CHAPTER 5

Geodesic E-convex Sets and Geodesic E-convex Functions on Riemannian Manifolds

The important thing is not to stop questioning.

-Albert Einstein

5.1. Introduction

Convex analysis is one of the best mathematical tools that is used in many mathematical disciplines. Convex sets played a dominant role in convex analysis. Several attempts have been made to generalize convextiy by preserving some of the properties of convex functions.

In 1999, Youness [57] introduced a new class of sets and a new class of functions called E-convex sets and E-convex functions by relaxing the definitions of convex sets and convex functions and shown that this class is more general than the class of invex functions. The initial results of Youness [57] inspired a great deal of subsequent work which has greatly expanded the role of E-convexity in optimization theory; see [8,9,15,48,52]. However, Yang [52] shows that some of the results obtained by Youness [57] are incorrect. Later, Syau and Lee [48] introduced the concept of E-quasiconvex functions and discuss some properties of E-convex and E-quasiconvex functions. Recently, Chen [7] introduced a new class of functions namely semi-E-convex functions and discuss some of its basic properties.

In this chapter, motivated by the earlier research works of Fulga and Preda [15], Syau and Lee [48], Yang [52] and Youness [57] and by the impor-
tance of generalized convexity, we introduce a new class of sets in Riemannian manifolds and named as geodesic E-convex sets. Two different classes of functions named as geodesic E-convex and geodesic semi-E-convex functions are introduced on them. In addition, we define epigraph and E-epigraph and discuss some of their properties. An optimization problem is considered in the last section of this chapter.

5.2. Geodesic E-convex Sets and Geodesic E-convex Functions

In this section, we introduce geodesic E-convex sets and geodesic E-convex functions on a Riemannian manifold, which are the extension of convex sets and geodesic convex functions defined by Udriste [49] (See Definition 1.3.1.).

Let \((M; g)\) be a complete n-dimensional Riemannian manifold with Riemannian connection \(\nabla\). Let \(x\) and \(y\) be two points in \(M\) and \(\gamma_{x,y}: [0; 1] \to M\) be a geodesic joining the points \(x\) and \(y\), i.e., \(\gamma_{x,y}(0) = y\), \(\gamma_{x,y}(1) = x\).

Definition 5.2.1. Let \(E: M \to M\) be a map. A subset \(A\) of \(M\) is said to be geodesic E-convex if and only if there exists a unique geodesic \(\gamma_{E(x); E(y)}(t)\) of length \(d(x; y)\), which belongs to \(A\), for each \(x; y \in A\) and \(t \in [0; 1]\).

Youness [57] has proved some properties for \(E\)-convex sets. We extend these results to geodesic \(E\)-convex sets of Riemannian manifolds.

Proposition 5.2.1. Every convex set \(A \subseteq M\) is geodesic \(E\)-convex.

Proof. The proof is obvious by taking a map \(E: M \to M\) as the identity map.

Proposition 5.2.2. Let \(A\) be a subset of \(M\). If \(A\) is geodesic \(E\)-convex, then \(E(A) \subseteq A\).

Proof. Since \(A\) is geodesic \(E\)-convex, for each \(x; y \in A\) and \(t \in [0; 1]\), we have \(\gamma_{E(x); E(y)}(t) \in A\). For \(t = 0\), we have \(\gamma_{E(x); E(y)}(0) = E(y) \subseteq A\). Hence, \(E(A) \subseteq A\).
Proposition 5.2.3. Let $E(A)$ be convex and $E(A) \subseteq A$. Then, $A$ is geodesic $E$-convex.

Proof. Let $x, y \in A$, then $E(x); E(y) \subseteq E(A)$. Since $E(A)$ is convex, for all $t \in [0; 1]$, we have $E(x); E(y)(t) \subseteq E(A) \subseteq A$. Hence, $A$ is geodesic $E$-convex.

Now, we give the following examples of geodesic $E$-convex sets which are not convex.

Example 5.2.1. Let $M$ be a Cartan-Hadamard manifold and $x_0; y_0 \in M$, $x_0 \neq y_0$. Let $B(x_0; r_1) \setminus B(y_0; r_2) = ;$ for some $0 < r_1 < r_2 < \frac{1}{2}d(x_0, y_0)$, where $B(x; r) = \{ y \in M : d(x, y) < r \}$ is an open ball with center $x$ and radius $r$. We define

$$ A := B(x_0; r_1) \setminus B(y_0; r_2), $$

Then, $A$ is not a convex set because every geodesic curve passing through $x_0$ and $y_0$ does not completely lie in $A$.

Let $E : M \rightarrow M$ be defined as $E(x) = f(y) \in L_{xy} : d(x_0; y) = \frac{1}{2} r$; $8x \in A$, where $L_{xy}$ denotes the geodesic joining $x$ and $y$ whose existence is ensured in Cartan-Hadamard manifolds, then $A$ is geodesic $E$-convex.

Example 5.2.2. Consider the two dimensional sphere $S^2 = \{ f(x; y; z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$; $U = f(x; y; z) \in S^2 : z > 0$ and $V = f(x; y; z) \in S^2 : z < 0$ are two open sets of $S^2$.

Define $A = U \setminus V$ and let $E : S^2 \rightarrow S^2$ be defined as $E(x; y; z) = (x; y; j x; j y; j z)$. Then, it is obvious that $A$ is geodesic $E$-convex but not convex.

Theorem 5.2.1. If $f_{A_1g_{21}}$ is an arbitrary collection of geodesic $E$-convex subsets of $M$ with respect to the mapping $E : M \rightarrow M$, then their intersection $\bigcap_{i \in I^2} A_i$ is a geodesic $E$-convex subset of $M$.

Proof. Let $f_{A_1g_{21}}$ be a collection of geodesic $E$-convex subsets of $M$ with respect to the mapping $E : M \rightarrow M$. If $\bigcap_{i \in I^2} A_i$ is an empty set then the result
is obvious. Let \( x; y \in E^i A_i \), then \( x; y \in E^i A_i \) for each \( i \in I \). By the geodesic E-convexity of \( A_i \), we have \( \circ_{E(x); E(y)}(t) \in E^i A_i \), for each \( i \in I \) and \( 0 \leq t \leq 1 \). Which implies that \( \circ_{E(x); E(y)}(t) \in E^i A_i \), for each \( i \in I \) and \( 0 \leq t \leq 1 \).

