CHAPTER-III
INEQUALITIES BETWEEN THE MOMENTS OF CONTINUOUS DISTRIBUTIONS
1. INTRODUCTION

For a continuous random variate $x$ which varies over the interval $[a,b]$ with arbitrary probability density function $\phi(x)$ the $r$th order moment $\mu_r$ about the origin is defined as

$$\mu_r = \int_a^b x^r \phi(x) \, dx \quad (3.1.1)$$

In particular

$$\mu_0 = \int_a^b \phi(x) \, dx = 1, \quad (3.1.2)$$

and

$$\mu_1 = \int_a^b x \phi(x) \, dx \quad (3.1.3)$$

be the mean of random variate $x$.

The power mean of order $r$ is defined as

$$M_r = (\mu_r)^{\frac{1}{r}} \quad r \neq 0 \quad (3.1.4)$$

For $r = 0$, $M_r$ defines geometric mean,

$$\log M_0 = \int_a^b \log x \phi(x) \, dx \quad (3.1.5)$$

where $M_0$ is geometric mean of the random variate $x$. For $r = -1$, $M_r$ defines harmonic mean,

$$M_{-1} = \frac{1}{\int_a^b \frac{1}{x} \phi(x) \, dx} \quad (3.1.6)$$

For $r = 2$, $M_r$ defines root mean square,

$$M_2 = \left[ \int_a^b x^2 \phi(x) \, dx \right]^{\frac{1}{2}} \quad (3.1.7)$$

In chapter I and chapter II we have discussed only the case when the variate takes discrete set of values. We can also obtain inequalities for the case when continuous random variate $x$ takes values in the given finite interval $[a,b]$. Inequalities between the moments and between the means of a continuous random variate can be proved in similar ways as proved for the case of discrete random variate in chapter - I. In this chapter we therefore present only the summary of the results. The detailed derivation of each result is omitted. Further, Kapur [8] has reported the following minimum and maximum values of the $r$th order moment $\mu_r$.
when the sth order moment $\mu'_r$ is prescribed ($r > s$) and the random variate takes continuous values in a given finite positive real interval $[a, b]$:

$$\mu'_r \leq \frac{(b' - a')\mu'_r + a' b' - a^s b'}{b' - a'} \tag{3.1.8}$$

and

$$\mu'_r \geq \left(\mu'_s\right)^{\frac{r}{s}} \tag{3.1.9}$$

He was motivated by the consideration of maximizing the entropy function subjected to certain constraints. But before maximizing entropy function we have to see whether the given moment values are consistent or not. That is whether there is any probability distribution which correspond to the given values of moments. If there is no such distribution then the efforts of finding out maximum probability distribution will not produce any result. This is the primary objective and motivation of the work considered in present chapter which concerns with continuous case of Kapur’s inequality and its further related developments.

In the present chapter we shall try to generalize inequalities (3.1.8) and (3.1.9) for the case when $r$ and $s$ can assume any real values and continuous random variate $x$ takes values in the given finite interval $[a, b]$. This will help us in deducing the bounds for power means and also provide alternate proofs of inequalities (3.1.8) and (3.1.9). In addition minimum and maximum values of the ratio and difference of $\mu'_r$ and $\left(\mu'_s\right)^{\frac{r}{s}}$ are also obtained here when continuous random variate takes values in a given finite interval $[a, b]$.

2. INEQUALITIES BETWEEN MEANS

Inequalities between means of a continuous random variate which takes values in the interval $[a,b]$, $a \geq 0$, have been obtained by kapur [10]. Here we show that these inequalities can be deduced from algebraic inequalities. As an example we consider inequalities between mean and harmonic mean.

If continuous random variate $x$ takes values in the interval $[a, b]$, then

$$(x - a) (x - b) \leq 0$$

$$\Rightarrow \quad x^2 \leq (a + b) x - ab$$

For $x > 0$, this gives

$$x \leq a + b - \frac{ab}{x} \tag{3.2.1}$$
Multiplying both sides of inequality (3.2.1) by probability density function \( \phi(x) \) we get, on using properties of definite integral, the following inequality:

\[
M \leq a + b - \frac{ab}{H}
\]

(3.2.2)

where \( M \) and \( H \) are respectively arithmetic mean and harmonic mean of the continuous variable \( x \). We know that \( M \geq H \), \( [14] \). Thus we have

\[
H \leq M \leq a + b - \frac{ab}{H}
\]

(3.2.3)

It may be noted here that in case of discrete probability distribution we have been able to tighten the inequality between \( M \) and \( H \). But in case of continuous probability distribution inequality (3.2.3) is same as obtained by Kapur\([10]\). Only alternate proof of inequality (3.2.3) is given here. From inequality (3.2.3), we also have

\[
\frac{ab}{a + b - M} \leq H \leq M
\]

(3.2.4)

Similarly, we can deduce inequalities between other means from the corresponding algebraic inequalities.

Minimum and maximum values of the ratio of two means when continuous random variate \( x \) takes values in the interval \([a, b]\) can be obtained in the similar ways as obtained for the case of discrete random variate in chapter I. Here we try to give alternate proofs of some of these inequalities.

Substituting \( M = KH \) in inequality (3.2.2), we get

\[
KH^2 - (a+b) H + ab \leq 0
\]

(3.2.5)

From inequality (3.2.3) we have \( M \geq H \) therefore it is clear that \( K \geq 1 \). Further since \( H \) is a positive real number it follows that the discriminant of the quadratic equation given by (3.2.5) must be positive. Therefore, we must have,

\[
K \leq \frac{(a + b)^2}{4ab}
\]

Hence

\[
1 \leq K \leq \frac{(a + b)^2}{4ab}
\]

or

\[
1 \leq \frac{M}{H} \leq \frac{(a + b)^2}{4ab}
\]

(3.2.6)
Inequality (3.2.6) gives the bound for the ratio $M/H$ when continuous random variate takes values in the interval $[a, b]$.

For a continuous variate $x$ which takes values in the interval $[a, b]$ we have

$$\mu_1^2 \leq \mu_2' \leq (a + b)\mu_1' - ab$$  \hspace{1cm} (3.2.7)

Put $\mu_2' = k\mu_1^2$ in inequality (3.2.7). we get

$$1 \leq k$$

and

$$k\mu_1^2 - (a + b)\mu_1' + ab \leq 0$$  \hspace{1cm} (3.2.8)

The discriminant of quadratic equation given by (3.2.8) must be positive. Therefore we have

$$K \leq \frac{(a + b)^2}{4ab}$$

Hence

$$1 \leq K \leq \frac{(a + b)^2}{4ab}$$

or

$$1 \leq \frac{S^2}{M^2} \leq \frac{(a + b)^2}{4ab}$$  \hspace{1cm} (3.2.9)

Inequality (3.2.9) gives the bound for the ratio $S/M$ when continuous random variate takes values in the interval $[a, b]$.

Thus we see that the bounds on the ratio of two means when continuous random variate $x$ takes values in the interval $[a, b]$ can be obtained either by the method explained above or by the method as discussed in chapter-1.

