Introduction

Finite temperature behaviour of quantum field systems has become a very relevant field of investigation with the advent of Grand Unified Theories (GUTS) [69,72,131,132]. Since the grand unification scale is cosmologically high \( T > 10^{15} \) GeV or \( t < 10^{-35} \) s, it should not come as a surprise that grand unified interactions may have played an important role in the 'earliest' stages of the evolution of the universe. In fact the unification of strong, weak and electromagnetic interactions into a unified field theory, can only occur at very high energies which cannot be reached, at present, in laboratories [66,67].

In the early universe, characterised by very high temperatures, the gauge symmetry of electroweak and grand unified theories were unbroken. As the universe cooled, a series of first order phase transitions may have taken place, which brought these symmetries into their present spontaneously broken form. The most popular mechanism [206-208] for the breaking of these symmetries is based on the Coleman-Weinberg model [158]. The phase transition in
Coleman-Weinberg type models is first order, but the rate of transition is very low [208–210]. For electroweak theory, Witten [210] has shown that the phase transition is driven by unexpected sources. He used a clever method by which he made the $\phi^4$ coupling term negative and temperature-dependent, which is the driving force for the phase transition. The same mechanism is used for GUTS too [206–8, 211]. Recent studies of the SU(5) model have shown that the first order phase transition occurs at a temperature of approximately 1 GeV, indicating extreme supercooling. In their calculations the temperature dependence of the coupling constant has not been taken into account. If the temperature dependence is taken into consideration, the amount of supercooling may be drastically reduced and the transition temperature may be raised to $2 \times 10^{10}$ GeV [211].

Renormalisation group studies reveal that the coupling constants vary with $q^2$ (the momentum transfer square), at which they are probed. The strong interaction coupling constant decreases as $q^2$ increases, while U(1) coupling constant increases slightly with $q^2$ (fig. 6.1). Collins and Perry [212] showed, using renormalisation group arguments, that the non-abelian gauge coupling constant of SU(3)$_c$ field theory of quarks and gluons decreases with density of the quark soup, and that perturbation theory becomes more reliable under such conditions. In the present
study we make an attempt to investigate the temperature
debehaviour of coupling constants. We have chosen two
models: i) $\phi^4$ theory ii) scalar electrodynamics (SED).
In the case of $\phi^4$ theory we find that when the mass square
is positive, the coupling constant decreases with tempera­
ture, while when $m^2$ is negative the coupling constant
increases with temperature. The calculations are verified
using the renormalisation group approach. In the case of
SED, the gauge coupling constant is found to increase with
temperature.

$\phi^4$ Theory

We consider a massive scalar particle theory defined by
the Lagrangian

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \right)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 \quad , \quad \lambda > 0 \quad (6.1)$$

The parameters appearing in the Lagrangian are bare or
unrenormalised. The renormalisation of these quantities
at zero temperature has been discussed elsewhere [213].
In the present study we are interested only on the renorma­
lisation of the coupling constant $\lambda$. The renormalised
coupling constant can be obtained by the vertex renormali­
sation procedure. The theory contains one kind of vertex
only (fig.6.2) which is of order $\lambda$. This single vertex can be represented by

$$\Gamma^1 = -i \lambda (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - k_4). \quad (6.2)$$

The vertex correction, in the lowest order, comes from three graphs (fig.6.3), and the correction is of order $\lambda^2$. The contributions from these graphs are identical, and hence it is sufficient to consider any one of them. Using the Feynman rules (Appendix-B), the contribution from the first graph is

$$\Gamma^2 = \frac{(-i\lambda)^2}{2} \int \frac{d^4k_5}{(2\pi)^4} \frac{d^4k_6}{(2\pi)^4} \frac{i}{(k_5^2 - m^2)} \frac{i}{(k_6^2 - m^2)} \delta^4(k_1 + k_2 - k_5 - k_6). \delta^4(k_5 + k_6 - k_3 - k_4). \quad (6.3)$$

