3.1. INTRODUCTION

Several authors have analysed the case of M/G/1 queueing system with bulk arrivals. Gaver (1959) seems to be the first to take up queues with bulk arrivals. In a way, the study of bulk queues may be said to have begun with Erlang's investigation of the model M/Bk/1; for its solution contains implicitly the solution of the model M^k/M/1, a Poisson queue where arrivals are in groups of size k. Gaver considers the case of a queueing system with compound Poisson input and general service time and finds transform of the steady state queue length distribution. Bhat (1968) analyses this system in detail in his monograph. Chaudhry (1979) discusses limiting queue size distribution at three different epochs; random epoch, epoch just before an arrival and epoch just after a departure, using supplementary variable technique.

In all the queueing models referred to above, an idle server remains alert awaiting a new arrival and will commence service immediately upon the customer's arrival.
The effect of vacation periods in queueing models is studied by several authors. Scholl and Kleinrock (1983) analyses an $M/G/1$ queueing system with vacations to the server. Assuming steady state exists, Fuhrmann and Cooper (1985) shows that for a class of $M/G/l$ queueing system with generalized vacations to the server, the 'decomposition property' holds. An $M/G/l$ model in which the server is required to search for customers is analysed by Neuts and Ramalhoto (1984). Keilson and Servi (1986b) analyses the case of blocking probability for $M/G/1$ vacation system with occupancy level dependent schedules. For more details on queueing systems with vacations to the server, one may refer to Doshi (1986).

All the above mentioned articles analyse the case of steady state distribution. Jacob and Krishnamoorthy (1987) using renewal theoretic arguments gives transient solution for a finite capacity $M/G/1$ queueing system with vacations to the server. Time dependent solution for a finite capacity $M/G^{a,b}/1$ queueing system with vacations to the server is given by Jacob and Madhusoodanan (1988). In this chapter, we extend these results to an infinite capacity $M/G/1$ queueing system with group arrivals and vacations to the server.
Here we consider a service facility with only one server. Customers arrive in groups of size \( G_n \) (\( n=1, 2, \ldots \)) with distribution

\[
\Pr \{ G_n = j \} = p_j, \quad j = 1, 2, \ldots \tag{3.1}
\]

These group arrivals occur according to a Poisson process with parameter \( \mu \). Service is one by one and the service times are independent and identically distributed random variables with distribution function \( G(.) \) having density function \( g(.) \). Whenever the system becomes empty, server goes for vacation for a random length of time. Vacation periods are independent and identically distributed random variables with distribution function \( H(.) \) having density function \( h(.) \).

Let us suppose that, at time zero the system starts with 'a' (\( > 0 \)) units in the waiting room. The server takes all the 'a' units to the service station and serves them one by one. When all the 'a' units are served, server goes back to the waiting room. If there is at least one unit waiting, server takes all of them to the service station and starts service. If there is nobody waiting for service, server goes for vacation for a random duration. On completion of this vacation, if there is at least one unit present in the
system server starts serving them. On the otherhand, if there is no unit in the waiting room, the server extends his vacation for one more period having the same probability distribution. This process is continued until there is at least one unit in the system waiting for service. According to the terminology of Doshi (1986), this vacation is known as multiple vacation. Since the server serves the customers in a continuous manner until all the customers are exhausted, the service discipline is exhaustive. Thus we have an $M^X/G/1$ multiple vacation system with exhaustive service discipline.

3.2. BASIC RESULTS

Let $A(t)$ denote the number of arrivals during $(o,t]$.

Let $\varphi(z) = \sum_{i=1}^{\infty} p_i z^i$ with $0 < \varphi'(1) < \infty$

and $p_j^{(k)} = \text{the coefficient of } z^j \text{ in } [\varphi(z)]^k$

This says that $p_j^{(k)}$ is the k-fold convolution of $\{p_j\}$ with itself

and $p_j^{(o)} = 0$ for $j > 0$

$= 1$ for $j = 0$
Therefore, for \( j = 0,1,2, \ldots \)

\[
\mu_j(t) = \Pr \{ A(t) = j \} = \sum_{k=0}^{j} \frac{e^{-\mu t} \mu^j}{j!} p_j(k)
\]

For \( i = 1,2, \ldots \), \( j = 0,1,2, \ldots \), we define the transition probability density functions as follows. Let \( f_{ij}(x)dx \) be the probability that starting at time zero, the service of \( i \) units is over in the interval \((x,x+dx)\) and there are \( j \) arrivals during the interval \((0,x]\).