Minimum and maximum values of the difference between two means when continuous random variate $x$ takes values in the interval $[a, b]$ can be obtained in the similar ways as obtained for the case of discrete random variate in chapter – I. Here we try to give alternate proofs of some of these inequalities.

From inequality (3.2.3), we have

$$d \leq a + b - \frac{ab}{H} - H$$  \hspace{1cm} (3.2.10)

where

$$d = M - H$$  \hspace{1cm} (3.2.11)

Inequality (3.2.10) yields
\[ H^2 - (a + b - d) H + ab \leq 0 \]  
(3.2.12)

The discriminant of the quadratic equation given by (3.2.12) must be positive, therefore we have

\[
(a + b - d)^2 - 4ab \geq 0
\]

\[
\Rightarrow (a + b - d)^2 \geq 4ab
\]

\[
\Rightarrow a + b - d \geq 2 \sqrt[2]{ab}
\]

\[
\Rightarrow d \leq \left( \sqrt[2]{b} - \sqrt[2]{a} \right)^2
\]

We see that the difference of Arithmetic mean and Harmonic mean is bounded by the following inequality

\[ 0 \leq M - H \leq (\sqrt[2]{b} - \sqrt[2]{a})^2 \]  
(3.2.13)

We now find minimum and maximum values of the difference \( \mu_2' - \mu_1' \). From inequality (3.2.7), we have

\[
d \leq (a + b) \mu_1' - ab - \mu_1' \]

where

\[
d = \mu_2' - \mu_1'
\]

(3.2.15)

Inequality (3.2.14) yields

\[
\mu_1' - (a + b)\mu_1' + ab + d \leq 0
\]

(3.2.16)

The discriminant of the quadratic equation (3.2.16) must be positive, therefore we have

\[
(a + b)^2 - 4(ab + d) \geq 0
\]

\[
\Rightarrow d \leq \left( \frac{b - a}{2} \right)^2
\]

The difference "d" defines the variance of the continuous random variate \( x \). Thus we see that bounds for the difference between two means when continuous variate \( x \) takes values in the interval \([a, b]\) can be obtained either by the method explained above or by the method as discussed in chapter 1.

We now present a geometrical significance of the bounds for the difference between two means.

For a continuous random variable \( x \) which varies over the interval \([a, b]\) we have

\[
\mu_2 \leq (a + b)\mu_1' - ab
\]

and

\[ 109 \]
\[ \mu_2 \geq \mu_1^{'} \]  \hspace{1cm} (3.2.18)

This means that the point \((\mu_1, \mu_2^{'})\) in \(\mu_1, \mu_2^{'}\)-plane lies on or above the parabola given by equation (3.2.18) and below the straight line given by equation (3.2.17). The parabola and straight line intersect at two points namely \((a, a^2)\) and \((b, b^2)\). This is shown in figure 3.1.

Consider a function \(f(\mu_1^{'})\) defined by

\[ f(\mu_1^{'}) = \mu_1^{'}^2 \]  \hspace{1cm} (3.2.19)

The function \(f(\mu_1^{'})\) is continuous in the interval \([a, b]\) and derivable in the interval \((a, b)\) therefore by Lagrange’s mean value theorem there exists a point \(c\) in the interval \((a, b)\) such that

\[ f^{'}(c) = \frac{f(b) - f(a)}{b - a} \]

\[ \Rightarrow c = \frac{a + b}{2} \]

The tangent to the curve \(f(\mu_1^{'})\) at the point \(P\) \(\left(\frac{a + b}{2}, \frac{a^2 + b^2}{2}\right)\) is therefore parallel to the line passing through the points \((a, a^2)\) and \((b, b^2)\). The line perpendicular to \(\mu_1^{'}\) axis and passing through the point \(P\) meets the straight line given by equation (3.2.17) at the point \(Q\) where

\[ Q \leftrightarrow \left(\frac{a + b}{2}, \frac{a^2 + b^2}{2}\right) \]

The distance between the points \(P\) and \(Q\) is given by

\[ PQ = \left(\frac{b - a}{2}\right)^2 \]

From this we see that the distance between the points \(P\) and \(Q\) gives the maximum of the difference between \(\mu_2\) and \(\mu_1^{'}\). Further since parabola and straight line are intersecting curves it follows that the minimum value of the difference \(\mu_2 - \mu_1^{'}\) will be zero. It may be noted here that equation of tangent line to the curve \(\mu_2^{'} = \mu_1^{'}\) at the point \(P\) is given by

\[ \mu_2 = (a + b)\mu_1 - \left(\frac{a + b}{2}\right)^2 \]

This is explained in figure 3.1.
Figure 3.1: The point \((\mu_1', \mu_2')\) in \(\mu_1 \mu_2\)-plane lies in a region bounded by the straight line \(\mu_2' = (a + b)\mu_1' - ab\) and parabola \(\mu_2' = \mu_1^2\). The straight line and parabola intersect at two points namely \((a, a^2)\) and \((b, b^2)\). The abscissa of the point P is obtained from equation \((b-a)f'(\mu_1') = f(b) - f(a)\), where \(f(\mu_1') = \mu_1^2\). The point Q has same abscissa and the distance PQ gives the maximum of the difference \(\mu_2' - \mu_1''\). The curve \(\mu_2' = \mu_1^2\) is concave upward in the interval \([a,b]\). The tangent line passing through the point P is parallel to the line passing through the points R and S. It is also clear from the figure that minimum value of the difference \(\mu_2' - \mu_1''\) is zero.

\[
P \leftrightarrow \left( \frac{a + b}{2}, \frac{(a + b)^2}{2} \right)
\]

\[
Q \leftrightarrow \left( \frac{a + b}{2}, \frac{a^2 + b^2}{2} \right)
\]
For a continuous random variate $x$ which varies over the interval $[a,b]$ we have

$$M \leq a + b - \frac{ab}{H} \quad (3.2.20)$$

and

$$M \geq H \quad (3.2.21)$$

This means that the point $(H,M)$ in $HM$-plane lies on or above the straight line given by equation (3.2.21) and on or below the curve given by equation (3.2.20). The straight line and curve intersect at two points namely $(a,a)$ and $(b,b)$. This is shown in figure 3.2.

Consider a function $f(H)$ defined by

$$f(H) = a + b - \frac{ab}{H} \quad (3.2.22)$$

This gives

$$f'(H) = \frac{ab}{H^2}$$

and

$$f''(H) = -\frac{2ab}{H^3}$$

The second derivative of $f(H)$ is negative in the interval $[a,b]$ Therefore graph of the function $f(H)$ in the interval $[a,b]$ is concave downward. By Lagrange’s mean value theorem we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad c = \sqrt{ab}$$