The above theorem is not true in general for the union of geodesic E-convex subsets of \( M \). However the composition of two geodesic E-convex sets are again a geodesic E-convex set as proved in the following lemma:

Lemma 5.2.1. Let \( A \subseteq M \) be \( E_1 \)-convex and \( E_2 \)-convex set. Then, \( A \) is \((E_1(E_2)) \) and \((E_2(E_1)) \)-convex set.

Proof. Let \( A \subseteq M \) be \( E_1 \)-convex and \( E_2 \)-convex set and \( x; y \in E A \). Let \( \circ_{E_1(E_2)(x); E_1(E_2)(y)}(t) \not\in A \) for some \( t \in [0; 1] \), that is, \( \circ_{E_1(E_2)(x); E_1(E_2)(y)}(t) \not\in E A \), for some \( t \in (0; 1] \).

Since, from proposition 5.2.2, \( E_1(x); E_2(y) \in E A \), then \( \circ_{E_1(E_2)(x); E_1(E_2)(y)}(t) \not\in E A \) contradicts the \( E_1 \)-convexity of \( A \). Hence, \( A \) is \((E_1(E_2)) \)-convex set. Similarly, \( A \) is \((E_2(E_1)) \)-convex set.

Now, we generalize the De\(^{n}\)nition 1.3.3 of geodesic convex function to geodesic E-convex function on a geodesic E-convex subset of a Riemannian manifold.

De\(^{n}\)nition 5.2.2. Let \( M \) be a complete \( n \)-dimensional Riemannian manifold and \( A \subseteq M \) be a geodesic E-convex set in \( M \). A real valued function \( f : A \to \mathbb{R} \) is said to be geodesic E-convex if

\[
  f \left( \circ_{E(x); E(y)}(t) \right) \leq (1 - t)f(E(x)) + tf(E(y));
\]

for all \( x; y \in E A \) and \( 0 \leq t \leq 1 \).

If the above inequality is strict for all \( x; y \in E A \), \( E(x) \notin E(y) \) and \( 0 \leq t \leq 1 \), then \( f \) is called strictly geodesic E-convex.

Remark 5.2.1. Every geodesic convex function \( f \) on a convex set \( A \) is geodesic E-convex function, where \( E \) is the identity map.

We now have some properties (Theorem 5.2.2 and Theorem 5.2.3) which hold for geodesic convex functions also hold good for geodesic E-convex func-
Theorem 5.2.2. Let \( A \subseteq M \) be a geodesic \( E \)-convex set and \( f : A \to \mathbb{R} \) be a geodesic \( E \)-convex function. If \( h : [0, 1] \to \mathbb{R} \) be an increasing convex function such that range \( f \) \( \subseteq [0, 1] \). Then, the composite function \( h(f) \) is geodesic \( E \)-convex on \( A \).

Proof. Since \( f \) is geodesic \( E \)-convex function, we have

\[
f(0_{E(x); E(y)}(t)) \leq (1 \cdot t)f(0_{E(y)}) + tf(0_{E(x)});
\]

for all \( x, y \in A \) and \( t \in [0; 1] \). Since \( h \) is an increasing convex function, we get

\[
h[f(0_{E(x); E(y)}(t))] \leq h[(1 \cdot t)f(0_{E(y)}) + tf(0_{E(x)})]
\]

\[\leq (1 \cdot t)h[f(0_{E(y)})] + th[f(0_{E(x)})]
\]

\[= (1 \cdot t)(h(f))(0_{E(y)}) + t(h(f))(0_{E(x)});
\]

which shows that \( h(f) \) is geodesic \( E \)-convex on \( A \). Similarly, we can show that \( h(f) \) is strictly geodesic \( E \)-convex function by considering \( h \) to be a strictly increasing convex function.

Theorem 5.2.3. Let \( A \subseteq M \) be a geodesic \( E \)-convex set and \( f_i : A \to \mathbb{R} ; \ i = 1; 2; \ldots ; p \) be geodesic \( E \)-convex functions. Then,

\[
f = \sum_{i=1}^{p} f_i ; \ \text{for all } x, y \in A ; \ x, y \in A ; \ i = 1; 2; \ldots ; p;
\]

is geodesic \( E \)-convex on \( A \).

Proof. The proof is obvious from the definition.

The concept of quasiconvex functions on Riemannian manifold was introduced by Udriste, given in Definition 1.3.6. We generalize the concept and define geodesic \( E \)-quasiconvex functions on Riemannian manifold and study some of their properties.

Definition 5.2.3. Let \( A \subseteq M \) be a nonempty geodesic \( E \)-convex set. A function \( f : A \to \mathbb{R} \) is said to be:
(i) geodesic $E$-quasiconvex if for all $x; y \in A$, $t \in [0; 1]$, 
\[ f(\circ_{E(x)}E(y)(t)) \leq \max f(E(x)) f(E(y)) \]
(ii) strictly geodesic $E$-quasiconvex if for all $x; y \in A$ with $E(x) \notin E(y)$ and $8 t 2 (0; 1)$,
\[ f(\circ_{E(x)}E(y)(t)) < \min f(E(x)) f(E(y)) \]

Theorem 5.2.4. Let $A \subseteq M$ be a geodesic $E$-convex set and $f_{i}g_{2i}$ be a family of real valued functions defined on $A$ such that $\sup f_{i}(x)$ exists in $R$, for all $x \in A$. Let $f : A \rightarrow R$ be a real function defined by $f(x) = \sup_{i_{2i}} f_{i}(x)$; $8 x \in A$:

(i) If $f_{i} : A \rightarrow R$; for any $i \in I$, are geodesic $E$-convex functions on $A$, then the function $f$ is geodesic $E$-convex on $A$.

(ii) If $f_{i} : A \rightarrow R$; for any $i \in I$, are geodesic $E$-quasiconvex functions on $A$, then the function $f$ is geodesic $E$-quasiconvex on $A$.

Proof. (i) If $f_{i} : A \rightarrow R$; for any $i \in I$, are geodesic $E$-convex functions on $A$, then for every $x; y \in A$ and $8 t 2 [0; 1]$, 
\[ f_{i}(\circ_{E(x)}E(y)(t)) \leq (1 i t)f_{i}(E(y)) + tf_{i}(E(x)) : \]
\[ \sup_{i_{2i}} f_{i}(\circ_{E(x)}E(y)(t)) \leq \sup_{i_{2i}} [(1 i t)f_{i}(E(y)) + tf_{i}(E(x))] \]
\[ = (1 i t)\sup_{i_{2i}} f_{i}(E(y)) + t \sup_{i_{2i}} f_{i}(E(x)) \]
\[ = (1 i t)f(E(y)) + tf(E(x)) ; \]
or
\[ f(\circ_{E(x)}E(y)(t)) \leq (1 i t)f(E(y)) + tf(E(x)) ; \]
Hence, $f$ is geodesic $E$-convex on $A$.

