1, t_2) = R(s_1, t_2) R(t_1, t_2)$$  \hspace{1cm} (3.52)

where

$$R(t_2) = P[X_2 > t_2]$$

Integrating (3.52) with respect to $s_1$

$$\int_0^\infty f(t_1 + s_1 | X > t) ds_1 = \int_0^\infty f(s_1 | X_2 > t_2) ds_1$$
which is the same as

$$\int_{t_1}^{\infty} (s_1-t_1) f(s_1 \mid X > t) \, ds_1 = \int_{0}^{\infty} s_1 f(s_1 \mid X_2 > t_2) \, ds_1$$

or

$$E[X_1-t_1 \mid X > t] = E[X_1 \mid X_2 > t_2]$$

as stated in equation (3.48).

The proof for \(i=2\) is similar. The converse follows by retracing the steps and this completes our assertion.

We notice that

(i) for \(i=1\), and \(t_2, s_2\) tending to zero (3.51) becomes

$$P[X_1 > t_1 + s_1 \mid X_1 > t_1] = P[X_1 > s_1]$$

the lack of memory property of the random variable \(X_1\).

(ii) The condition (3.48) is weaker than (3.51) as the former requires only the knowledge of the expected values while the latter requires the entire truncated distribution.

(iii) In (3.48) the existence of the mean is a necessity while in (3.50) only the distribution function need be known.
3.3.2 Properties of truncated moments.

The property (3.27) that characterizes the bivariate exponential distribution requires that it must be true for every positive integer \( k \) which appears to be a rather stringent condition when one wishes to verify the property to identify the distribution in a practical situation. It is therefore of some interest to enquire whether a relaxation of the requirement can be accomplished. An investigation in this direction necessitates a more detailed study of the properties of bivariate truncated moments. To begin with we establish a recurrence relation satisfied by truncated moments \( \varnothing^{r,s}(t_1, t_2) \) defined in Section 3.1.3.

Theorem 3.3

The truncated moments \( \varnothing^{r,s} \) satisfy the recurrence relation

\[
(\varnothing^{r,s})^{-2} \left[ D\varnothing^{r,s} - rs \cdot E \varnothing^{r,s} \right] = (\varnothing^{r-1,s-1})^{-2} \\
\left[ D\varnothing^{r-1,s-1} - (r-1)(s-1) \right] \\
E\varnothing^{r-1,s-1}
\]

for \( r, s \geq 2 \).
where

\[
D = \begin{bmatrix}
1 & \frac{\partial}{\partial t_1} \\
\frac{\partial}{\partial t_2} & \frac{\partial^2}{\partial t_1 \partial t_2}
\end{bmatrix}
\quad ; \quad E = \begin{bmatrix}
E_{00} & E_{01} \\
E_{10} & E_{11}
\end{bmatrix}
\quad (3.54)
\]

and \( E_{mn} \phi^{r,s} = \phi^{r-m,s-m} \)

Proof

From the definition in equation (3.15)

\[
R(t_1,t_2) \phi^{r,s} = \int_t^\infty \int_t^\infty (x_1-t_1)^r (x_2-t_2)^s \frac{\partial^2 R}{\partial x_1 \partial x_2} \, dx_1 \, dx_2
\]

\[
= rs \int_t^\infty \int_t^\infty (x_1-t_1)^{r-1} (x_2-t_2)^{s-1} \, R(x_1,x_2) \, dx_1 \, dx_2 \quad (3.55)
\]

by partial integration. Logarithmic differentiation in (3.55) with respect to \( t_1 \) and then differentiation of the resulting expression with respect to \( t_2 \) yields after some involved algebra to the equation

\[
(\phi^{r,s})^{-2} \left\{ \phi^{r,s} \frac{\partial^2 \phi^{r,s}}{\partial t_1 \partial t_2} - \phi^{r,s} \frac{\partial^2 \phi^{r,s}}{\partial t_1 \partial t_2} \right\}
\]

\[
- rs[ \phi^{r,s} \phi^{r-1,s-1} \phi^{r-1,s} \phi^{r,s-1} ]
\]

\[
= R^{-2} \left[ R \frac{\partial^2 R}{\partial t_1 \partial t_2} - \frac{\partial R}{\partial t_1} \frac{\partial R}{\partial t_2} \right] \quad (3.56)
\]
Since the right hand side is independent of both \( r \) and \( s \), the recurrence relation (3.53) is immediate if we change \( r \) and \( s \) respectively to \( (r-1) \) and \( (s-1) \) and subtract the resulting equation from (3.57).