The tangent to the curve $f(H)$ at the point $P \left(\sqrt{ab}, a + b - \sqrt{ab}\right)$ is therefore parallel to the line given by equation (3.2.21). The line perpendicular to $H$ axis and passing through the point $P$ meets the straight line $M = H$ at the point $Q$ where

$$Q \leftrightarrow \left(\sqrt{ab}, \sqrt{ab}\right)$$

The distance between the points $P$ and $Q$ is given by

$$PQ = \left(\sqrt{b} - \sqrt{a}\right)^2$$

From this we see that the distance between the points $P$ and $Q$ gives the maximum of the difference $M - H$. Further since straight line and curve are intersecting curves it follows that the minimum value of the difference $M - H$ will be zero. Thus we see that the geometrical significance of the bound for the difference between two means follows from Lagrange’s mean value theorem.
Figure 3.2: The point \((H,M)\) in HM-plane lies in a region bounded by the straight line \(M=H\) and the curve \(MH = (a+b)H - ab\). The straight line and curve intersect at two points namely \((a,a)\) and \((b,b)\). The abscissa of the point \(P\) is obtained from equation \((b-a)f'(H) = f(b) - f(a)\), where \(f(H) = a+b - \frac{ab}{H}\). The point \(Q\) has same abscissa and the distance \(PQ\) gives the maximum of the difference “\(M-H\)”. The curve \(MH = (a+b)H -ab\) is concave downward in the interval \([a,b]\). Equation of tangent line passing through the point \(P\) is given by \(M = H + \left(\sqrt{b} - \sqrt{a}\right)\) and we note that this line is parallel to the line \(M=H\). It is also clear from the figure that the minimum value of the difference “\(M-H\)” is zero.
3. GENERALIZED INEQUALITIES FOR CONTINUOUS RANDOM VARIATE

Result 3.3.1

Let r and s be positive real numbers with r > s. If a continuous random variate takes values in the interval \([a, b]\) with \(a > 0\) then \(\mu'_r\) is bounded below by the following inequality:

\[
\mu'_r \geq (\mu'_s)^{\frac{r}{s}}
\]  
(3.3.1)

Proof

If \(\alpha\) and \(\beta\) are positive real numbers and \(\alpha + \beta = 1\) then Holder's inequality states that

\[
\int f^\alpha(x)g^\beta(x)dx \leq \left(\int f(x)dx\right)^\alpha \left(\int g(x)dx\right)^\beta
\]  
(3.3.2)

For \(r > s > 0\) if we put \(s = r\alpha\) then \(0 < \alpha < 1\). Let

\[
x' \phi(x) = f(x)
\]  
(3.3.3)

and

\[
\phi(x) = g(x)
\]  
(3.3.4)

From (3.3.3) and (3.3.4) we have \(f(x) \geq 0\), \(g(x) \geq 0\) and

\[
x'^\alpha \phi(x) = x'^\alpha \phi(x)
\]

\[
= (x' \phi(x))^{\alpha} (\phi(x))^{1-\alpha}
\]

\[
= f^\alpha(x) g^{1-\alpha}(x)
\]  
(3.3.5)

From (3.3.2), we have

\[
\int f^\alpha(x)g^{1-\alpha}(x)dx \leq \left(\int f(x)dx\right)^\alpha \left(\int g(x)dx\right)^{1-\alpha}
\]

Substituting values from (3.3.3), (3.3.4) and (3.3.5), we get

\[
\int x'^\alpha \phi(x)dx \leq \left(\int x' \phi(x)dx\right)^\alpha \left(\int \phi(x)dx\right)^{1-\alpha}
\]

\[
\Rightarrow \quad \mu'_r \leq (\mu'_s)^{\frac{r}{s}}
\]

\[
\Rightarrow \quad \mu'_r \geq (\mu'_s)^{\frac{r}{s}}, \quad r > s
\]

Hence proved.

Result 3.3.2

Let r and s be negative real numbers with \(r > s\). If a continuous random variate takes values in the interval \([a, b]\) with \(a > 0\) then \(\mu'_r\) is bounded above by the following inequality:

\[
\mu'_r \leq (\mu'_s)^{\frac{r}{s}}
\]  
(3.3.6)
Proof
For $0 > r > s$ if we put $r = s = \alpha$ then $0 < \alpha < 1$. Let
\[ x^\alpha \phi(x) = f(x) \] (3.3.7)
and
\[ \phi(x) = g(x) \] (3.3.8)
We note that $f(x) \geq 0$, $g(x) \geq 0$ and
\[ x^r \phi(x) = x^{\alpha \alpha} \phi(x) \\
= [x^\phi(x)]^\alpha \phi^{1-\alpha}(x) \\
= f^\alpha(x) g^{1-\alpha}(x) \] (3.3.9)
Substituting values from (3.3.7), (3.3.8) and (3.3.9) in (3.3.2), we get
\[ \int x^r \phi(x) dx \leq \left( \int x^r \phi(x) dx \right)^\alpha \left( \int \phi(x) dx \right)^{1-\alpha} \]
\[ \Rightarrow \mu_r \leq \left( \mu_r \right)^{\alpha} \]
Hence proved.

Result 3.3.3
For a continuous random variate which takes values in the interval $[a,b]$, with $a > 0$, we have
\[ \mu_r \geq (M_0)^r \] (3.3.10)
where $r$ is any real number and
\[ \log M_0 = \int_a^b \log \phi(x) \, dx \] (3.3.11)
Proof
We first consider the case when $r > 0$. From Result (3.3.1) we have for $s > 0$
\[ \mu_r \geq \left( \mu_s \right)^r, \quad r > s \]
or
\[ \left( \mu_r \right)^r \geq \left( \mu_r \right)^r, \quad r > s \] (3.3.12)
From (3.3.12) we see that $\left( \mu_s \right)^r$ is an increasing function of $r$ for $r > 0$. We conclude that
\[ \left( \mu_r \right)^r \geq \lim_{s \to 0} \left( \mu_s \right)^r = M_0 \]
This gives
\[ \mu_r' \geq M_0' \]

It may be noted here that

\[ \lim_{s \to 0} \left( \int_s^1 x^r \phi(x) \, dx \right)^{\frac{1}{s}} = P \quad (3.3.13) \]

\[ \Rightarrow \quad \log P = \lim_{s \to 0} \frac{b}{s} \int_s^b x^r \phi(x) \, dx \]

Using Hospital's rule, we see that

\[ \log P = \lim_{s \to 0} \frac{b}{s} \int_s^b x^r \phi(x) \, dx \]

\[ = \int_a^b x^r \phi(x) \, dx \]

or

\[ P = e^{\int_a^b x^r \phi(x) \, dx} \quad (3.3.14) \]

From (3.3.11), (3.3.13) and (3.3.14) we conclude that

\[ \lim_{s \to 0} \left( \int_s^1 x^r \phi(x) \, dx \right)^{\frac{1}{s}} = M_0 \quad (3.3.15) \]

Consider now the case when \( r < 0 \). From Result (3.3.2) we have for \( s < 0 \),

\[ \mu_r' \leq \left( \mu_0' \right)^r \quad r > s \]

or

\[ \left( \mu_0' \right)^r \leq \left( \mu_r' \right)^{\frac{1}{r}} \quad r > s \]

Therefore \( \mu_r' \) increases as \( r \) increases and we have that

\[ \left( \mu_r' \right)^r \leq \lim_{r \to 0} \left( \mu_r' \right) = M_0 \]

This gives

\[ \Rightarrow \quad \mu_r' \geq \left( M_0' \right)^r \]

Hence proved.
Result 3.3.4

Let \( r \) be a positive real number and \( s \) be a negative real number. If a continuous random variate takes values in the interval \([a, b]\) with \( a > 0 \) then \( \mu'_r \) is bounded below by the following inequality

\[
\mu'_r \geq (\mu'_s)^{r}
\]  

(3.3.16)

Proof

Let \( r \) be a positive real number, from Result (3.3.3) we see that

\[
\mu'_r \geq M'_0
\]  

(3.3.17)

Further for a negative real number \( s \), we have

\[
\mu'_s \geq M'_0
\]  

(3.3.18)

or

\[
M_0 \geq (\mu'_s)^{r}
\]  

(3.3.19)

From (3.3.17) and (3.3.19), we get

\[
\mu'_r \geq (\mu'_s)^{r}
\]

Hence proved.