(ii) Since $f_{i} : A \rightarrow R$, for any $i \in I$, are geodesic $E$-quasiconvex functions on $A$, we have
\[ f(\circ_{E(x)}E(y)(t)) = \sup_{i_{2i}} f_{i}(\circ_{E(x)}E(y)(t)) ; \text{for all } x; y \in A ; 8 t 2 [0; 1] \]
\[ \leq \sup_{i_{2i}} \max f_{i}(E(x)) f_{i}(E(y)) \]
\[ = \max f_{i}(E(x)) \sup f_{i}(E(y)) \sup f_{i}(E(y)) \]
\[ = \max f_{i}(E(x)) f(E(y)) ; \]
Hence, \( f \) is geodesic \( E \)-quasiconvex on \( A \).

Let \( A \subseteq M \) be a nonempty geodesic \( E \)-convex set. It follows from Proposition 5.2.2, \( E(A) \subseteq A \). Hence, for any \( f : A \to \mathbb{R} \), the restriction \( f^\ast : E(A) \to \mathbb{R} \) of \( f : A \to \mathbb{R} \) to \( E(A) \) defined by

\[
f^\ast(x) = f(x) \quad \text{for all } x \in E(A)
\]
is well defined. And we have

**Theorem 5.2.5.** Let \( A \subseteq M \) be a geodesic \( E \)-convex set and \( f : A \to \mathbb{R} \) be a geodesic \( E \)-quasiconvex function on \( A \). Then the restriction, \( f^\ast : C \to \mathbb{R} \) of \( f \) to any nonempty convex subset \( C \) of \( E(A) \) is a quasiconvex on \( C \).

**Proof.** Let \( x, y \in C \subseteq E(A) \), then there exist \( x^0, y^0 \in A \) such that \( x = E(x^0) \) and \( y = E(y^0) \). Since, \( C \) is convex subset, we have

\[
\circ_{x,y}(t) = \circ_{E(x^0);E(y^0)}(t) 2 C; 8 t 2 [0;1].
\]

Therefore, we have

\[
f^\ast(\circ_{x,y}(t)) = f(\circ_{E(x^0);E(y^0)}(t)) \leq \max f(\circ_{E(x^0);E(y^0)}(t)) = \max f(x); f(y)g = \max f(x); f(y)g.
\]

**Theorem 5.2.6.** Let \( A \subseteq M \) be a geodesic \( E \)-convex set and \( f : A \to \mathbb{R} \) be a real function. If \( E(A) \) is a convex set, then \( f \) is a geodesic \( E \)-quasiconvex on \( A \), if and only if its restriction \( f^\ast : E(A) \to \mathbb{R} \) to \( E(A) \) is quasiconvex on \( E(A) \).

**Proof.** The if condition is true due to the Theorem 5.2.5. Conversely, if \( f^\ast : E(A) \to \mathbb{R} \) is quasiconvex, then for \( x, y \in A \), \( E(x); E(y) \in E(A) \) and \( \circ_{E(x);E(y)}(t) 2 E(A) \subseteq A; 8 t 2 [0;1] \). Since \( E(A) \subseteq A \), we have

\[
f(\circ_{E(x);E(y)}(t)) = f(\circ_{E(x);E(y)}(t)) \leq \max f(\circ_{E(x);E(y)}(t)) = \max f(x); f(y)g.
\]

We now have the analogous results to Theorem 5.2.5 and Theorem 5.2.6 for the geodesic \( E \)-convex case.
Theorem 5.2.7. Let $A \subseteq M$ be a geodesic $\mathcal{E}$-convex set and $f: A \to \mathbb{R}$ be a geodesic $\mathcal{E}$-convex function on $A$. Then, the restriction $f^*: C \to \mathbb{R}$ of $f$ to any nonempty convex subset $C$ of $\mathcal{E}(A)$ is a geodesic convex function.

Theorem 5.2.8. Let $A \subseteq M$ be a geodesic $\mathcal{E}$-convex set and $f: A \to \mathbb{R}$ be a real function. If $\mathcal{E}(A)$ is a convex set, then $f$ is a geodesic $\mathcal{E}$-convex on $A$ if and only if its restriction $f^*: \mathcal{E}(A) \to \mathbb{R}$; to $\mathcal{E}(A)$ is geodesic convex function.

For any real number $r$, the lower level set, $L_r(f \pm \mathcal{E})$, of $f \pm \mathcal{E}: A \to \mathbb{R}$ is defined as

$$L_r(f \pm \mathcal{E}) = \{ x \in A : (f \pm \mathcal{E})(x) = f(\mathcal{E}(x)) \leq rg \}.$$ 

The lower level set, $L_r(f^*)$, of $f^*: \mathcal{E}(A) \to \mathbb{R}$ is defined as

$$L_r(f^*) = \{ x \in \mathcal{E}(A) : f(x) \leq rg \}.$$ 

In the following theorem we give an important characterization of geodesic $\mathcal{E}$-quasiconvex function in terms of the lower level sets of its restriction to $\mathcal{E}(A)$.

Theorem 5.2.9. Let $\mathcal{E}(A)$ be a convex set and $f: A \to \mathbb{R}$ be a real valued function. Then, $f$ is geodesic $\mathcal{E}$-quasiconvex if and only if the lower level set $L_r(f^*)$ of its restriction $f^*: \mathcal{E}(A) \to \mathbb{R}$ is convex for each $r \geq 2$.

Proof. Let $x, y \in \mathcal{E}(A)$, then $\mathcal{E}(x); \mathcal{E}(y) \geq 2$ and $\mathcal{E}(x); \mathcal{E}(y)(t) \geq \mathcal{E}(A)$, since $\mathcal{E}(A)$ is a convex set.

Let $x = \mathcal{E}(x)$, $y = \mathcal{E}(y)$ and $x, y \in L_r(f^*)$, then $f(x) \leq r$ and $f(y) \leq r$.

$$f^*(x, y(t)) = f^*(\mathcal{E}(x); \mathcal{E}(y)(t)) \leq \max f(\mathcal{E}(x)); f(\mathcal{E}(y))g$$

$$= \max f(x); f(y)g$$

$$\leq r;$$

which implies that $\mathcal{E}(x, y(t)) \geq L_r(f^*)$. Hence, $L_r(f^*)$ is convex.