**Theorem 3.4**

The truncated moments \( \phi^{r,0}(t_1, t_2) \) satisfy the recurrence relation

\[
\frac{\partial}{\partial t_1} \psi_{r,0} - (r-1) \left( \frac{\partial \psi_{r,0}}{\partial t_1} / \psi_{r-1,0} \right) + r = 0 \quad (3.58)
\]

where

\[
\psi_{r,0} = \frac{\phi^{r,0}}{\phi^{r-1,0}}
\]
Proof:

By definition,

\[ R(t_1, t_2) \varphi^{r, o}(t_1, t_2) = \int_{t_1}^{t_2} \int_{t_1}^{t_2} (x_1 - t_1)^r \frac{\partial^2 R}{\partial x_1 \partial x_2} \, dx_1 \, dx_2 \]

\[ = \int_{t_1}^{t_2} (x_1 - t_1)^r \frac{\partial R}{\partial x_1} \, dx_1 \]

\[ = \int_{t_1}^{t_2} r(x_1 - t_1)^{r-1} R(x_1, t_2) \, dx_1 \]

Logarithmic differentiation yields

\[ \left( \frac{\partial}{\partial t_1} \varphi^{r, o} + r \varphi^{r, o} \right) / \varphi^{r, o} = - \left( \frac{\partial R}{\partial t_1} \right) / R \]  (3.59)

Changing \( r \) to \( (r-1) \) in (3.59) and subtracting the resulting equation from (3.59) leads to (3.58) after simplification.

In an identical fashion one can also prove that

\[ \frac{\partial}{\partial t_2} \psi_o, s - (s-1) (\psi_o, s / \psi_o(s-1)) + s = 0, \text{ } s \geq 2 \]  (3.60)

Theorem 3.5.

When \( X \) is a random vector which satisfies the conditions in theorem (3.1), \( X \) follow the Gumbel
distribution if for two specific integers \( r \) and \( s \)
\(( r, s \geq 2)\)

\[
\frac{\phi^{r,0}(t_1, t_2)}{r \phi^{r-1,0}(t_1, t_2)} = a_1(t_2) \tag{3.61}
\]

and

\[
\frac{\phi^{0,s}(t_1, t_2)}{s \phi^{0,s-1}(t_1, t_2)} = a_2(t_1) \tag{3.62}
\]

where \( a_i(t_{3-i}) \) are non-increasing in \( t_{3-i}, i=1,2 \).

Proof:

The recurrence relation in theorem 3.4 can be written as

\[
\frac{\partial \phi^{r,0}}{\partial t_1} + r \phi^{r-1,0} = \frac{\partial \phi^{r-1,0}}{\partial t_1} + (r-1) \phi^{r-2,0} \tag{3.63}
\]

Using (3.63) in (3.61)

\[
\frac{\phi^{r-1,0}}{(r-1) \phi^{r-2,0}} = a_1(t_2)
\]
and hence
\[ \varphi^{0,1}(t_1, t_2) = a_1(t_2) \]

Similarly we can show that
\[ \varphi^{0,1}(t_1, t_2) = a_2(t_1) \]

Thus by corollary 1 to theorem 3.1, the distribution of \( X \) is Gumbel's bivariate exponential distribution. By virtue of theorem 3.5 it becomes evident that the ratios of consecutive higher order truncated moments satisfy the specified functional form is sufficient to guarantee the bivariate exponential distribution, in relaxation to the conditions in theorem 3.1.

The constancy of the coefficient of variation of the residual life \( X-t|X > t \) of a continuous non-negative random variable \( X \) is cited as a unique property exhibited by the univariate exponential distribution in several investigations eg. Nagaraja (1975), Mukherjee and Roy (1986) and Gupta and Gupta (1983). While this result focusses on a property of relative measure of dispersion of the residual life, similar characterizations exist if we consider various absolute measures of dispersion also. Johnson and Kotz (1970) points out the following results due to
Guerrieri.

(a) the variance of the conditional distribution, given that the variable takes values exceeding x does not depend on x.

(b) As for (a) "mean deviation" replacing "variance".

(c) As for (a) "mean difference" replacing "variance".