In chapter II we have obtained some elementary inequalities in Results (2.2.3- 2.2.6) and (2.3.1- 2.3.8). Here we present alternate proofs of these inequalities.

Result 3.3.5

If \( r \) is a positive real number and \( s \) is any non zero real number with \( r > s \) then for \( a < x < b \), with \( a > 0 \), we have

\[
\mu'_r \leq \frac{(b' - a')x' + a'b' - a'b'}{b' - a'}
\]  

(3.3.20)

and for \( x \) lying outside \((a, b)\) we have

\[
\mu'_r \geq \frac{(b' - a')x' + a'b' - a'b'}{b' - a'}
\]  

(3.3.21)

If \( r \) is a negative real number with \( r > s \) then inequality (3.3.20) holds for \( x \) lying outside \((a,b)\) and inequality (3.3.21) holds for \( a \leq x \leq b \).

Proof

Consider the following function \( f(x) \) for positive real values of \( x \):

\[
f(x) = x' - \frac{b' - a' \cdot x' + a^2 b' - a'b'}{b' - a'}
\]  

(3.3.22)
where \(r\) and \(s\) are real numbers such that \(r > s\) and \(s \neq 0\). The function \(f(x)\) is continuous in the interval \([a, b]\) with \(a > 0\). Then \(f'(x)\) is given by
\[
f'(x) = x^{r-1} \left[ \frac{rx^{r-1} - s \left( \frac{b' - a'}{b'^r - a'^r} \right)}{b' - a'} \right]
\]
\(f'(x)\) vanishes at \(x = 0\) and \(c\), where
\[
c = \left[ \frac{s \left( \frac{b' - a'}{b'^r - a'^r} \right)}{r} \right]^{1/r-1}
\]
(3.3.23)
By Rolle’s theorem we have that \(c\) lies in the interval \((a, b)\).

If \(r\) is a positive real number and \(s\) is negative real number with \(r > s\) then \(f'(x) \leq 0\) iff \(x \leq c\). This means that \(f(x)\) decreases in the interval \((0, c)\) and increases in the interval \((c, \infty)\). Further since \(c\) lies in the interval \((a, b)\) and \(f(a) = f(b) = 0\), it follows that
\[
f(x) \leq 0 \quad \text{for} \quad a \leq x \leq b
\]
and for \(x\) lying outside \((a, b)\)
\[
f(x) > 0
\]
(3.3.25)
On substituting value of \(f(x)\) from equation (3.3.22) in inequalities (3.3.24) and (3.3.25) we respectively get inequalities (3.3.20) and (3.3.21).

If \(r\) is a negative real number with \(r > s\) then \(f'(x) < 0\) iff \(x > c\). This means that \(f(x)\) increases in the interval \((0, c)\) and decreases in the interval \((c, \infty)\). Since \(c\) lies in the interval \((a, b)\) and \(f(a) = f(b) = 0\) it follows that inequality (3.3.25) holds for \(a \leq x \leq b\) while inequality (3.3.24) holds for \(x\) lying outside \((a, b)\) and thus we get inequalities for the case when \(r\) is negative real number.

Result 3.3.6

For \(a \leq x \leq b\) with \(a > 0\), we have
\[
x \geq \frac{(b' - a') \log x + a' \log b - b' \log a}{\log b - \log a}
\]
(3.3.26)
and for \(x\) lying outside \((a, b)\), we have
\[
x \geq \frac{(b' - a') \log x + a' \log b - b' \log a}{\log b - \log a}
\]
(3.3.27)
where \(r\) is a real number.

Proof

Consider the following function \(f(x)\) defined for positive real values of \(x\),
The function \( f(x) \) is continuous in the interval \([a, b]\) where \( a > 0 \). Then \( f'(x) \) is given by

\[
f'(x) = \frac{1}{x} \left[ r x^r - \frac{b^r - a^r}{\log b - \log a} \right]
\]

We have \( f'(x) = 0 \) at \( x = c \) where

\[
c = \left[ \frac{b^r - a^r}{r(\log b - \log a)} \right]^{1/r}
\]

By Rolle's theorem we have that \( c \) lies in the interval \((a, b)\). Also \( f'(x) \leq 0 \) iff \( x \leq c \). This means that \( f(x) \) decreases in interval \((0, c)\) and increases in the interval \((c, \infty)\). Further, since \( c \) lies in the interval \((a, b)\) and \( f(a) = f(b) = 0 \) it follows that

\[
f(x) \leq 0 \quad \text{for } a \leq x \leq b \quad (3.3.31)
\]

and for \( x \) lying outside \((a, b)\) we have

\[
f(x) \geq 0. \quad (3.3.32)
\]

On substituting value of \( f(x) \) from equation (3.3.28) in inequalities (3.3.31) and (3.3.32) we respectively get inequalities (3.3.26) and (3.3.27).

We now prove inequalities between moments of a continuous random variate which varies over the interval \([a, b]\) with \( a > 0 \). Here we prefer to present the proofs of these inequalities in such a way that in addition we also get the alternate proofs of these inequalities for the case of discrete random variate. We shall also try to show that the inequalities obtained in Result (3.3.1),(3.3.2),(3.3.3) and (3.3.4) can also be deduced from the corresponding inequalities for discrete random variate.

**Result 3.3.7**

Let \( r \) be a positive real number and \( s \) be any non-zero real number with \( r > s \). If a positive random variate takes values \( x_i \) \((i = 1, 2, \ldots, n)\) in the interval \([a, b]\), with \( a > 0 \), then we have

\[
\mu' = \frac{\left(b^r - a^r\right)\mu' + a^r b^s - a^r b^s}{b^r - a^r} \quad (3.3.33)
\]

and

\[
\mu' \geq \frac{(x_j - x_{j+1})\mu' + x_{j+1}^r x_j^s - x_j^r x_{j+1}}{x_j^r - x_{j+1}^s} \quad (3.3.34)
\]

where \( j = 2, 3, \ldots, n \).
If a continuous random variate takes values in the interval \([a, b]\), with \(a > 0\), then the upper bound for \(\mu'_i\) is given by the inequality (3.3.33) whereas the lower bound is given by the following inequality:

\[
\mu'_i \geq \left(\mu'_s\right)^s
\]  \hspace{1cm} (3.3.35)