Conversely, let $L_r(f^*)$ is convex for each $r \geq 2$, i.e., for all $x, y \in L_r(f^*)$, we
have $\circ_{x,y}(t) \leq L_r(f^t)$ and $r = \max f(x); f(y)g$.

$$f(\circ_{E(x),E(y)}(t)) = f(\circ_{E(x);E(y)}(t)) = f(\circ_{x,y}(t)) \leq r$$

$$= \max f(x); f(y)g$$

$$= \max f(E(x)); f(E(y))g$$

Hence, $f$ is geodesic $E$-quasiconvex.

Next, we have

**Theorem 5.2.10.** Let $A \subseteq M$ be a nonempty geodesic $E$-convex set and $f : A \rightarrow R$ be a geodesic $E$-quasiconvex on $A$. Suppose that $A : R \rightarrow R$ is a non-decreasing function. Then, $A(f)$ is geodesic $E$-quasiconvex on $A$.

**Proof.** Since $f : A \rightarrow R$ is a geodesic $E$-quasiconvex on $A$ and $A : R \rightarrow R$ is a non-decreasing function, for all $x, y \in S$ and $t \in [0; 1]$, we have

$$(A(f))(\circ_{E(x),E(y)}(t)) = A(f(\circ_{E(x),E(y)}(t)))$$

$$\leq A(\max f(E(x)); f(E(y))g)$$

$$= \max A(f(E(x))); A(f(E(y))g)$$

$$= \max f(\circ f(E(x))); (A(f))(E(y))g$$

which shows that the composite function $A(f)$ is geodesic $E$-quasiconvex on $A$.

In the following result we establish a relationship between geodesic $E$-convex and geodesic $E$-quasiconvex functions.

**Theorem 5.2.11.** If the function $f$ is geodesic $E$-convex on $A$, then $f$ is geodesic $E$-quasiconvex on $A$.

**Proof.** Let $f$ be geodesic $E$-convex on $A$. Then, for every $x, y \in A$ and $t \in [0; 1]$, we have

$$f(\circ_{E(x),E(y)}(t)) \leq (1 + t)f(E(y)) + tf(E(x))$$

$$\leq (1 + t)\max f(E(x)); f(E(y))g + t\max f(E(x)); f(E(y))g$$

$$= \max f(E(x)); f(E(y))g.$$
Hence, the result.

5.3. Geodesic Semi-E-convex Functions

In this section, we introduce a new class of functions called geodesic semi-E-convex functions and discuss some of their properties.

Definition 5.3.1. Let $M$ be a complete $n$-dimensional Riemannian manifold and $A \subseteq M$ be a geodesic $E$-convex set in $M$. A function $f : A \to \mathbb{R}$ is said to be geodesic semi-E-convex if

$$f(t^* E(x); E(y))(t) \leq (1 - t)f(y) + tf(x);$$

for all $x, y \in A$ and $0 \leq t \leq 1$.

If the above inequality is strict for all $x, y \in A$, $x \notin y$ and $0 \leq t < 1$, then $f$ is called strictly geodesic semi-E-convex.

Remark 5.3.1. In particular, if we take $E$ to be the identity map, then every geodesic convex function on a geodesic convex set is geodesic semi-E-convex.

Proposition 5.3.1. If the function $f : A \to \mathbb{R}$ is geodesic semi-E-convex on a geodesic $E$-convex set $A$, then $f(E(x)) \leq f(x)$ for all $x \in A$.

Proof. Since $f$ is geodesic semi-E-convex on a geodesic $E$-convex set $A$, we have $f(t^* E(x); E(y))(t) \leq f(y) + tf(x)$ for each $x, y \in A$ and $0 \leq t \leq 1$ and

$$f(t^* E(x); E(y))(t) \leq (1 - t)f(y) + tf(x);$$

If we take $t = 1$, then $f(E(x)) \leq f(x)$.

A geodesic $E$-convex function on geodesic $E$-convex set is not necessarily a geodesic semi-E-convex function. To enforce this statement we consider the following example.

Example 5.3.1. We consider the function $f : \mathbb{R}^n \to \mathbb{R}$ such that
\[ f(x) = i \cdot j \times j, \text{ and let } E : \mathbb{R} ! \mathbb{R} \text{ be defined as} \]
\[ E(x) = \alpha x; \quad 0 < \alpha \leq 1; \quad 8x \in \mathbb{R} \]

Now we consider the geodesic \( ^{o}E(x;E(y)(t)) \) such that

\[
\begin{align*}
^{o}E(x;E(y)(t)) & = \begin{cases} 
8 & E(y) + t(E(x) \cdot E(y)); \quad xy \neq 0; \\
\frac{1}{8} & E(y) + t(E(y) \cdot E(x)); \quad xy < 0; \\
\end{cases} \\
& = y + t(y \cdot x); \quad xy < 0;
\end{align*}
\]

If \( x; y \neq 0; \) then
\[ f^{o}E(x;E(y)(t)) = f(y + t(x \cdot y)) \]
\[ = i \cdot j \cdot y + t(x \cdot y) \cdot j \]
\[ = i \cdot j \cdot (1 + t)y + tx \cdot j = i \cdot [(1 + t)y + tx]: \]

On the other hand
\[ (1 + t)f(E(y)) + tf(E(x)) = (1 + t)f(\alpha y) + tf(\alpha x) \]
\[ = i \cdot (1 + t) \cdot y \cdot j + t \cdot \alpha j \]
\[ = i \cdot \alpha f(1 + t)y + tx g \cdot 8 \cdot t \in [0; 1]: \]

Hence
\[ f^{o}E(x;E(y)(t)) \leq (1 + t)f(E(y)) + tf(E(x)); \quad 8 \cdot t \in [0; 1]; 0 < \alpha \leq 1: \]

Similarly, we can show this inequality holds good when \( x; y < 0: \)

Now we take \( x < 0 \) and \( y > 0, \) then
\[ f^{o}E(x;E(y)(t)) = f(y + t(y \cdot x)) \]
\[ = i \cdot j \cdot y + t(y \cdot x) \cdot j \]
\[ = i \cdot j \cdot (1 + t)y \cdot i \cdot tx \cdot j = i \cdot [(1 + t)y \cdot i \cdot tx]: \]

On the other hand
\[ (1 + t)f(E(y)) + tf(E(x)) = (1 + t)f(\alpha y) + tf(\alpha x) \]
\[ = i \cdot \alpha f(1 + t)y \cdot j \cdot j + t \cdot j \cdot x \cdot j \]
\[ = i \cdot \alpha f(1 + t)y \cdot j \cdot tx \cdot j: \]

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It follows that
\[
f(\circ_{E(x);E(y)}(t)) \leq (1 + t)f(E(y)) + tf(E(x))
\]
if and only if
\[
i [(1 + t)y; tx] \leq i \circ(1 + t)y; tx]
\]
if and only if
\[
y(i 1 + t + \circ(1 + t)) + xt(1 + \circ) \leq 0;
\]
which is always true for all \( t \in [0; 1] \) and \( 0 < \circ \leq 1 \). Similarly we can show this inequality also holds good for \( x > 0, y < 0 \). Thus \( f \) is geodesic \( E \)-convex function on \( R \).