In the remainder of this section we establish some bivariate analogous of these results that characterize the bivariate exponential distribution.

Theorem 3.6.

Let \( X = (X_1, X_2) \) be a random vector admitting a non-degenerate distribution function in \( \mathbb{R}_2^+ \) and \( t = (t_1, t_2) \) be a vector of non-negative real numbers. Then \( X \) follow the Gumbel distribution if and only if

\[
\frac{\varphi^{2,0}(t_1, t_2)}{[\varphi^{1,0}(t_1, t_2)]^2} = \frac{\varphi^{0,2}(t_1, t_2)}{[\varphi^{0,1}(t_1, t_2)]^2} = 2 \quad (3.64)
\]

Proof:

When \( r=2 \), the recurrence relation in theorem 3.4 takes the form
Introducing (3.64) into (3.65)

\[
\frac{\delta \phi^{1,0}}{\delta t_1} = -1 + \frac{\delta}{\delta t_1} \left[ \frac{2(\phi^{1,0})^2}{\phi^{1,0}} \right] + \frac{2[\phi^{1,0}]^2}{\phi^{1,0}}
\]

\[
= 2 \frac{\delta \phi^{1,0}}{\delta t_1}
\]  

(3.66)

For equation (3.66) to be true we must have

\[
\frac{\delta \phi^{1,0}}{\delta t_1} = 0
\]

Integrating

\[
\phi^{1,0}(t_1,t_2) = c_1(t_2)
\]

(3.67)

where \(c_1(t_2)\) is independent of \(t_1\). Proceeding on similar lines one can also show that

\[
\phi^{0,1}(t_1,t_2) = c_2(t_1)
\]

(3.68)

Noticing that (3.67) and (3.68) are essentially same as (3.27) with \(k=1\), the sufficiency follows. Conversely
when $X$ has distribution (3.1), utilising the expression for truncated moments in (3.18) we arrive at (3.64).

Theorem 3.7

Let $X = (X_1, X_2)$ be a continuous non-negative random variable admitting absolutely continuous distribution with $V(X_1) < \infty$.

Denoting

$$V(X_1 | X > t) = V_1(t_1, t_2) \quad (3.69)$$

$X$ follows the Gumbel distribution if and only if

$$V_1(t_1, t_2) = V_1(t_{3-i}) \quad (3.70)$$

where $V_1(t_{3-i})$ are non-increasing functions, independent of $t_1$, with

$$V_1(o) = \alpha_i^{-2} ; i = 1,2.$$

Proof:

When (3.70) holds for $i=1$

$$V(X_1-t_1 | X > t) = V_1(t_2)$$

implies
Introducing the transformation

\[ h(t_1) = \int_{t_1}^{\infty} (x_1-t_1)[F_1(x_1)-F(x_1,t_2)] dx_1 \]  

(3.72)

as a parameter and writing

\[ g(h) = (\frac{dh}{dt_1})^2 \]  

(3.73)

we get

\[ \frac{d^2 h}{dt_1^2} = R(t_1,t_2) \]  

(3.74)

and

\[ \frac{dg}{dh} = 2 \frac{d^2 h}{dt_1^2} \]

Equation (3.71) after effecting the transformation reduces to
\[ v_1(t_2) \left( \frac{d^2 h}{dt_1^2} \right)^2 = 2 \left( \frac{d^2 h}{dt_1^2} \right) h - \left( \frac{dh}{dt_1} \right)^2 \]

or

\[ \left( \frac{d g}{dh} \right)^2 - \frac{4}{v_1(t_2)} h(t_1) \left( \frac{d g}{dh} \right) + \frac{4}{v_1(t_2)} g = 0 \]  \hspace{1cm} (3.75)

This is Clairant's equation with solutions

\[ h_1(t_1) = c \exp \left( \pm t_1 \sqrt{\frac{1}{v_1(t_2)}} \right) \]

and

\[ h_2(t_1) = -\frac{4}{v_1(t_2)} (2 t_1 + c_1)^2 - c_2 \]

Of these, the only solution which meets the requirements of a probability density function is

\[ h(t_1) = c(t_2) \exp \left[ -t_1 [V(t_2)]^{-\frac{1}{2}} \right] \]

where \( c(t_2) \) is a constant independent of \( t_1 \).