**Proof**

It is seen that \(\mu'_i\) can be expressed in terms of \(\mu'_s\) in the following form:

\[
\mu'_i = \left(\frac{x^i - x^j}{x^j - x^k}\right) \mu'_s + \sum_{i=1}^{n} p_i \left[ x^j - x^i - x^s + \frac{x^j - x^s - x^j x^k}{x^j - x^s} \right] \]  \hspace{1cm} (3.3.36)

where \(\alpha\) and \(\beta\) take one of the values among 1, 2, ..., \(n\) with \(\alpha \neq \beta\). Without loss of generality we can arrange values of the variate such that \(a = x_1 < x_2 < \ldots < x_n = b\). If we take \(\alpha = 1\) and \(\beta = n\) then \(x_i < x_{i-1} < x_n\) for \(i = 1, 2, \ldots, n\). It follows from (3.3.20) that the last term in equation (3.3.36) is negative and we conclude that the upper bound for \(\mu'_i\) is given by inequality (3.3.33). Further if \(x_{n-1} = x_{j-1}\) and \(x_j = x_{j+1}\) then each \(x_i\) lies outside \((x_{j-1}, x_j)\) and it follows from (3.3.21) that the last term in equation (3.3.36) is positive and we conclude that the lower bound for \(\mu'_i\) is given by inequality (3.3.34). It is also clear that equality sign in inequalities (3.3.33) and (3.3.34) holds iff \(n = 2\).

If the value of \(\mu'_i\) coincides with one of \(x^s_{j-1}\) or \(x^s_j\) then from inequality (3.3.34) we have

\[
\mu'_i \geq \left(\mu'_s\right)^s
\]  \hspace{1cm} (3.3.37)

Also if \(x_{j-1}\) approaches \(x_j\) we get inequality (3.3.37) and we conclude that for continuous random variate the lower bound for \(\mu'_i\) is given by inequality (3.3.37). The upper bound for \(\mu'_i\) can be deduced from Result (3.3.5). Multiplying both sides of inequality (3.3.20) by pdf \(\phi(x)\) we get, on using properties of definite integral, inequality (3.3.33).

**Result 3.3.8**

Let \(r\) and \(s\) are negative real numbers with \(r > s\). If a positive random variate takes values \(x_i (i = 1, 2, \ldots, n)\) in the interval \([a, b]\), with \(a > 0\), we have

\[
\mu'_i \geq \left(\frac{b' - a'}{b' - a}\right) \mu'_s + a'b' - a'b' \]  \hspace{1cm} (3.3.38)

and
\[
\mu'_i \leq \frac{(x'_j - x'_{j-1})\mu'_s + x'_j x'_s - x'_s x'_j}{x'_j - x'_{j-1}} \tag{3.3.39}
\]

where \( j = 2, 3, \ldots, n \).

If a continuous random variate takes values in the interval \([a, b]\), with \(a > 0\), the lower bound for \(\mu'_i\) is given by inequality (3.3.38) whereas upper bound for \(\mu'_i\) is given by following inequality:

\[
\mu'_i \leq (\mu'_i)^{\gamma} \tag{3.3.40}
\]

Proof

We again consider equation (3.3.36). If we take \(a = 1\) and \(\beta = n\) then \(x_i \leq x_i \leq x_a\) for \(i = 1, 2, \ldots, n\). It follows from Result (3.3.5) that last term in equation (3.3.36) is positive and we conclude that the lower bound for \(\mu'_i\) is given by inequality (3.3.38). Also if \(x_a = x_{j-1}\) and \(x_b = x_j\), \(j = 2, 3, \ldots, n\) then each \(x_i\) lies outside \((x_{j-1}, x_j)\). It follows from Result (3.3.5) that the last term in equation (3.3.36) is negative and we conclude that the upper bound for \(\mu'_i\) is given by inequality (3.3.39). Also if \(x_{j-1}\) approaches \(x_j\) we get inequality (3.3.40). The lower bound for \(\mu'_i\) can be deduced from Result (3.3.5). Multiplying both sides of inequality (3.3.21) by \(pdf \phi(x)\) we get, on using properties of definite integral, inequality (3.3.38).

Result 3.3.9

For a random variate which takes values \(x_i (i = 1, 2, \ldots, n)\) in the interval \([a, b]\), with \(a > 0\), we have

\[
\mu'_r \leq \frac{(b^r - a^r)\log M_0 + a^r \log b - b^r \log a}{\log b - \log a} \tag{3.3.41}
\]

and

\[
\mu'_r \geq \frac{(x'_j - x'_{j-1})\log M_0 + x'_j x'_s - x'_s x'_j}{\log x_j - \log x_{j-1}} \tag{3.3.42}
\]

where \( j = 2, 3, \ldots, n\), \(r\) is a real number and

\[
M_0 = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n} \tag{3.3.43}
\]

For a continuous random variate which takes values in the interval \([a, b]\) with \(a > 0\) the upper bound for \(\mu'_r\) is given by inequality (3.3.41) whereas the lower bound for \(\mu'_r\) is given by the following inequality:

\[
\mu'_r \geq (M_0)^r \tag{3.3.44}
\]
Proof

It is seen that $\mu_i'$ can be expressed in terms of $\log M_0$ in the following form:

$$\mu_i' = \frac{x_i^\beta - x_a^\beta}{\log x_\beta - \log x_a} \log M_0 + \frac{x_i^\alpha \log x_\beta - x_i^\beta \log x_a}{\log x_\beta - \log x_a}$$

$$+ \sum_{i=1}^{n} p_i \left[ \frac{x_i^\beta - x_a^\beta}{\log x_\beta - \log x_a} \log x_i - \frac{x_i^\alpha \log x_\beta - x_i^\beta \log x_a}{\log x_\beta - \log x_a} \right]$$  \hspace{1cm} (3.3.45)

Without loss of generality we can arrange values of the variate such that $a = x_1 \leq x_2 \leq \ldots \leq x_n = b$. If we take $\alpha = 1$ and $\beta = n$ then $x_1 \leq x_i \leq x_n$ for $i = 1, 2, \ldots, n$. It follows from Result (3.3.6) that last term in equation (3.3.45) is negative and we conclude that the upper bound for $\mu_i'$ is given by inequality (3.3.41). Also if $x_a = x_{j-1}$ and $x_\beta = x_j$, $j = 2, 3, \ldots, n$ then each $x_i$ lies outside $(x_{j-1}, x_j)$. It follows from Result (3.3.6) that the last term in equation (3.3.45) is positive and we conclude that the lower bound for $\mu_i'$ is given by inequality (3.3.42).

If the value of $M_0$ coincides with one of $x_{j-1}$ or $x_j$ then from inequality (3.3.42) we have

$$\mu_i' \geq (M_0)^r$$  \hspace{1cm} (3.3.46)

Also if $x_{j-1}$ approaches $x_j$ we get inequality (3.3.46) and we conclude that for continuous random variate the lower bound for $\mu_i'$ is given by inequality (3.3.46). The upper bound for $\mu_i'$ can be deduced from Result (3.3.6). Multiplying both sides of inequality (3.3.26) by pdf $\phi(x)$ we get, on using properties of definite integral, inequality (3.3.41).