Since, \( f(E(1)) = f(\circ) = i \circ > f(1) = i 1 \); for \( \circ = \frac{1}{2} \), then from Proposition 5.3.1., it follows that \( f \) is not geodesic semi-\( E \)-convex on \( R \).

Remark 5.3.2. It can be seen easily that Theorem 5.2.2, Theorem 5.2.3 and Theorem 5.2.4 (i) also hold good for geodesic semi-\( E \)-convex functions.

Proposition 5.3.2. Let \( A \subseteq M \) be a geodesic \( E \)-convex set and \( f : A \to R \) be a geodesic semi-\( E \)-convex function, then for any real number \( \circ \), lower level set \( L_\circ = \{ x \in A : f(x) \leq \circ \} \) is geodesic \( E \)-convex.

Proof. Let \( x; y \in L_\circ \). Then \( f(x) \leq \circ, f(y) \leq \circ \). Since \( f \) is geodesic semi-\( E \)-convex on \( A \subseteq M \), we have
\[
f(\circ_{E(x);E(y)}(t)) \leq (1 + t)f(y) + tf(x) \leq \circ;
\]
which implies that \( \circ_{E(x);E(y)}(t) \in L_\circ \). Hence, \( L_\circ \) is geodesic \( E \)-convex.

Proposition 5.3.3. Let the function \( f : A \to R \) be geodesic \( E \)-convex on a geodesic \( E \)-convex set \( A \subseteq M \). Then, \( f \) is geodesic semi-\( E \)-convex on \( A \) if and only if \( f(E(x)) \leq f(x) \) for each \( x \in A \).

Proof. Let \( f : A \to R \) be a geodesic \( E \)-convex function on geodesic \( E \)-convex set \( A \) and \( f(E(x)) \leq f(x) \) for each \( x \in A \), then for any \( x; y \in A \) and \( t \in [0; 1] \), we have
\[
f(\circ_{E(x);E(y)}(t)) \leq (1 + t)f(E(y)) + tf(E(x))
\]
\[
\leq (1 + t)f(y) + tf(x);
\]
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Hence, \( f \) is geodesic semi-E-convex on \( A \). Converse is obvious from Proposition 5.3.1.

Remark 5.3.3. From Proposition 5.3.3., it follows that geodesic E-convex function \( f \) on a geodesic E-convex set \( A \subseteq M \) with the property \( f(E(x)) \leq f(x) \) for each \( x \in A \) is geodesic semi-E-convex but the converse need not be true. For, in the Example 5.3.1, if we take

\[
E(x) = @x; \; @ > 1; \; 8x \in R.
\]

Then the function \( f(x) \) is geodesic semi-E-convex on the geodesic E-convex set \( R \).

However, if we take \( x = 1, y = 1 \) and \( t = \frac{1}{2} \), we have

\[
f(\circ E(x); E(y)(t)) = f(1 + \frac{1}{2}(1 \cdot 1)) = f(1) = 1
\]

\[
> \frac{1}{2}f(E(1)) + \frac{1}{2}f(E(1))
\]

\[
= \frac{1}{2}f(\oplus) + \frac{1}{2}f(\oplus)
\]

\[
= f(\oplus) = i \oplus; \; \text{for } \oplus = 2:
\]

Which shows that the function \( f(x) \) is not geodesic E-convex on the geodesic E-convex set \( R \).

The concept of E-quasiconvex functions on \( R^n \) was introduced by Syau and Lee [48]. We have generalized this concept and defined geodesic E-quasiconvex functions on Riemannian manifolds. Now, we define geodesic semi-E-quasiconvex functions on Riemannian manifolds and discuss some of their properties.

Definition 5.3.2. Let \( A \subseteq M \) be a geodesic E-convex set. A function \( f : A \to R \) is said to be geodesic semi-E-quasiconvex on \( A \), if

\[
f(\circ E(x); E(y)(t)) \leq \max f(x); f(y)g;
\]

for all \( x; y \in A, t \in [0; 1] \).
In view of above definition we see that Theorem 5.2.4 (ii) also holds
good for geodesic semi-E-quasiconvex functions. Next, we give a characteri-
zation of geodesic semi-E-quasiconvex functions in terms of its lower level set.

Proposition 5.3.4. Let A ⊆ M be a geodesic E-convex set. Then the
function f : A → R is geodesic semi-E-quasiconvex if and only if the lower
level set L_@ = f x 2 A : f (x) ≤ @g is geodesic E-convex for each @ 2 R.

Proof. Let f be geodesic semi-E-quasiconvex on A. Then for any x; y 2 L_@
we have f (x) ≤ @; f (y) ≤ @ and

f (°_E(x):E(y)(t)) ≤ max f (x); f (y) ≤ @;

which implies that °_E(x):E(y)(t) 2 L_@ and hence the set L_@ is geodesic E-
convex.

Conversely, let L_@ is geodesic E-convex for each @ 2 R, and let @ =
max f (x); f (y)g; 8x; y 2 A. Then, x 2 L_@; y 2 L_@ implies that °_E(x):E(y)(t) 2 L_@, it follows that

f (°_E(x):E(y)(t)) ≤ @ = max f (x); f (y)g;

Which shows that f is geodesic semi-E-quasiconvex.

Proposition 5.3.5. Let g_i : M → R, i = 1; 2;...; n be geodesic semi-E-
quasiconvex on M. Then the set A = f x 2 M | g_i(x) ≤ 0; i = 1; 2;...; n is geodesic E-convex.

Proof. From Proposition 5.3.4., it follows that the set A_i = f x 2 M | g_i(x) ≤ 0, i = 1; 2;...; n is geodesic E-convex and hence A = \overset{n}{\underset{i=1}{\bigcap}} A_i geodesic E-convex.