Using (3.74)

\[ R(t_1, t_2) = c(t_2) \left[ V_1(t_2) \right]^{-1} \exp \left[ -t_1 [V_1(t_2)]^{-\frac{1}{2}} \right] \]  \hspace{1cm} (3.76)

As \( t_1 \) tends to zero

\[ c(t_2) = \left[ 1 - F_2(t_2) \right] V_1(t_2) \]
Equation (3.76), therefore turns out to be

\[ R(t_1,t_2) = \left[1 - F_2(t_2)\right] \exp \left[-t_1(V_1(t_2))^{\frac{1}{2}}\right] \] (3.77)

Similarly for \(i=2\),

\[ R(t_1,t_2) = \left[1 - F_1(t_1)\right] \exp \left[-t_2(V_2(t_1))^{\frac{1}{2}}\right] \] (3.78)

In (3.77) and (3.78) as \(t_2\) and \(t_1\) respectively tends to zero

\[ F_1(t_1) = 1 - \exp(-\alpha_1 t_1) \] (3.79)

leading to the functional equation

\[ \alpha_1 t_1 - \alpha_2 t_2 = t_1(V_1(t_2))^{\frac{1}{2}} - t_2(V_2(t_1))^{\frac{1}{2}} \]

or

\[ \frac{t_1[V_2(t_1)]^{\frac{1}{2}}}{1-\alpha_2[V_2(t_1)]^{\frac{1}{2}}} = \frac{t_2[V_1(t_2)]^{\frac{1}{2}}}{1-\alpha_1[V_1(t_2)]^{\frac{1}{2}}} \] (3.80)

for all \(t_1\) and \(t_2\). Proceeding as in (3.45), the only solutions that satisfy (3.80) are

\[ V_i(t_{3-i}) = \frac{1}{(\alpha_1 + \Omega t_{3-i})^2} \] (3.81)
Substituting in (3.77) or (3.78) we get

\[ R(t_1,t_2) = \exp(-\alpha_1 t_1 - \alpha_2 t_2 - \Theta t_1 t_2). \]

The sufficiency follows from the fact that when X has the proposed distribution

\[ V(X_1|X>t) = \frac{1}{(\alpha_1 + \Theta t_{3-1})^2} \]

which satisfy the conditions of the theorem.

3.4 Characterizations based on Geometric Compounding

Let X_1, X_2 ... be independent and identically distributed random variables with common distribution function F(x) and N be a random variable following the geometric law

\[ P(N=n) = pq^{n-1}, \quad n=1,2,3... \quad (3.82) \]

independently of the X_i's. If F*(x) is the distribution function of S* defined by

\[ pS^* = X_1 + X_2 + \ldots + X_N, \quad (3.83) \]

the point of interest in geometric compounding models is
the relation between $F^*(x)$ and $F(x)$. When the common distribution of the $X_i$'s is exponential, Arnold (1973) has established that the distribution of $pS^*$ is identical with that of $X_1$. A detailed exposition of the geometric compounding model and its relationship with the rarefaction models of Renyi (1956) in renewal processes and the damage models introduced by Rao and Rubin (1964) is discussed in Galambos and Kotz (1978). Although these models are of wide applicability in biology, analysis of incomes, under reporting of accidents etc, there have been only a few investigations (see Talwalker (1970) and Patil and Ratnaparkhi (1975)) that extend such ideas to higher dimensions. Our aim in the present section is to generalise the concept of geometric compounding to two dimensions by using the bivariate exponential distribution.

Theorem 3.8*

Let $(X_k)$ be a sequence of non-degenerate, independent and identically distributed random variables admitting probability density function with respect to Lebesgue measure, with components $X_k = (X_{1k}, X_{2k})$ and support $R^+_2 = \{(x, y) \mid x, y > 0\}$ such that the conditional * forms part of the paper "Characterizations of the Gumbel's bivariate exponential distribution" Statistics, Vol. 21: (1990) (Reference 39)
expectations

\[ m_j(t_{3-j}) = E[X_{jk} \mid X_{3-j,k} > t_{3-j}], \quad j=1,2 \]  

exist for all real \( t_1, t_2 > 0 \) and are non-increasing.