4. MINIMUM AND MAXIMUM VALUES OF THE RATIO $\frac{\mu_i'}{(\mu_i')^r}$

Result 3.4.1

For a continuous random variate which takes values in the interval $[a,b]$ where $a > 0$, we have

$$1 \leq \frac{\mu_i'}{\mu_i'^r} \leq \frac{s}{r} \left[ \left( \frac{b'}{b-a'} \right) \left( \frac{r-s}{r(b'-a')} \right) \right]^{r-1}$$  \hspace{1cm} (3.4.1)

where $r$ is a positive real number and $s$ is any non zero real number such that $r > s$.

Result 3.4.2

For a continuous random variate which takes values in the interval $[a,b]$ where $a > 0$, we have
\[
\frac{\varepsilon \left( \frac{b' - a'}{b'' - a''} \right) \left( r - s \right) \left( b' - a' \right)^{r-s}}{1} \leq \mu_i \leq 1
\] 
\[
\frac{\varepsilon \left( \frac{b' - a'}{b'' - a''} \right) \left( r - s \right) \left( b' - a' \right)^{r-s}}{1}
\]

where \( r \) and \( s \) are negative real numbers such that \( r > s \).

**Result 3.4.3**

For a continuous random variate which takes values in the interval \([a,b]\) where \( a > 0 \), we have

\[
1 \leq \frac{\mu_i}{M_0} \leq \frac{b' - a'}{r \log b - \log a} \left[ \frac{1}{b'' - a''} \left( 1 - a'' \log b'' - b'' \log a'' \right) \right]^{1/r}
\]

where \( r \) is any non zero real number.

Inequalities (3.4.1), (3.4.2) and (3.4.3) can be proved in the similar ways as corresponding inequalities are proved for the case of discrete distributions in chapter II. We now discuss the equality signs in these inequalities.

Let the random variable \( x \) takes values \( a \) and \( b \) with probability \( p \) and \( 1 - p = q \) and takes the value zero elsewhere then the probability density function \( \phi(x) \) can be written as

\[
\phi(x) = p\delta(x - a) + q\delta(x - b)
\]

where \( \delta(x) \) denotes the Dirac delta function and \( p \) and \( q \) may have any positive value subject to their sum being unity. In this case we have

\[
\mu_s = pa^s + qb^s
\]

and

\[
\mu_i = pa^i + qb^i
\]

so that

\[
\frac{(b' - a')\mu_s + a'b^s - a'b^i}{b' - a'} = pa^i + qb^i = \mu_i
\]

It follows from (3.4.7) that inequalities (3.3.33) and (3.3.38) reduce to equalities when \( \phi(x) = p\delta(x - a) + q\delta(x - b) \).

When the probability mass is concentrated at one point of the interval \([a,b]\), we have

\[
\phi(x) = \delta(x - c), a \leq c \leq b
\]
In this case we have
\[ \mu_r' = c' \] (3.4.9)
and
\[ \mu_s' = c^s \] (3.4.10)
so that
\[ \mu_r' = \mu_s' \] (3.4.11)
It follows from (3.4.11) that inequalities (3.3.35) and (3.3.40) reduce to equalities when \( \phi(x) = \delta (x-c) \).

Further when probability density function is given by (3.4.4), we have
\[ M_0 = a^p b^q \] (3.4.12)
so that
\[ \frac{(b' - a') \log M_0 + a' \log b - b' \log a}{\log b - \log a} = \mu_r' \] (3.4.13)
It follows from (3.4.13) that inequality (3.3.41) reduces to equality when \( \phi(x) = p \delta (x-a) + q \delta (x-b) \). When \( \phi(x) \) is given by (3.4.8) we have
\[ M_0 = c \] (3.4.14)
so that
\[ \mu_r' = M_0' \] (3.4.15)
Therefore inequality (3.3.44) reduces to equality when \( \phi(x) = \delta (x-c) \). It is clear from the above discussion that lower bound in inequality (3.4.1) is attained for the case when \( \phi(x) = \delta (x-c) \). The upper bound in inequality (3.4.1) is attained for the case when
\[ \phi(x) = p \delta (x-a) + q \delta (x-b) \]
where
\[ p = \frac{r}{r-s} \left( \frac{b'}{b' - a'} \right) - \frac{s}{r-s} \left( \frac{b^s}{b^s - a^s} \right) \] (3.4.16)
and
\[ q = \frac{s}{r-s} \left( \frac{a^s}{b^s - a^s} \right) - \frac{r}{r-s} \left( \frac{a^r}{b^r - a^r} \right) \] (3.4.17)
In this case
\[ \mu_r' = \frac{s}{r-s} \left( \frac{a^s b' - a^r b^s}{b^r - a^r} \right) \] (3.4.18)
\[
\mu'_r = \frac{r}{r-s} \left( \frac{a^r b'^r - a'^r b^r}{b'^r - a'^r} \right)
\] (3.4.19)

so that

\[
\frac{\mu'_r}{\mu'_s} = \frac{s}{r} \left( \frac{b'^r - a'^r}{b^r - a^r} \right) \left[ \frac{(r-s)(b'^r - a'^r)}{r(a^r b'^r - a'^r b^r)} \right]^{\frac{1}{r-1}}
\] (3.4.20)

It also follows from (3.4.20) that lower bound in inequality (3.4.2) is attained for \( \phi(x) = p \delta (x-a) + q \delta (x-b) \) when \( p \) and \( q \) are respectively given by (3.4.16) and (3.4.17). The upper bound in inequality (3.4.2) is attained for the case when \( \phi(x) = \delta (x-c) \). The lower bound in inequality (3.4.3) is attained when \( \phi(x) = \delta (x-c) \) and the upper bound is attained for the case when

\[\phi(x) = p \delta (x-a) + q \delta (x-b)\]

where

\[
p = \frac{b'}{b' - a'} - \frac{1}{r(\log b - \log a)}
\] (3.4.21)

and

\[
q = \frac{1}{r(\log b - \log a)} - \frac{a'}{b' - a'}
\] (3.4.22)

In this case we have

\[
\mu'_r = \frac{b'^r - a'^r}{r(\log b - \log a)}
\] (3.4.23)

and

\[
M_0 = e \left( \frac{b'-a'-r(a' \log b-b' \log a)}{r(b'-a')} \right)
\] (3.4.24)

so that

\[
\frac{\mu'_r}{M'_0} = \frac{b'-a'}{r(\log b - \log a)} \left[ \frac{1}{e \left( \frac{b'-a'-r(a' \log b-b' \log a)}{r(b'-a')} \right)^r} \right]
\] (3.4.25)
5. MINIMUM AND MAXIMUM VALUES OF THE DIFFERENCE $\mu_i' - \mu_s'$

**Result 3.5.1**

For a continuous random variate which takes values in the interval $[a,b]$ where $a > 0$, we have

$$0 \leq \mu_i' - \mu_s' \leq \frac{a'b^r - a^r b'}{a^r - b^r} + \frac{r-s}{r} \left( \frac{s}{b^r - a^r} b^r - a^r \right)^{\frac{r}{r-s}}$$

(3.5.1)

where $r$ is a positive real number and $s$ is any non zero real number such that $r > s$.