Every geodesic semi-E-convex function is geodesic semi-E-quasiconvex
on A.

Proposition 5.3.6. Let f : A → R be geodesic semi-E-convex on A, then
f is also geodesic semi-E-quasiconvex on A.
Proof. Let \( f : A \to R \) be geodesic semi-\( E \)-convex on \( A \). Then, for every \( x, y \in A \) and \( t \in [0; 1] \), we have

\[
f(\circ_{E(x);E(y)}(t)) \leq (1 + t)f(y) + tf(x)
\]

\[
\leq (1 + t)\max f(x); f(y)g + t\max f(x); f(y)g
\]

\[
= \max f(x); f(y)g;
\]

Hence the result.

Definition 5.3.3. Let \( A \subseteq M \) be a nonempty geodesic \( E \)-convex set. A function \( f : A \to R \) is said to be geodesic semi-\( E \)-pseudoconvex on \( A \), if there exists a strictly positive function \( b : A \to A \to R \) such that,

\[
f(x) < f(y) \iff f(\circ_{E(x);E(y)}(t)) \leq f(y) + t(t - 1)b(x; y);
\]

for all \( x, y \in A \) and \( t \in (0; 1) \).

Proposition 5.3.7. If \( f : A \to R \) is geodesic semi-\( E \)-convex function on \( A \), then \( f \) is geodesic semi-\( E \)-pseudoconvex on \( A \).

Proof. Since \( f(x) < f(y) \) and \( f \) is geodesic semi-\( E \)-convex function on \( A \), then for all \( x, y \in A \) and \( t \in (0; 1) \), we have

\[
f(\circ_{E(x);E(y)}(t)) \leq (1 + t)f(y) + tf(x)
\]

\[
= f(y) + t(f(x) - f(y))
\]

\[
< f(y) + t(f(x) - f(y)) = f(y) + t(t - 1)(f(y) - f(x))
\]

\[
= f(y) + t(t - 1)b(x; y);
\]

where \( b(x; y) = f(y) - f(x) > 0 \), and hence the result.

Proposition 5.3.8. If \( f : A \to R \) is geodesic semi-\( E \)-pseudoconvex function on \( A \), then \( f \) is geodesic semi-\( E \)-quasiconvex on \( A \).

Proof. Let \( f(x) < f(y) \). Since \( f \) is geodesic semi-\( E \)-pseudoconvex on \( A \) for all \( x, y \in A \) and \( t \in (0; 1) \), we have

\[
f(\circ_{E(x);E(y)}(t)) \leq f(y) + t(t - 1)b(x; y)
\]

\[
< f(y)
\]

\[
= \max f(x); f(y)g;
\]

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Hence, $f$ is geodesic semi-$E$-quasiconvex on $A$.

### 5.4. $E$-epigraph and Optimality

This section presents the definitions, relations and some properties of epigraphs, $E$-epigraphs, geodesic $E$-convex and geodesic semi-$E$-convex functions. A characterization of geodesic semi-$E$-convex function in terms of its epigraph is discussed and an optimization problem is considered in the end of this section.

**Definition 5.4.1.** If $A \subseteq M \subseteq \mathbb{R}$ and $E : M \rightarrow M$, then the set $A$ is said to be geodesic $E$-convex if $(x; \circledcirc; (y; \circledast)) \subseteq A$ imply

$$\left( E(x) \circledast E(y)(t); (1 + t) \circledast + t \circledcirc \right) \subseteq A; \quad 0 \leq t \leq 1$$

Now, we define an epigraph of $f$ as follows:

$$\text{epi}(f) = \{ (x; \circledcirc); \text{x} \in A, \circledast \subseteq \mathbb{R} ; f(x) \leq \circledast \}$$

While $E$-epigraph of $f$ is given by

$$\text{epi}_E(f) = \{ (E(x); \circledcirc); \text{x} \in A, \circledast \subseteq \mathbb{R} ; f(E(x)) \leq \circledast \}$$

**Theorem 5.4.1.** Let $A_i$, where $i \in \mathbb{I}$, be a family of geodesic $E$-convex sets. Then their intersection $\bigcap_{i \in \mathbb{I}} A_i$ is a geodesic $E$-convex set.

**Proof.** Let $(x; \circledcirc); (y; \circledast) \subseteq A_i$. Then for each $i \in \mathbb{I}$, $(x; \circledcirc); (y; \circledast) \subseteq A_i$. By the geodesic $E$-convexity of $A_i$, for each $i \in \mathbb{I}$, it follows that

$$\left( E(x) \circledast E(y)(t); (1 + t) \circledast + t \circledcirc \right) \subseteq A_i; \quad 0 \leq t \leq 1$$

Thus,

$$\left( E(x) \circledast E(y)(t); (1 + t) \circledast + t \circledcirc \right) \subseteq \bigcap_{i \in \mathbb{I}} A_i; \quad 0 \leq t \leq 1$$

Hence, $\bigcap_{i \in \mathbb{I}} A_i$ is a geodesic $E$-convex set.
The following theorem gives a sufficient condition for $f$ to be a geodesic semi-$E$-convex function.

**Theorem 5.4.2.** Let $A \subseteq M$ be a geodesic $E$-convex set and $f : A \to \mathbb{R}$ be a real valued function. If $\text{epi}(f)$ is a geodesic $E$-convex set, then $f$ is a geodesic semi-$E$-convex function on $A$.

**Proof.** Let $x, y \in A$ and $(x; f(x)); (y; f(y)) \in \text{epi}(f)$. Since $\text{epi}(f)$ is geodesic $E$-convex, we have

$$(^oE_{(x)}E_{(y)}(t); (1 \cdot t)f(y) + tf(x)) \in \text{epi}(f);$$

so,

$$f(\text{epi}(E_{(x)}E_{(y)}(t))) \leq (1 \cdot t)f(y) + tf(x)$$

and hence $f$ is geodesic $E$-convex.

Now, in particular if we consider $E : M \to M$ to be the idempotent map, we have the following results:

**Theorem 5.4.3.** Assume that $E : M \to M$ is an idempotent map. Let $A \subseteq M$ be a geodesic $E$-convex set and $f : A \to \mathbb{R}$ be a real valued function. If $\text{epi}_E(f)$ is a geodesic $E$-convex set, then $f$ is a geodesic $E$-convex function on $A$.