The relations,

\[ \Pr[ pS_{jN} > x \mid X_{3-j,k} > t_{3-j}, 1 \leq k \leq N ] = \Pr[ X_{jk} > x \mid X_{3-j,k} > t_{3-j}] \]  

where \( S_{jN} = \sum_{k=1}^{N} X_{jk} \) and \( N \) is a random variable following the geometric law

\[ \Pr[N=n] = p(1-p)^{n-1}, \quad n=1,2,3\ldots \]

independently of \( X_{jk} \), are satisfied for all real \( x > 0 \) and \( t_1, t_2 > 0 \) if and only if, \( X_k \) has bivariate exponential distribution (3.1) with \( \alpha_j = [m_j(t_0)]^{-1} \).

Proof:

The logic in the proof is the same as in the univariate case (see Azlarov and Volodin (1986)) appropriately adapted to the bivariate situation.
When $X_k$ has the bivariate exponential distribution, the density function of $X_{1k}$ given $X_{2k} > t_2$ is

$$f(x|X_{2k} > t_2) = (\alpha_1 + \Theta t_2) \exp[-(\alpha_1 + \Theta t_2)x], x > 0$$

with characteristic function

$$A(s,t_2) = [1 - is(\alpha_1 + \Theta t_2)]^{-1} \quad (3.86)$$

Now, if $B(s,t_2)$ is the characteristic function of $S_{jN}$ given $X_{2k} > t_2$ for $1 \leq k \leq N$, we have

$$B(s,t_2) = \sum_{n=1}^{\infty} P(N=n) [A(ps,t_2)]^n$$

$$= \sum_{n=1}^{\infty} p(1-p)^{n-1} [A(ps,t_2)]^n$$

$$= pA(ps,t_2) \left[1-(1-p)A(ps,t_2)\right]^{-1} \quad (3.87)$$

Substituting (3.86) into (3.87) we find

$$A(s,t_2) = B(s,t_2) \text{ for all } s, t_2 > 0 \quad (3.88)$$

proving (3.85) for $j=1$. The proof for $j=2$ follows by symmetry.
Conversely if (3.85) holds for \( j = 1 \), (3.88) is true and therefore from (3.87)

\[
A(s, t_2) = p^n A(p^n s, t_2)[1-(1-p^n)A(p^n s, t_2)]^{-1}
\]  

(3.89)

for all \( n = 1, 2, 3, \ldots \). Following the proof in the univariate case given in Azlarov and Volodin (1986) we write (3.89) in the form

\[
[1-A(s, t_2)] A(s, t_2) = [1-A(p^n s, t_2)] p^{-n}[A(p^n s, t_2)]^{-1}
\]

Taking limits as \( n \) tends to infinity

\[
\frac{1-A(s, t_2)}{A(s, t_2)} = \lim_{p^n s \to 0} \frac{1-A(p^n s, t_2)}{p^n s A(p^n s, t_2)}
\]

\[
= - \lim_{p^n s \to 0} \frac{1}{A(p^n s, t_2)} \frac{A(p^n s, t_2)-1}{p^n s}
\]

\[
= - A'(o, t_2), \text{ since } A(o, t_2) = 1
\]

\[
= \text{im}_1(t_2)
\]

Thus \( A(s, t_2) = [1 - \text{is}_{m_1}(t_2)]^{-1} \)
and whence the density function of $X_{1k}$ given $X_{2k} > x_2$ is

$$ f(x_1 | X_{2k} > x_2) = [m_1(x_2)]^{-1} \exp[-m_1(x_2)]^{-1} x_1 \quad (3.90) $$

If we consider the relation (3.85) for $j=2$ arguments similar to the above leads to

$$ f(x_2 | X_{1k} > x_1) = [m_2(x_1)]^{-1} \exp [m_2(x_1)]^{-1} x_2 \quad (3.91) $$

As $x_1$ tends to zero in (3.91)

$$ P [X_{2k} > x_2] = \exp [ -\alpha_2 x_2 ] $$

Accordingly

$$ R(x_1, x_2) = \int_{x_1}^{\infty} [m_1(x_2)]^{-1} \exp[-m_1(x_2)]^{-1} x_1 \exp[-\alpha_2 x_2] \, dx_1 $$

$$ = \exp \left\{ -\alpha_2 x_2 - [m_1(x_2)]^{-1} x_2 \right\} \quad (3.92) $$

Similarly

$$ R(x_1, x_2) = \exp \left\{ -\alpha_1 x_1 - [m_2(x_1)]^{-1} x_1 \right\} \quad (3.93) $$

leading to the functional equation

$$ (\alpha_1 x_1 - \alpha_2 x_2) m_1(x_2) m_2(x_1) = x_1 m_2(x_1) - x_2 m_1(x_2) \quad (3.94) $$
Under conditions imposed on the m's in the theorem, proceeding as in theorem 3.1, the unique form of the solution of (3.94) is

\[ m_j(x_{3-j}) = (a_j + \Theta x_{3-j})^{-1}, \quad j=1,2 \quad (3.95) \]