The aim of the present study is to obtain an expression for the temperature-dependent coupling constant. This is achieved by translating the zero temperature formalism to finite temperature formalism by using the prescriptions stated in chapter 1. Thus we can find that $\Gamma^2$ is temperature independent while $\Gamma^1$ is temperature dependent.
Rewriting $\Gamma^2$ in the form

$$\Gamma_\beta = \frac{(-i\lambda)^2}{2} \frac{1}{(-i\beta)} \sum \sum \int \frac{d^3k_5}{(2\pi)^3} \frac{d^3k_6}{(2\pi)^3} \frac{1}{(\omega_{n_1}^2-k_5^2-m^2)} \frac{1}{(\omega_{n_2}^2-k_6^2-m^2)}$$

$$(2\pi)^6 (-i\beta)^2 \delta^3(k_1 + k_2 - k_5 - k_6) \delta(\omega_{n_1} + \omega_{k_2} - \omega_{k_5} - \omega_k).$$

$$\delta^3(k_5 + k_6 - k_3 - k_4) \delta(\omega_{n_5} + \omega_{k_6} - \omega_{k_3} - \omega_{k_4}).$$

(6.4)

Here $\beta \sim \frac{1}{T}$ (the inverse of temperature). The calculations can be simplified by putting the external momenta to zero. Thus we find

$$\Gamma_\beta = \frac{(-i\lambda)^2}{2} \frac{1}{(-i\beta)} \sum \int \frac{d^3k_3}{(2\pi)^3} \frac{1}{(\omega_{n_1}^2-k_3^2-m^2)^2}.$$ 

$$\delta^3(k_1 + k_2 - k_3 - k_4) \delta(\omega_{n_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4}).$$

(6.5)

Expressing $\Gamma_\beta$ as

$$\Gamma_\beta = \Gamma_\beta^1 + \Gamma_\beta^2.$$
where

$$\lambda_\beta = -i\lambda_\beta (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - k_4).$$

$\lambda_\beta$ is identified to be the temperature-dependent coupling constant. Thus the temperature-dependent coupling constant $\lambda_\beta$ in the one-loop approximation, taking into account the contributions from the remaining graphs also,

$$\lambda_\beta = \lambda - \frac{3\lambda^2}{2} \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\omega_n^2 - k^2 - m^2)^2}$$

(6.6)

Setting $\omega_n^2 = k^2 + m^2$, and substituting for $\omega_n$ as

$$\omega_n = \frac{2\pi \hbar}{\beta} (n = 0, \pm 1, \pm 2, \ldots),$$

we obtain

$$\lambda_\beta = \lambda - \frac{3\lambda^2}{2\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{(4\pi \hbar^2 + \omega_n^2)^2}$$

(6.7)

The summation is done first in the following way

$$\sum_n \frac{1}{(4\pi \hbar^2 + \omega_n^2)^2} = \frac{1}{2E_k} \frac{dE_k}{dE_k} \left[ \sum_n \frac{1}{(4\pi \hbar^2 + \omega_n^2)^2} \right]$$

(6.8)

The identity [191]

$$\sum_{n=1}^{\infty} \frac{1}{(n^2 + a^2)} = - \frac{1}{2a^2} + \frac{\pi}{2a} \coth(\pi a)$$
gives

\[
\sum_{n=\infty}^{\infty} \frac{1}{\left(\frac{4\pi^2 n^2}{\beta^2} + E_k^2\right)} = \beta \left( \frac{1}{2E_k} + \frac{1}{E_k(e^{\beta E_k} - 1)} \right)
\]

This enables us to write (6.8) in the following form

\[
\sum_{n=\infty}^{\infty} \frac{1}{\left(\frac{4\pi^2 n^2}{\beta^2} + E_k^2\right)^2} = \beta \left[ -\frac{1}{4E_k^3} + \frac{1}{2E_k(e^{\beta E_k} - 1)} \right]
\]

Thus (6.6) can be cast into the form

\[
\Lambda = \lambda - \frac{3\lambda^2}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{4E_k^3} + \frac{1}{2E_k^3(e^{\beta E_k} - 1)} \right]
\]

The temperature-independent term cancels with the renormalisation counter term at zero temperature. Hence the \( \Lambda \) appearing on the r.h.s of (6.10) will be renormalised one. Thus

\[
\Lambda = \lambda - \frac{3\lambda^2}{4} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{E_k^3(e^{\beta E_k} - 1)} + \frac{\beta e^{\beta E_k}}{E_k^2(e^{\beta E_k} - 1)^2} \right]
\]
Putting \( x^2 = \beta^2 k^2 \), \( \beta m = a \), we find

\[
\lambda_\beta = \lambda + \frac{3\lambda^2}{4\pi^2} \int_0^\infty k^2 dk \frac{d}{dk^2} \left[ \frac{1}{\left(k^2 + m^2\right)^{1/2}} \right] \frac{\beta \left(k^2 + m^2\right)^{1/2}}{\left(k^2 + m^2\right)^{1/2} - 1}
\]