**Proof.** Let $x, y \in A$ and $(E(x); f(E(x))); (E(y); f(E(y))) \in \text{epi}_E(f)$. Since $\text{epi}_E(f)$ is geodesic $E$-convex, we have

$$(^oE_{(E(x))}E_{(E(y))}(t); (1 \cdot t)f(E(y)) + tf(E(x)) \in \text{epi}_E(f);$$

which implies,

$$f(\text{epi}(E_{(E(x))}E_{(E(y))}(t))) \leq (1 \cdot t)f(E(y)) + tf(E(x))$$

or

$$f(\text{epi}_E(E_{(E(x))}E_{(E(y))}(t))) \leq (1 \cdot t)f(E(y)) + tf(E(x));$$

Since $E : M \to M$ is an idempotent map, the above inequality becomes

$$f(\text{epi}(E_{(E(x))}E_{(E(y))}(t))) \leq (1 \cdot t)f(E(y)) + tf(E(x));$$

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Hence, $f$ is geodesic $E$-convex.

Theorem 5.4.4. Let $E : M \to M$ be an idempotent map and let $(f_i)_{i \in I}$ be a family of real-valued functions defined on a geodesic $E$-convex set $A \subseteq M$ which are bounded from above. If the $E$-epigraphs $\text{epi}_E(f_i)$ are geodesic $E$-convex sets, then the function $f$ defined by $f(x) = \sup_{i \in I} f_i(x)$, $8x 2 A$ is geodesic $E$-convex on $A$.

Proof. Since $E$-epigraphs,

$$\text{epi}_E(f_i) = f(E(x); @) : x 2 A; @ 2 R; f_i(E(x)) \leq @; i \in I$$

are geodesic $E$-convex sets in $A \subseteq R$. Therefore, their intersection

$$\text{epi}_E(f_i) = f(E(x); @) : x 2 A; @ 2 R; f_i(E(x)) \leq @; i \in I$$

$$= f(E(x); @) : x 2 A; @ 2 R; f(E(x)) \leq @;$$

where $f(E(x)) = \sup_{i \in I} f_i(E(x))$, is also a geodesic $E$-convex set. We can see that this intersection is an $E$-epigraph; hence by Theorem 5.4.3, $f$ is a geodesic $E$-convex function on $A$.

Now for $E : M \to M$, let the mapping $E \in : M \subseteq M \subseteq R$ to be

$$(E \in A)(x; t) = (E(x); A); \quad \text{for } (x; A) 2 M \subseteq R$$

It is obvious that $A \subseteq M$ is geodesic $E$-convex if $A \subseteq R$ is geodesic $E \subseteq I$-convex. In the following Proposition we give a characterization of geodesic $E$-convex function in terms of its $\text{epi}(f)$.

Proposition 5.4.1. Let $A \subseteq M$ be a geodesic $E$-convex set, then $f : A \to R$ is geodesic semi-$E$-convex on $A$ if and only if its $\text{epi}(f)$ is geodesic $E \subseteq I$-convex on $A \subseteq R$.

Proof. Let $f$ be geodesic semi-$E$-convex on $A$ and $(x; @), (y; -) 2 \text{epi}(f)$, then we have $\circ_{E(x); E(y)}(t) 2 A$ and $8t 2 [0; 1]$ and

$$f(\circ_{E(x); E(y)}(t)) \leq (1_t f(y) + tf(x) \leq (1_t) - t^@$$
thus,

\((^\circ E_{(x); E(y)}(t);(1 \cdot t) + t \otimes) 2 \text{epi}(f)\);

which implies that epi(f) is geodesic E $\otimes 1$-convex on $A \otimes R$.

Conversely, let epi(f) is geodesic E $\otimes 1$-convex on $A \otimes R$. Let $x; y \leq A$, then $(x; f(x)) 2 \text{epi}(f)$ and $(y; f(y)) 2 \text{epi}(f)$. Since epi(f) is geodesic E $\otimes 1$-convex on $A \otimes R$, we have

\((^\circ E_{(x); E(y)}(t);(1 \cdot t)f(y) + tf(x)) 2 \text{epi}(f)\);

which implies that

\(f(^\circ E_{(x); E(y)}(t)) \leq (1 \cdot t)f(y) + tf(x)\);

and hence f is geodesic semi-E-convex function on A.

Theorem 5.4.5. Let \((f_i)_{i = 1} \leq A\) be a family of real valued functions which are bounded from above on a geodesic E-convex set $A \subseteq M$ and let their epigraphs $E(f_i)$ be geodesic E-convex sets. Then, the function $f(x) = \sup_{i = 1} f_i(x)$ is geodesic semi-E-convex on A.

Proof. Since epigraphs,

\[\text{epi}(f_i) = f(x; \otimes): x \leq A; \otimes 2 R; f_i(x) \leq \otimes,\]

are geodesic E-convex sets in $A \otimes R$. Therefore, their intersection

\[\bigcap_{i = 1} \text{epi}(f_i) = f(x; \otimes): x \leq A; \otimes 2 R; f_i(x) \leq \otimes; i \leq 1 \leq g\]

is also a geodesic E-convex set. We can see that this intersection is an E-epigraph; hence by Theorem 5.4.2, f is geodesic semi-E-convex function on A.

Now, we consider the following problem:

\((P) \text{ Min } f(x); \text{ such that } x \leq A = fx \leq M \text{ j } g_i(x) \leq 0; i = 1; 2; \text{.\textunderscore\textunderscore\textunderscore mg; }\)
where \( f : M \to \mathbb{R} \) and \( g_i : M \to \mathbb{R}; i = 1, 2, \ldots, m \), are real valued functions on a geodesic \( E \)-convex set.

**Theorem 5.4.6.** Let \( A \subseteq M \) be a geodesic \( E \)-convex set and \( f (E(x)) \leq f(x) \) for each \( x \in A \). If \( \dot{x} \) is a solution of the problem:

\[
(P_E) \quad \min \{ f(x) \}; \quad \text{such that } x \in A;
\]

then \( E(\dot{x}) \) is a solution of the problem \( (P) \).

**Proof.** Let \( E(\dot{x}) \) is not a solution of the problem \( (P) \), then there exists \( y \in A \) such that \( f(y) < f(E(\dot{x})) \), then \( f(E(y)) \leq f(y) < f(E(\dot{x})) \), which contradicts the optimality of \( \dot{x} \) of problem \( (P_E) \). Hence, \( E(\dot{x}) \) is a solution of the problem \( (P) \).