Using (3.95) in (3.92) or (3.93) we get the desired form of the bivariate exponential distribution. The result of Theorem 3.8 concerns the geometric sum of the components of \((X_k)\) that are independent and identically distributed satisfying condition (3.85).

The question that arises now is, what can we say about the distribution of \((X_k)\) if we have two such partial geometric sums that are identically distributed? The only answer to the problem turns out to be that each \((X_k)\) has Gumbel's bivariate exponential distribution. This we establish in

Theorem 3.9.

If \((X_k)\) be the sequence of random variables in theorem 3.8, then the conditions

\[ P[p_1 s_{jN_1} > x_j | x_{3-j,k} > t_{3-j}; 1 \leq k \leq N_1] \]

\[ = P[p_2 s_{jN_2} > x_j | x_{3-j,k} > t_{3-j}, 1 \leq k \leq N_2](3.96) \]

\[ j = 1,2. \]
holds for all \( t_j, x_j > 0 \) if and only if the common distribution of \( (X_k) \) is the bivariate exponential distribution (3.1) where \( N_j, j = 1, 2 \) are geometric variables with

\[
P(N_j = n_j) = p_j(1-p_j)^{n_j-1}, \quad n_j = 1, 2, 3... \quad (3.97)
\]

independently of \( X_{jk} \).

Proof:

To prove the necessary part, following the notation in the proof of theorem 3.8, the characteristic function \( A(s, t_2) \) of \( X_{1k} \) given \( X_{2k} > t_2 \), when \( X_k \) has distribution (3.1) is by (3.8)

\[
A(s, t_2) = \left[ 1-i(\alpha_1 + \Theta t_2)s \right]^{-1}
\]

The characteristic function \( B_1(s, t_2) \) of \( p_1 S_{1N} \) given \( X_{2k} > t_2 \) for \( 1 \leq k \leq N \) is given by (3.87) as

\[
B_1(s, t_2) = p_1 A(p_1 s, t_2) \left[ 1-(1-p_1) A(p_1 s, t_2) \right]^{-1}
\]

We see that

\[
B_1(s, t_2) = A(s, t_2) \quad \text{for all} \quad t_2.
\]
Likewise, the characteristic function of $p_2 S_{1N_2}$ given $X_{2k} > t_2, B_2(s, t_2)$, reduces to $A(s, t_2)$, proving (3.96) for $j=1$. Since the proof for $j=2$ similar, the necessity of the condition follows.

Conversely condition (3.96) implies that for $j=1$

$$B_1(s, t_2) = B_2(s, t_2)$$

That is

$$p_1 A(p_1s, t_2) [1-(1-p_1)A(p_1s, t_2)]^{-1}$$

$$= p_2 A(p_2s, t_2) [1-(1-p_2)A(p_2s, t_2)]^{-1}$$

or

$$\frac{p_1 s A(p_1s, t_2)}{1-(1-p_1)A(p_1s, t_2)} = \frac{p_2 s A(p_2s, t_2)}{1-(1-p_2)A(p_2s, t_2)} \quad (3.98)$$

Equation (3.98) can be rearranged as

$$\frac{1}{1p_2s} - \frac{1}{1p_2s} A(p_2s, t_2) = \frac{1}{1p_1s} - \frac{1}{1p_1s} A(p_1s, t_2)$$

for all $0 < p_1, p_2 < 1$ so that
\[ \frac{1}{ip_j} s - \frac{1}{ip_j} s \frac{1}{A(p_j, s, t_2)} = k_1(t_2) \]

where \( k_1(.) \) is independent of \( p_j, j=1,2 \). This gives

\[ A(p_j, t_2) = \frac{1}{1-k_1(t_2)ip_j} \]

or

\[ A(s, t_2) = \frac{1}{1-isk_1(t_2)} \]

This leads to

\[ P[X_{1k} > t_1 | X_{2k} > t_2] = \exp(-k_1(t_2)t_1) \quad (3.99) \]

and

\[ P[X_{1k} > t_1] = \exp(-\alpha_1 t_1) \]

where \( \alpha_1 = k_1(o) \).