So we get

\[
f_{ij}(x) = g^i(x) \mu_j(x)
\]

Let us define an infinite vector

\[
f_i(x) = (f_{i1}(x), f_{i2}(x), \ldots) \text{ for } i = 1,2, \ldots
\]

and \( f_0(x) = (f_{10}(x), f_{20}(x), \ldots)^T \)

where \( T \) denotes transpose.
Now we can define a matrix $F$ given by

$$F(x) = \begin{bmatrix}
  f_{11}(x) & f_{12}(x) & \cdots \\
  f_{21}(x) & f_{22}(x) & \cdots \\
  \vdots & \vdots & \ddots \\
  \vdots & \vdots & \ddots & \ddots
\end{bmatrix}$$

For $i = 1, 2, \ldots$, $(f_{1i} * \sum_{n=0}^{\infty} F^{*n})(x)$ will be a row vector of infinite order, where the vector $f_{1i}$ is convoluted with the columns of the matrix $\sum_{n=0}^{\infty} F^{*n}$. Now for $i, n = 1, 2, \ldots$, let

$$M_{1}^{\eta}(x) = \eta \text{th coordinate of the vector}(f_{1i} * \sum_{n=0}^{\infty} F^{*n})(x) \quad (3.4)$$

Also let,

$$K_{1}(x) = (f_{1i} * \sum_{n=0}^{\infty} F^{*n} * f_{0})(x) \quad (3.5)$$

Now we can find out the probability distribution of a busy period. Let $F_{1}(x)$ denote the probability density function of a busy period initiated by $i$ customers.

Then for $i = 1, 2, \ldots$, we have

$$F_{i}(x) = f_{i0}(x) + K_{i}(x) \quad (3.6)$$
For \( j = 1, 2, \ldots \), let \( b_j(x)dx \) be the probability that after a busy period, server goes for vacation at time zero and after one or more vacations, the next busy period starts in the interval \((x, x+dx)\) when there are \( j \) units present in the system.

Then for \( j = 1, 2, \ldots \),

\[
b_j(x) = \int_0^x \sum_{m=0}^{\infty} h^m(u) \mu_0(u) h(x-u) \mu_j(x-u) du \quad (3.7)
\]

The renewal points of the process are those time points at which the server goes for vacation after a busy period. Let \( Z \) be the time between two such renewal points. Then the probability density function of \( Z \) is given by

\[
k(t) = \int_0^t \sum_{j=1}^{\infty} b_j(u) F_j(t-u) du \quad (3.8)
\]

Then the renewal density function of the delayed renewal process is given by

\[
M(u) = \sum_{n=0}^{\infty} (F a^u k^n)(u) \quad (3.9)
\]

In the next section, we derive explicit expressions for the
time dependent system size probabilities at arbitrary epochs.

3.3. SYSTEM SIZE PROBABILITIES

Let $P_i(t)$ denote the probability that at time $t$, there are $i$ customers in the system including the one being served. Considering all the mutually exclusive and exhaustive cases, we have the following equations.

$$P_0(t) = \int_0^t M(u) \int_0^\infty h^m(v-u)[1-H(t-v)]\mu_0(t-v)dv \, du \quad (3.10)$$

For $i = 1, 2, \ldots, a$, we have

$$P_i(t) = \sum_{j=1}^{i} [G^{*a-j}(t) - G^{*a-j+1}(t)] \mu_{1-j}(t)$$

$$+ \int_0^t \sum_{k=1}^{i-1} M^k(u) \sum_{j=1}^{k} [G^{*k-j}(t-u) - G^{*k-j+1}(t-u)] \mu_{1-j}(t-u) \, du$$

$$+ \int_0^t \sum_{k=i}^{\infty} M^k(u) \sum_{j=1}^{i} [G^{*k-j}(t-u) - G^{*k-j+1}(t-u)] \mu_{1-j}(t-u) \, du$$
\[ + \int M(u) \int_0^t \sum_{k=1}^{i-l} b_k(v-u) \sum_{j=1}^k \left[ G^{*k-j}(t-v) - G^{*k-j+1}(t-v) \right] \]
\[ \times \mu_{i-j}(t-v) dv \, du \]

\[ + \int M(u) \int_0^t \sum_{k=1}^{i-l} b_k(v-u) \sum_{j=1}^k \left[ G^{*k-j}(t-v) - G^{*k-j+1}(t-v) \right] \]
\[ \times \mu_{i-j}(t-v) dv \, du \]