(6.13)

Integrating by parts, we get

\[
\lambda_\beta = \lambda - \frac{3\lambda^2}{8\pi^2} \int_0^\infty dx \left[ \frac{1}{\left(x^2 + a^2\right)^{1/2}} \right] \frac{\beta \left(x^2 + a^2\right)^{1/2}}{\left(x^2 + a^2\right)^{1/2} - 1}
\]

This integral can be evaluated in the high temperature limit \((T >> m)\), yielding

\[
\lambda_\beta = \lambda - \frac{3\lambda^2}{8\pi^2} \left[ \frac{\pi}{2m\beta} + \frac{1}{2} \ln\left(\frac{\beta m}{4\beta}\right) \right]
\]
This may be rewritten as

\[
\lambda_\beta = \lambda - \frac{3\lambda^2}{8\pi^2} \left[ \frac{m^2}{2m} + \frac{1}{2} \ln \left( \frac{m}{4\pi T} \right) \right] \quad (6.14)
\]

\[
= \lambda \left[ 1 - \frac{3\lambda}{8\pi^2} \left( \frac{m}{2m} \right) + \frac{1}{2} \ln \left( \frac{m}{4\pi T} \right) \right].
\]

This shows that in the absence of SSB (i.e., \( m^2 > 0 \) in (6.1)), the scalar coupling constant is seen to decrease with temperature. If we neglect the \( \ln T \) term—which is justified at the high temperature—then, a critical temperature \( T_a \) may be defined corresponding to the vanishing of the coupling constant:

\[
T_a = \frac{16\pi m}{3\lambda} - \frac{m}{\pi} \ln \left( \frac{m}{4\pi} \right) \quad (6.15)
\]

The temperature \( T_a \) signals the emergence of a non-interacting phase for the field system. It is known that at zero temperature the model can possess asymptotic freedom only for a negative value of \( \lambda \) [214]. The present result must be contrasted with this because it holds for the physically interesting case of positive \( \lambda \). If the onset of 'no-interaction' is a genuine phase transition, then it should also work in the reverse. It is not difficult to glean a few illustrations from physics where
forces get weakened with rise of temperature. The rupturing of chemical bonds and disappearance of the phonon field picture in solids under thermal agitation are examples that bring out this mechanism at least in a qualitative manner. However, it is hoped that a clear and specific quantitative comparison regarding the behaviour of scalar coupling constants can be made by studying the anharmonic vibration in the context of a self-coupled phonon field at finite temperature.

We encounter an imaginary mass \( m^2 < 0 \) in (6.1) for a theory with SSB. However the imaginary terms may be separated, and it is hoped that they will disappear when higher order effects are taken into account [112], the resulting expression for \( \lambda_\beta \) now reads

\[
\lambda_\beta = \lambda + \frac{3\lambda^2}{16\pi^2} \ln(4\pi T),
\]  

(6.16)

It is seen that with SSB, \( \lambda \) increases uniformly with temperature. This provides a justification for Witten's well-known recipe [210]. The present result may be applied to the Ginzburg-Landau model for superconductivity which predicts a variation of the penetration depth \( \delta \) with the quadratic coupling constant \( \lambda : \delta \sim \sqrt{\lambda} \) [215]. At high temperatures, since \( \lambda \) increases, the penetration depth \( \delta \) increases and as a result, superconductivity is inevitably lost.
Renormalisation Group Approach

The above calculations can be checked using the renormalisation group [216]. We shall use a temperature-dependent, mass-independent renormalisation programme [217]. The RG equation for the present problem can be written in the form

\[ [T \frac{\partial}{\partial T} + \beta \frac{\partial}{\partial \lambda} + (1 + \gamma_m) \frac{\partial}{\partial m} + \gamma \int d^4x \frac{\partial}{\partial \phi_c} \left( \phi_c(x) \frac{\delta}{\delta \phi_c(x)} \right) \] \[ \frac{\delta}{\delta \phi_c(x)} \Gamma = 0 \]

(6.17)

where \( \beta, \gamma \) and \( \gamma_m \) are the RG functions. Following Coleman and Weinberg [155], the above equation can be recast into the following forms:

\[ (T \frac{\partial}{\partial T} + \beta \frac{\partial}{\partial \lambda} + (1 + \gamma_m) \frac{\partial}{\partial m} + \gamma \frac{\partial \phi_c}{\partial \phi_c} )v^T(\phi_c) = 0, \]