Similarly the analysis for \( j=2 \) leaves

\[ P[X_{2k} > t_2 | X_{1k} > t_1] = \exp(-k_2(t_1)t_2) \quad (3.100) \]

and

\[ P[X_{2k} > t_2] = \exp(-\alpha_2 t_2), \quad \alpha_2 = k_2(o) \]

The rest of the proof is similar to that in the previous theorem and the result is established.
In the last two theorems, while taking the geometric sum of random variables, the parameter was confined to a fixed value in the interval (0, 1). We presently examine the possibility of relaxing this assumption by permitting p to be the value of a random variable in (0, 1) when the $X^i_k$'s follow distribution (3.1).

**Theorem 3.10.**

Let the sequence $(X^i_k)$ and the random variable $N$ be as in Theorem 3.8. If $p$ is a random variable with distribution function $G(p)$ in (0, 1), then the random variables $X^i_{jk}$ and $p S_j N$ have the same distribution if $(X^i_k)$ follow bivariate exponential distribution (3.1).

**Proof:**

Assuming $X^i_k$ has bivariate exponential distribution, in the notations used in theorem 3.8,

$$B(s, t_2) = \frac{1}{0} p A(ps, t_2) [1-(1-p) A(ps, t_2)]^{-1} dG(p)$$

$$= \frac{1}{0} \frac{p[1-ips(\alpha_1 + \Theta t_2)]^{-1}}{1-(1-p)[1-ips(\alpha_1 + \Theta t_2)]^{-1}} dG(p)$$

$$= \frac{1}{0} \frac{1}{1-ips(\alpha_1 + \Theta t_2)} dG(p)$$

$$= A(s, t_2)$$
3.5 Characterization by form of the conditional distributions

It is well known that a bivariate distribution is not uniquely determined by its marginal distributions. The best illustration of this fact is provided by the bivariate exponential models reviewed in the previous chapter. However, the form of marginal distributions can be taken as the basis of constructing bivariate versions as seen in the works of Morgenstern (1956) and Farlie (1960). However if we turn attention from marginals to the conditional distributions, there is possibility of uniquely determining the bivariate model with specified conditional densities. Abrahams and Thomas (1984) has shown that the conditional densities $f_1(x|y)$ and $f_2(y|x)$ determines uniquely a bivariate density $f(x,y)$ if and only if

$$\frac{f_1(x|y)}{f_2(x|y)} = \frac{g(x)}{h(y)} \quad (3.101)$$

where $g(.)$ and $h(.)$ are non-negative integrable functions with equal marginals. In the case of distribution (3.1) we see that
\[
\begin{align*}
\frac{f_1(x_1|x_2)}{f_2(x_2|x_1)} &= \frac{[(\alpha_1 + \Theta x_2)(\alpha_2 + \Theta x_1) - \Theta] \exp[-(\alpha_1 + \Theta x_2)x_1]}{[(\alpha_1 + \Theta x_2)(\alpha_2 + \Theta x_1) - \Theta] \exp[-(\alpha_2 + \Theta x_1)x_2]} \\
&= \frac{\exp(-\alpha_1 x_1)}{\exp(-\alpha_2 x_2)} 
\end{align*}
\] (3.102)

with the terms on the right side satisfy the required conditions. Thus the two conditional densities confirm to the model (3.1). However, we note that the form of the conditionals are neither exponential nor reducible to any well known standard model to be of any practical interest. On the other hand if we consider the conditional densities \( f(x_i|x_j>x_j) \) presented in equation (3.9) which are exponential, a characterization in terms of them could be more useful. In this section we present a general result on a necessary and sufficient condition that enable the determination of the joint density \( f(x_1,x_2) \) in terms of the conditional densities \( f(x_i|x_j>x_j) \) and then use it to characterize (3.1).