**Remark 5.4.1.** The above result also holds good for geodesic semi-\( E \)-convex function \( f : M \to \mathbb{R} \).

In the following we show that every local minimum of the problem \( (P) \) is a global minimum and that if \( f \) is a strictly geodesic semi-\( E \)-convex function on geodesic \( E \)-convex set \( A \), then the global optimal solution of problem \( (P) \) is unique.

**Theorem 5.4.7.** Let \( A \subseteq M \) be a geodesic \( E \)-convex set, \( f : M \to \mathbb{R} \) be geodesic \( E \)-convex on \( A \), and \( f(E(x)) \leq f(x) \) for each \( x \in A \). If \( x^0 = E(z^0) \) \( E(A) \) is a local minimum of the problem \( (P) \), then \( x^0 \) is global minimum of problem \( (P) \) on \( A \).

**Proof.** Let \( x^0 = E(z^0) \) \( E(A) \) be a nonglobal minimum of the problem \( (P) \) on \( A \), then there is \( y \in A \) such that \( f(y) < f(x^0) = f(E(z^0)) \), since function \( f : M \to \mathbb{R} \) is geodesic \( E \)-convex and \( f(E(x)) \leq f(x) \) for each \( x \in A \), we have

\[
f(x^0) \leq (1 - t)f(y) + tf(z^0)
\]

for any \( t \in (0, 1) \), which contradicts the local optimality of \( x^0 \) for problem
Hence \( x^0 \) is a global minimum of problem (P) on \( A \).

**Theorem 5.4.8.** Let \( f : M \to R \) be strictly geodesic semi-E-convex on geodesic E-convex set \( A \subseteq M \), then the global optimal solutions of problem (P) is unique.

**Proof.** Let \( x_1, x_2 \in A \) be two different global optimum solutions of problem (P). Then, \( f(x_1) = f(x_2) \). Since \( f \) is strictly geodesic semi-E-convex on \( A \), we have

\[
f(\circ_{E(x_1);E(x_2)}(t)) < (1-t)f(x_2) + tf(x_1) = f(x_1)
\]

for each \( t \in (0;1) \). Which contradicts the optimality of \( x_1 \) of problem (P). Hence, the global optimal solution of problem (P) is unique.

**Theorem 5.4.9.** Let \( f : M \to R \) be geodesic semi-E-quasiconvex on geodesic E-convex set \( A \subseteq M \), and \( \mathcal{O} = \min f(x) \). Then, the set \( X = \{ x \in A | f(x) = \mathcal{O} \} \) of optimal solutions of problem (P) is geodesic E-convex. If \( f \) is strictly geodesic semi-E-quasiconvex on \( A \), then the set \( X \) is a singleton.

**Proof.** Let \( x_1, x_2 \in A \) be two different global solutions of problem (P). Then \( f(x_1) = \mathcal{O} \) and \( f(x_2) = \mathcal{O} \). Since \( f : M \to R \) is geodesic semi-E-quasiconvex on \( A \), we have

\[
f(\circ_{E(x_1);E(x_2)}(t)) \leq \max f(x_1); f(x_2) \mathcal{O} = \mathcal{O}
\]

which implies that \( \circ_{E(x_1);E(x_2)}(t) \in X \), it follows that \( X \) is geodesic E-convex.

Now, to show that \( X \) is singleton, on contrary we suppose that \( x_1, x_2 \in X \) for \( t \in (0;1) \), \( \circ_{E(x_1);E(x_2)}(t) \in A \). Further, since \( f \) is strictly geodesic semi-E-quasiconvex on \( A \), we have

\[
f(\circ_{E(x_1);E(x_2)}(t)) < \max f(x_1); f(x_2) \mathcal{O} = \mathcal{O}
\]

which contradicts our supposition that \( \mathcal{O} = \min f(x) \), and hence \( X \) must be a singleton.

By using geodesic semi-E-convexity of the function \( f \) we prove the following result.
Theorem 5.4.10. If $f : M \to R$ is geodesic semi-$E$-convex on geodesic $E$-convex set $A \subseteq M$, then the set of optimal solutions of problem (P) is geodesic $E$-convex.

Proof. Let $x^m$ be optimal solution of problem (P), and let $\bar{\Xi} = f(x^m)$. Let $X$ be the set of optimal solutions for problem (P), $X = \{ x \in A | f(x) \leq \bar{\Xi} \}$ for any $x_1, x_2 \in X, x_1 \neq x_2$, and $0 \leq t \leq 1$. Since $f : M \to R$ is geodesic semi-$E$-convex on geodesic $E$-convex set $A \subseteq M$, we have

$$f(\circ_{E(x_1):E(x_2)}(t)) \leq (1 + t)f(x_2) + tf(x_1) \leq \bar{\Xi}$$

Thus, $\circ_{E(x_1):E(x_2)}(t) \in X$; it follows that $X$ is geodesic $E$-convex.

From Proposition 5.3.5 and Theorem 5.4.9, we have the following result.

Theorem 5.4.11. If $f : M \to R$ and $g_i : M \to R, i = 1; 2; \ldots; m$ are geodesic semi-$E$-quasiconvex on $M$. Then the set of optimal solutions of problem (P) is geodesic $E$-convex.

Corollary 5.4.1. If $f : M \to R$ and $g_i : M \to R, i = 1; 2; \ldots; m$ are geodesic semi-$E$-convex on $M$. Then the set of optimal solutions of problem (P) is geodesic $E$-convex.

If $f : A \to R$ be differentiable on $A$, then we have the following result.

Theorem 5.4.12. Let $A \subseteq M$ be geodesic $E$-convex set and let $f : A \to R$ be a differentiable geodesic semi-$E$-convex function on $A$. Then,

$$\circ_{E(x):E(y)}(0)(f) \leq f(x) + f(y)$$

for all $x, y \in A$.

Proof. Since $f$ is geodesic semi-$E$-convex on $A$, we have

$$f(\circ_{E(x):E(y)}(t)) \leq (1 + t)f(y) + tf(x); \quad t \in (0; 1]$$

or

$$\frac{f(\circ_{E(x):E(y)}(t)) - f(y)}{t} \leq f(x) - f(y)$$
By taking limit $t \to 0$; we get

$$\frac{d}{dt} f \left( ^{\circ}_{E(x);E(y)}(t) \right) \bigg|_{t=0} \leq f(x) \cdot f(y)$$

Thus,

$$^{\circ}_{E(x);E(y)}(0)(f) \leq f(x) \cdot f(y)$$

for all $x, y \in A$. 