(6.18)

\[ (T \frac{\partial}{\partial T} + \beta \frac{\partial}{\partial \lambda} + (1 + \gamma_m) \frac{\partial}{\partial m} + \gamma \phi_c \frac{\partial}{\partial \phi_c} + 2\gamma )z = 0, \]

(6.19)

Where \( v^T(\phi_c) \) is the effective potential at finite temperature and \( z \) is the scale of the field. The temperature-dependent effective potential at the one-loop level evaluated in the high temperature limit is [112]

\[ v^T(\phi_c) = \frac{1}{2} m^2 \phi_c^2 + \frac{\lambda}{4} \phi_c^4 + \frac{\lambda^4}{64\pi^2} \ln(\frac{T^2}{4\pi^2 m^2}) + \frac{\lambda^2 T^2}{24} - \frac{\lambda^3 T}{12\pi} \]

(6.20)
where
\[ M^2 = m^2 + \frac{1}{2} \lambda \phi_c^2 \]

It is convenient to use
\[ V^{(4)} = \frac{\delta^4 V}{\delta \phi_c^4} \]
in place of \( V^T \) in (6.18). Introducing a scale parameter \( t \),
\[ t = \ln(T/m), \quad (6.21) \]
equations (6.15) and (6.16) can be rewritten as
\[ (- \frac{\partial}{\partial t} + \overline{\beta} \frac{\partial}{\partial \lambda} + \overline{\gamma} \phi_c \frac{\partial}{\partial \phi_c} + 4\overline{\gamma})V^{(4)} = 0, \quad (6.22) \]
\[ (- \frac{\partial}{\partial t} + \overline{\beta} \frac{\partial}{\partial \lambda} + \overline{\gamma} \phi_c \frac{\partial}{\partial \phi_c} + 2\overline{\gamma})Z = 0, \quad (6.23) \]
where
\[ \overline{\beta} = \beta/\gamma_m \]
\[ \overline{\gamma} = \gamma/\gamma_m. \]

Imposing the renormalisation conditions [155],
we can find the zero-loop values for $V^{(4)}$ and $Z$:

$$V^{(4)}(0, \lambda) = \lambda \quad (6.24)$$

$$Z(0, \lambda) = 1 \quad (6.25)$$

Combining these equations with (6.22) and (6.23) we obtain

$$\bar{\gamma} = \frac{1}{2} \frac{\partial}{\partial t} Z(0, \lambda) \quad (6.26)$$

$$\bar{\beta} = \frac{\partial}{\partial t} V^{(4)}(0, \lambda) - 4\bar{\gamma} \lambda \quad (6.27)$$

Thus, we can find $\bar{\beta}$ and $\bar{\gamma}$ exactly, if we know the derivatives that occur on the right side of the above equations. The required result can be obtained if we go for the one-loop calculations. We will denote the temperature dependent coupling constant by $\lambda_{\beta}$ such that the RG function $\bar{\beta}$ satisfies the ordinary differential equation [218]

$$\frac{d}{dt} \lambda_{\beta} = \bar{\beta} (\lambda_{\beta}) \quad (6.28)$$

with the boundary condition $\lambda_{\beta}(t=0) = \lambda$. In the one-loop approximation the value of $V^{(4)}$ can be obtained from (6.20):

$$V^{(4)}(t, \lambda) = \lambda - \frac{3\lambda^2}{8\pi^2} \left( \frac{\pi}{2} \frac{T}{m} - \frac{1}{2} \ln(T/m) \right) \quad (6.29)$$
The one-loop correction to $Z$ vanishes; hence

$$Z = 1. \quad (6.30)$$

These results yield

$$\bar{\gamma} = 0, \quad (6.31)$$

and

$$\bar{\beta} = -\frac{3\lambda^2}{8\pi^2} (\frac{n}{2} e^t - \frac{1}{2} t). \quad (6.32)$$

The differential equation (6.28) now reads

$$\frac{d\lambda_{\beta}}{dt} = -\frac{3\lambda^2}{8\pi^2} (\frac{n}{2} e^t - \frac{1}{2} t) \quad (6.33)$$

On integrating this equation we find

$$\lambda_{\beta} = \frac{\lambda}{1 - \frac{3\lambda}{16\pi^2} \ln(T/m) + \frac{3\lambda}{16\pi} (T/m)} \quad (6.34)$$