\[ + \int M(u) \int_0^t \sum_{\gamma=1}^{\infty} b_\gamma(v-u) \int_0^t \sum_{k=1}^\gamma \left[ G^{*k}(w-v) \right] \]
\[ \times \sum_{j=1}^k \left[ G^{*k-j}(t-w) - G^{*k-j+1}(t-w) \right] \mu_{i-j}(t-w) dw \, dv \, du \]

\[ + \int M(u) \int_0^t \sum_{\gamma=1}^{\infty} b_\gamma(v-u) \int_0^t \sum_{k=1}^\gamma \left[ G^{*k}(w-v) \right] \]
\[ \times \sum_{j=1}^k \left[ G^{*k-j}(t-w) - G^{*k-j+1}(t-w) \right] \mu_{i-j}(t-w) dw \, dv \, du \]

\[ + \int M(u) \int_0^t \sum_{\gamma=1}^{\infty} h^{*\gamma}(v-u) \mu_0(v-u) \left[ 1 - H(t-v) \right] \]
\[ \times \mu_i(t-v) dv \, du \]

(3.11)
For \( i = a+1, a+2, \ldots, \) we have

\[
P_i(t) = \sum_{j=1}^{a} \left[ G^{*a-j}(t) - G^{*a-j+1}(t) \right] \mu_{i-j}(t)
\]

\[
+ \int_{0}^{t} \sum_{k=1}^{i-1} M_{k}(u) \sum_{j=1}^{k} \left[ G^{*k-j}(t-u) - G^{*k-j+1}(t-u) \right]
\]

\[
x \mu_{i-j}(t-u) du
\]

\[
+ \int_{0}^{t} \sum_{k=1}^{\infty} M_{k}(u) \sum_{j=1}^{i} \left[ G^{*k-j}(t-u) - G^{*k-j+1}(t-u) \right]
\]

\[
x \mu_{i-j}(t-u) du
\]

\[
+ \int_{0}^{t} M(u) \int_{0}^{t} \sum_{k=1}^{i-1} b_{k}(v-u) \sum_{j=1}^{k} \left[ G^{*k-j}(t-v) - G^{*k-j+1}(t-v) \right]
\]

\[
x \mu_{i-j}(t-v) dv \ du
\]

\[
+ \int_{0}^{t} M(u) \int_{0}^{t} \sum_{k=1}^{\infty} b_{k}(v-u) \sum_{j=1}^{i} \left[ G^{*k-j}(t-v) - G^{*k-j+1}(t-v) \right]
\]

\[
x \mu_{i-j}(t-v) dv \ du
\]
The virtual waiting time in the queue at time $t$ is defined as the waiting time of a customer in the queue if it were to arrive at time $t$. Let $W_t$ be the virtual waiting time at time $t$. Here we compute the probability distribution of $W_t$ conditional on the state of the system at time $t$ and it is enough because the system size probabilities are known. Here we assume that service is in the order of their arrivals.
We consider the following cases separately.

Case (i): At time $t$, there are $i, i=1, 2, \ldots$ units in the system and the server is working.

Then

$$
\Pr \{ W_t \leq x \} = \int_0^t \sum_{j=0}^{i-1} \sum_{k=1-j}^{\infty} M_k(u) \mu_j(t-u) G^{k+j}(t+x-u) du \\
+ \int_0^t M(u) \sum_{\gamma=1}^i \sum_{v=0}^{\infty} \sum_{j=0}^{\infty} M^\gamma(v-u) \int_0^t \int_0^\infty h(w-v) G(w-v-t+x-w) dw dv du \quad (3.13)
$$

Case (ii): At time $t$, there are $i, i=1, 2, \ldots$, units in the system and the server is on vacation.

Then

$$
\Pr \{ W_t \leq x \} = \int_0^t \int_0^\infty h^i(v-u) \mu_0(v-u) dw dv \int_0^t G^i(t+x-w) dw dv du \quad (3.14)
$$
Case (iii): At time $t$, the system is empty and the server is on vacation.

Then

$$
\Pr \{ W_t \leq x \} = \int_0^t M(u) \sum_{\mu=0}^{\infty} h^\mu(v-u)[H(t+x-v)-H(t-v)]dv du
$$

(3.15)

Remark:

The queueing system $\text{M}^X/G/1$ without vacations to the server can be obtained as a special case of this model by taking the vacation period distribution as the distribution of the idle period. That is, the same exponential distribution of the interarrival time of customers.