**Result 3.5.2**

For a continuous random variate which takes values in the interval $[a,b]$ where $a > 0$, we have

$$\frac{a'b^r - a^r b'}{a^r - b^r} + \frac{r-s}{r} \left( \frac{s}{b^r - a^r} b^r - a^r \right)^{\frac{r}{r-s}} \leq \mu_i' - \mu_s' \leq 0$$

(3.5.2)

where $r$ and $s$ are negative real numbers such that $r > s$.

**Result 3.5.3**

For a continuous random variate which takes values in the interval $[a,b]$ where $a > 0$, we have

$$0 \leq \mu_i' - M_b \leq \frac{a^r \log b - b^r \log a}{\log b - \log a} + \frac{1}{r} \left( \frac{b^r - a^r}{\log b - \log a} \right) \left( \log \frac{b^r}{a^r} - 1 \right)$$

(3.5.3)

where $r$ is any non zero real number.

Inequalities (3.5.1), (3.5.2) and (3.5.3) can be proved in the similar ways as corresponding inequalities are proved for the case of discrete distributions in chapter II.

We now present a geometrical interpretation of these inequalities.

Let $r > s > 0$ then equation of straight line passing through the points $(a^r, a')$ and $(b^r, b')$ in $\mu_s' \mu_i'$ plane can be written as

$$\mu_i' - a' = \frac{b' - a'}{b^r - a^r} (\mu_s' - a')$$

or

$$\mu_i' = \frac{(b' - a') \mu_s' + a'b^r - a'b'}{b^r - a^r}$$

(3.5.4)

We now consider a function $f(\mu_s')$ defined by

$$f(\mu_s') = \mu_i'^r$$

(3.5.5)
This gives

$$f'(\mu'_s) = \frac{r}{s} \mu'_s^{r-1}$$  \hspace{1cm} (3.5.6)

and

$$f''(\mu'_s) = \frac{r}{s} \left( \frac{r}{s} - 1 \right) \mu'_s^{r-2}$$  \hspace{1cm} (3.5.7)

The second derivative of $f(\mu'_s)$ is positive in the interval $[a^t, b^t]$ therefore graph of function $f(\mu'_s)$ in the interval $[a^t, b^t]$ is concave upward. By Lagrange's mean value theorem, we have

$$f'(c) = \frac{f(b^t) - f(a^t)}{b^t - a^t}$$

$$\Rightarrow c = \left[ \frac{s b^t - a^t}{r b^t - a^t} \right]^{\frac{1}{r-s}}$$  \hspace{1cm} (3.5.8)

The tangent to the curve $f(\mu'_s)$ at the point P(c,d) is therefore parallel to the line (3.5.4) where

$$d = \left[ \frac{s b^t - a^t}{r b^t - a^t} \right]^{\frac{1}{r-s}}$$

The line perpendicular to $y$ axis and passing through the point P meets the line (3.5.4) at the point Q where

$$Q \leftrightarrow \left[ c, \frac{b^t - a^t}{s b^t - a^t} \left[ \frac{s b^t - a^t}{r b^t - a^t} \right]^{\frac{1}{r-s}} + \frac{a^t b^t - a^t b^t}{b^t - a^t} \right]$$

The distance between the points P and Q is given by

$$PQ = \frac{a^t b^t - a^t b^t}{b^t - a^t} \left[ \frac{s b^t - a^t}{r b^t - a^t} \right]^{\frac{1}{r-s}} + \frac{b^t - a^t}{b^t - a^t} \left[ \frac{s b^t - a^t}{r b^t - a^t} \right]^{\frac{1}{r-s}}$$  \hspace{1cm} (3.5.9)

From this we see that the distance between the points P and Q gives the maximum of the difference between $\mu'_s$ and $\mu'_t$. Further since line (3.5.4) and curve (3.5.5) are intersecting curves it follows that minimum value of the difference is zero. This is explained in figure 3.3. The geometrical significance of inequality (3.5.1) for the case when $s < 0$ is essentially same.

We now give a geometrical significance of inequalities (3.5.1) and (3.5.2).
Figure 3.3: For $r > s > 0$ the function $f(\mu'_s)$ defined by (3.5.5) is concave upward in the interval $[a^b,b^*]$, $a > 0$. The point $(\mu'_s, \mu'_r)$ in $\mu'_s, \mu'_r$-plane lies in a region bounded by the curve and straight line respectively given by equations (3.5.5) and (3.5.4). The straight line and curve intersect at two points namely $R \leftrightarrow (a^s, a^r)$ and $S \leftrightarrow (b^s, b^r)$. The abscissa of the point $P$ is obtained on solving equation \((b^r-a^r)f'(c) = f(b^r) - f(a^r)\). The point $Q$ has same abscissa and the distance $PQ$ gives the maximum of the difference $\mu'_r - (\mu'_s)^\frac{1}{r}$. The tangent line passing through the point $P$ is parallel to the line passing through the points $R$ and $S$. It is also clear from the figure that minimum value of the difference $\mu'_r - (\mu'_s)^\frac{1}{r}$ is zero.
Consider a function \( f(h) \) defined by
\[
 f(h) = \frac{(b' - a') h^r + a' b^r - a b^r}{b^r - a^r} \tag{3.5.10}
\]
where
\[
 h = \mu_{r}^{s} \tag{3.5.11}
\]
Also
\[
 \mu_{r}^{s} = h \tag{3.5.12}
\]
The curve \( f(h) \) and straight line (3.5.12) intersect at two points \((a', a')\) and \((b', b')\) in \( h_{r}^{s} \)-plane. By Lagrange's mean value theorem we have
\[
 f'(c) = \frac{f(b') - f(a')}{b' - a'}
\]
\[
 \Rightarrow \quad c = \left[ \frac{r b^r - a^r}{s b^r - a^r} \right]^{s-r}
\]
The tangent to the curve \( f(h) \) at the point \( P(c, d) \) is therefore parallel to the line (3.5.12) where
\[
 d = \left( b' - a' \right) \left( \frac{r b^r - a^r}{b^r - a^r} \right)^{s-r} + \frac{a' b^r - a b^r}{b^r - a^r}
\]
The line perpendicular to \( h\)-axis and passing through the point \( P \) meet the line (3.5.12) at the point \( Q \) where
\[
 Q \leftrightarrow \left[ c, \left( \frac{r b^r - a^r}{s b^r - a^r} \right)^{s-r} \right]
\]
The distance between points \( P \) and \( Q \) is given by
\[
 PQ = \left( b' - a' \right) \left( \frac{r b^r - a^r}{b^r - a^r} \right)^{s-r} + \frac{a' b^r - a b^r}{b^r - a^r} - \left( \frac{r b^r - a^r}{s b^r - a^r} \right)^{s-r} \tag{3.5.13}
\]
From this we see that the distance between the points \( P \) and \( Q \) gives the maximum of the difference between \( \mu_{r}^{s} \) and \( \mu_{s}^{r} \). Further since line (3.5.12) and the curve (3.5.10) are intersecting curves it follows that minimum value of the difference is zero. This is explained in figure (3.4).
We now present a geometrical significance of inequality (3.5.3). Consider a function \( f(h) \) defined by

\[
f(h) = \frac{(b' - a') \log h + a' \log b - b' \log a}{\log b - \log a}
\]

(3.5.14)

where

\[ h = M'_\mu \]