Thus the RG calculation gives an improved expression for $\lambda_{\beta}$ - the temperature-dependent scalar coupling constant. But it is remarkable to note that both perturbation and RG calculations predict the same type of thermal behaviour for the coupling constant.
Scalar Electrodynamics

We shall now extend the above calculations to the case of gauge coupling constant in SED described by the Lagrangian

\[ \mathcal{L}_e = (\partial_\mu + i e A_\mu) \overline{\varphi} (\partial^\mu - i e A^\mu) \varphi - m^2 (\varphi^\dagger \varphi) - \lambda (\varphi^\dagger \varphi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]  

(6.35)

In this case two vertices are to be considered with respect to the gauge coupling constant $e$. The vertices are as shown in figs. 6.4 and 6.5 which are of orders $e$ and $e^2$ respectively. But it can be seen that the lowest order vertex correction at finite temperature to the vertex fig. 6.4 vanishes. The vertex represented in fig. 6.5 is expressed as

\[ \Gamma^1 = 2 e^2 g_{\mu\nu} (2\pi)^4 \delta^4 (k_1 + p_1 - p_2 - k_2). \]  

(6.36)

The lowest order correction to this vertex is given by fig. 6.6, and using the Feynman rules (Appendix-B) the contribution from this graph is

\[ \Gamma^2 = \frac{2 e^4 g_{\mu\nu}}{(2\pi)^4} \frac{d^4 k_3}{(2\pi)^4} \frac{d^4 k_4}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{(k_1 + k_3)(k_2 + k_4)}{(k_3^2 - m^2) q^2 (k_4^2 - m^2)} \]

\[ (2\pi)^8 \delta^4 (k_2 - k_4 - q) \delta^4 (k_3 - p_1 - p_2 - k_4). \]  

(6.37)
As before, $\Gamma^2$ is temperature-dependent and hence

$$\Gamma^2 = 2\varepsilon^4 g_{\mu\nu} \frac{1}{\lambda} \sum_n \frac{d^4k}{(2\pi)^3} \frac{1}{(\omega_n^2 - k^2 - m^2)^2}.$$ 

$$(2\pi)^3(-i\beta) \delta^3(k_1 + p_1 - p_2 - k_2) \delta(\omega_n + \omega_2 - \omega_3 - \omega_4). \quad (6.38)$$

Expressing $\Gamma_\beta$ as

$$\Gamma_\beta = \Gamma^4 + \Gamma^2_\beta,$$

where

$$\Gamma_\beta = 2ie^2 g_{\mu\nu}(2\pi)^4 \delta^4(k_1 + p_1 - p_2 - k_2). \quad (6.39)$$

Here $e_\beta$ is defined as the temperature-dependent gauge coupling constant. Thus we obtain

$$e^2_\beta = e^2[1 + \frac{e^2}{\beta} \sum_n \frac{d^3k}{(2\pi)^3} \frac{1}{(\omega_n^2 - k^2 - m^2)^2}] \quad (6.40)$$

It is now straightforward to see that (see (6.7)), in the high temperature limit, the temperature-dependent gauge coupling constant

$$e^2_\beta = e^2[1 + \frac{e^2}{4\pi^2} \frac{\pi T}{2m} - \frac{1}{2} \ln \left(\frac{4\pi T}{m}\right)] \quad (6.41)$$
Thus we can see that the gauge coupling constant in SED increases with temperature.

We have studied two models wherein the scalar and gauge coupling constants are temperature-dependent, SSB is the critical factor that determined the nature of the temperature variation. In the early universe when there was thermal equilibrium, the temperature of relativistic particles must have varied with time approximately as $T \sim (t)^{-1/2}$ [72]. This implies that the coupling constants could vary with time under conditions of thermal equilibrium, thus realising Dirac's [219,220] hypothesis of time variation of constants of nature.
Fig. 6.1 The qualitative behaviour of coupling constants with energy

![Graph showing coupling constants](image)

Fig. 6.2 The bare vertex in $\varphi^4$ theory

![Bare vertex diagram](image)

Fig. 6.3 Vertex correction in $\varphi^4$ theory at the one loop level

![Vertex correction diagrams](image)
Bare vertices in SED

Fig. 6.4

Fig. 6.5

Fig. 6.6 Vertex correction of the order of $e^2$ in SED