Theorem 3.11

Let \( X = (X_1,X_2) \) be a random vector possessing absolutely continuous distribution with respect to Lebesgue measure in the support of \( Q=\{(x_1,x_2)|x_i>0,i=1,2\} \)
\( t = (t_1, t_2) \) a vector of non-negative reals and

\[
R_i(t_1|t_j) = P[X_i > t_1 | X_j > t_j] \quad (3.103)
\]

\( i, j = 1, 2; \ i \neq j. \) The density function of \( X \) is uniquely determined by the survival functions \( R_1(t_1|t_2) \) and \( R_2(t_2|t_1) \) at those points for which these functions are non-zero if and only if

\[
\frac{R_1(t_1|t_2)}{R_2(t_2|t_1)} = \frac{g(t_1)}{h(t_2)} \quad (3.104)
\]

where \( g(.) \) and \( h(.) \) are non-negative real functions with continuous derivatives in the subspace \( Q_1 = \{ x | x > 0 \} \) satisfying

\[
g(o +) = h(o +)
\]

Proof:

To prove the sufficiency we note that under the conditions of the theorem, there exist functions \( u \) and \( v \) in \( Q_1 \) such that

\[
g(t_1) = \int_{t_1}^{\infty} u(y)dy
\]

and

\[
h(t_2) = \int_{t_2}^{\infty} v(y)dy
\]
Equation (3.104) is equivalent to

\[
\frac{R_1(t_1|t_2)}{R_2(t_2|t_1)} = \frac{\int_{t_1}^{\infty} u(y)dy}{\int_{t_2}^{\infty} v(y)dy} \tag{3.105}
\]

Since \(g(o^+)=h(o^+)\), (3.105) can be written as

\[
\frac{R_1(t_1|t_2)}{R_2(t_2|t_1)} = \frac{\int_{t_1}^{\infty} u(y)dy}{\int_{0}^{\infty} u(y)dy} \times \frac{\int_{t_2}^{\infty} v(y)dy}{\int_{0}^{\infty} v(y)dy} \tag{3.106}
\]

If we write

\[
S(t_1) = \frac{\int_{t_1}^{\infty} u(y)dy}{\int_{0}^{\infty} u(y)dy}
\]

We see that \(S\) is non-increasing, \(S(+\infty)=0\) and \(S(0)=1\). Further

\[
S(t_1+h) - S(t_1) = \frac{\int_{t_1}^{t_1+h} u(y)dy}{\int_{t_1}^{\infty} u(y)dy}
\]

so that \(S(t_1+o) = S(t_1)\), proving the right continuity of \(S\). Thus \(S\) is the survival function \(R_1(t_1)\) of the random variable \(X_1\) and similarly the denominator on the right hand side of (3.106) is the survival function.
\( R_2(t_2) \) of \( X_2 \). Thus the survival function of \( X \) is uniquely obtained as

\[
R(t_1, t_2) = R_1(t_1 | t_2) R_2(t_2)
\]

and hence the corresponding density \( f(x_1, x_2) \).

The necessary part is obtained by writing

\[
R(t_1 | t_2) = \frac{R(t_1, t_2)}{R_2(t_2)}
\]

\[
R(t_2 | t_1) = \frac{R(t_1, t_2)}{R_1(t_1)}
\]

taking their ratios and cancelling out common terms, if any, to \( R_1(t_1) \) and \( R_2(t_2) \) to arrive at \( g(.) \) and \( h(.) \).

**Corollary.**

\( X \) follows the bivariate exponential distribution (3.1) if and only if the conditional distributions of \( X_1 \) given \( X_j > t_j \) are exponential.

**Proof:**

We see from equation (3.8) that
\[
\frac{R(t_1|t_2)}{R(t_2|t_1)} = \frac{\exp \left[-(\alpha_1 + \Theta t_2)t_1\right]}{\exp \left[-(\alpha_2 + \Theta t_1)t_2\right]}
= \frac{\exp(-\alpha_1 t_1)}{\exp(-\alpha_2 t_2)}
= \frac{g(t_1)}{h(t_2)}
\]

where \(g\) and \(h\) satisfy the conditions of the Theorem.

Observations.

1. Theorem 3.11 is quite general in character and therefore applies to any bivariate distribution. It can be used to characterize other bivariate distributions as Pareto [Mardia, 1962], Lomax [Lindley and Singpurwalla, 1986] and Burr [Durling 1975]. Since these results do not come under the scope of the present thesis, they are not discussed here.

2. Unlike other characterizations presented in Sections 3.3 and 3.4 which are extensions of the corresponding univariate property in some sense, the results in this section apply solely to bivariate distributions.