(3.5.15)

Also

\[ \mu_\ne' \ge h \]

(3.5.16)

The curve \( f(h) \) and straight line (3.5.16) intersect at two points \( (a', a') \) and \( (b', b') \) in the \( h\mu_\ne' \)-plane. By Lagrange's mean value theorem we have

\[
f'(c) = \frac{f(b') - f(a')}{b' - a'}
\]

\[ \Rightarrow c = \frac{b' - a'}{r \left( \log b - \log a \right)}
\]

The tangent to the curve \( f(h) \) at the point \( P(c, d) \) is therefore parallel to the line (3.5.16) where

\[
d = \frac{b' - a'}{r \left[ \frac{(b' - a')}{r \left( \log b - \log a \right)} + a' \log b - b' \log a \right]} + a' \log b - b' \log a
\]

\[
\log b - \log a
\]

The line perpendicular to \( h \)-axis and passing through the point \( P \) meet the line (3.5.16) at the point \( Q \) where

\[ Q \leftrightarrow [c, c] \]

The distance between \( P \) and \( Q \) is given by

\[
PQ = \frac{b' - a'}{r \left[ \frac{(b' - a')}{r \left( \log b - \log a \right)} + a' \log b - b' \log a \right]} - \frac{b' - a'}{r \left( \log b - \log a \right)}
\]

130
Figure 3.4: For \( r > s > 0 \) or \( r > 0 \) and \( s < 0 \) the geometrical significance of the bounds for the difference \( \mu'_r - \left( \mu'_s \right)^r \) can be obtained on substituting \( \left( \mu'_s \right)^r = h \). We get,

\[
\mu'_r = f(h) = \frac{(b' - a') h^r + a'b' - a'b'}{b' - a'}
\]

and

\[
\mu'_r = \left( \mu'_s \right)^r = h
\]

The point \((h, \mu'_r)\) in \( h, \mu'_r \)-plane lies in a region bounded by straight line \( \mu'_r = h \) and the curve \( \mu'_r = f(h) \). The straight line and curve intersect at two points namely \((a', a')\) and \((b', b')\). The abscissa of the point \( P \) is obtained from equation \((b' - a') f'(c) = f(b') - f(a')\). The point \( Q \) has same abscissa and the distance \( PQ \) gives the maximum of the difference \( \mu'_r - \left( \mu'_s \right)^r \). It is also clear from the figure that minimum value of the difference \( \mu'_r - \left( \mu'_s \right)^r \) is zero.

Similarly we can discuss the geometrical significance of inequality (3.5.2). It may be noted that in this case the curve \( f(h) \) is concave upward.
From this we see that the distance between the points P and Q gives the maximum of the difference between \( \mu'_i \) and \( M'_0 \). Further since line (3.5.16) and curve (3.5.14) are intersecting curves it follows that minimum value of the difference is zero. This is shown in figure (3.5).

**WHEN DO INEQUALITIES (3.5.1), (3.5.2) AND (3.5.3) REDUCE TO EQUALITIES**

The lower bound in inequality (3.5.1) is attained for the case when probability density function \( \phi(x) \) is given by

\[
\phi(x) = \delta(x-c), \quad a \leq c \leq b
\]

that is when probability mass is concentrated at one point. The upper bound in inequality (3.5.1) is attained for the case when

\[
\phi(x) = p \delta(x-a) + q \delta(x-b)
\]

where

\[
p = \frac{b^s - \left[ \frac{s}{r} \left( \frac{b^t - a^t}{b^s - a^s} \right) \right]^{\frac{s}{r}}}{} (3.5.19)
\]

and

\[
q = \frac{\left[ \frac{s}{r} \left( \frac{b^t - a^t}{b^s - a^s} \right) \right]^{\frac{s}{r}} - a^s}{b^s - a^s} (3.5.20)
\]

In this case we have

\[
\mu'_1 = \left[ \frac{s}{r} \left( \frac{b^t - a^t}{b^s - a^s} \right) \right]^{\frac{s}{r}} (3.5.21)
\]

and

\[
\mu'_r = \frac{b^t - a^t}{b^s - a^s} \left[ \frac{s}{r} \left( \frac{b^t - a^t}{b^s - a^s} \right) \right]^{\frac{s}{r}} + \frac{a'b^s - a^sb^r}{b^r - a^s} \quad (3.5.22)
\]

so that

\[
\mu'_r - \mu'_s = \frac{a'b^s - a^sb^r}{b^s - a^s} + \frac{r - s(\frac{b^t - a^t}{b^s - a^s})^{\frac{s}{r}}}{r} \left( \frac{s}{r} \left( \frac{b^t - a^t}{b^s - a^s} \right) \right)^{\frac{s}{r}}
\]

It also follows from above discussion that the upper bound in inequality (3.5.2) is attained for the case when probability density function is given by (3.5.17). The lower bound in
Figure 3.5: When \( r \) is any real number and \( s = 0 \) the geometrical significance of the bound for the difference \( \mu'_r - M'_0 \) can be obtained on substituting \( M'_0 = h \). We get

\[
\mu'_r = f(h) = \left( \frac{b'_r - a'_r}{r} \right) \log h' + \frac{a'_r \log b - b'_r \log a}{\log b - \log a}
\]

and

\[
\mu'_r = M'_0 = h
\]

The point \((h, \mu'_r)\) in \( h\mu'_r \)-plane lies in a region bounded by straight line \( \mu'_r = h \) and the curve \( \mu'_r = f(h) \). The straight line and curve intersect at two points namely \((a'_r, a'_r)\) and \((b'_r, b'_r)\). The distance \( PQ \) gives the maximum of the difference \( \mu'_r - M'_0 \) and it is also clear from the figure that minimum value of the difference \( \mu'_r - \left( \mu'_r \right)^i \) is zero.
inequality (3.5.2) is attained for the case when probability density function is given by (3.5.18). The lower bound in inequality (3.5.3) is attained for the case when probability density function is given by (3.5.17). The upper bound in inequality (3.5.3) is attained for the case when probability density function is given by
\[ \phi(x) = p \delta(x-a) + q \delta(x-b) \]
where
\[ p = \frac{1}{\log b - \log a} \left[ \log b - \frac{1}{r} \log \frac{b' - a'}{r(\log b - \log a)} \right] \quad (3.5.23) \]
and
\[ q = \frac{1}{\log b - \log a} \left[ \frac{1}{r} \log \frac{b' - a'}{r(\log b - \log a)} - \log a \right] \quad (3.5.24) \]
In this case we have
\[ M_0 = \left[ \frac{b' - a'}{r(\log b - \log a)} \right] \quad (3.5.25) \]
and
\[ \mu' = \left[ a' \log b - b' \log a + \frac{b' - a'}{r} \log \frac{b' - a'}{r(\log b - \log a)} \right] \frac{1}{\log b - \log a} \quad (3.5.26) \]
so that
\[ \mu' - M_0 = \frac{a' \log b - b' \log a}{\log b - \log a} + \frac{1}{r} \left( \frac{b' - a'}{\log b - \log a} \right) \left[ \log \frac{b' - a'}{r(\log b - \log a)} - 1 \right] \]