P R E F A C E

This is a report of the investigations carried out by the author under the guidance of Dr. V.M. Nandakumaran in the Department of Physics, Cochin University of Science and Technology, during the period 1985-89. It essentially deals with the onset and characterisation of chaos in a nonlinear deterministic model.

The discovery that the evolution of low dimensional nonlinear systems with a set of completely deterministic equations can be very complex or 'chaotic' has triggered an upsurge of research interest among physicists today. The study of simple nonlinear models describing various natural phenomena has resulted in the emergence of a new area of research now known as 'deterministic chaos'. Within a short period of time, a number of important breakthroughs have been achieved including the establishment of universal properties at the transition to chaos and the development of the important concept of a 'strange attractor'. For
the time being most of our knowledge of chaotic behaviour comes from studies on discrete nonlinear mappings. An important example is the so called logistic equation given by

\[ x_{t+1} = 4 \lambda x_t (1 - x_t) \]

which maps the unit interval on the real line into itself and \( \lambda \in [0,1] \) is a control parameter determining the asymptotic behaviour of the system.

Now, eventhough one dimensional maps have been thoroughly investigated, studies on higher dimensional noninvertible maps are comparatively very few and their properties are less well understood today.

We introduce a logistic type model in two dimensions

\[ x_{t+1} = 4 \lambda_t x_t (1 - x_t) \]
\[ \lambda_{t+1} = 4 \mu \lambda_t (1 - \lambda_t) \]

where the role of the control parameter is played by \( \mu \in [0,1] \). In other words, we consider a situation where the parameter of the logistic map at any instant depends on its value at the previous instant in a nonlinear way. As suggested by Ruelle, many time evolutions occurring in nature are of this type with 'adiabatically fluctuating parameters'. The
work reported in the thesis is devoted to the study of this 'modulated' logistic map. We investigate this model thoroughly from the viewpoint of deterministic chaos and establish many interesting properties using analytical as well as numerical methods.

The thesis contains five chapters. First chapter provides a brief introduction to the subject of deterministic chaos. In this, we discuss the basic ideas of deterministic chaos and their importance in the study of nonlinear dynamical systems. A brief discussion of the period doubling and other commonly observed routes to chaos has been included. Moreover, the concepts of Lyapunov Exponent and fractal dimension, which are important quantitative measures of chaos are also introduced.

In Chapter II, certain simple models which have played a central role in the development of the subject are presented. We then give a brief review of the universal properties of one dimensional maps. Then the modulated logistic map is introduced which forms the subject matter for subsequent chapters. The importance of this model in understanding the chaotic behaviour of certain higher dimensional systems of current interest is also indicated.
In Chapter III, we study the onset of chaos in the modulated logistic map. It is shown that, as the parameter $\mu$ is increased, the map turns chaotic following an infinite cascade of period doubling bifurcations. By plotting the bifurcation structure of the map in the $(X_t, \mu)$ plane, we show that it is modified qualitatively as well as quantitatively from the first bifurcation onwards. A linear stability analysis is performed to determine the fixed points of the map analytically. A very interesting property of our map is the existence of the universal behaviour of Feigenbaum at the transition to chaos. By using the approximate renormalisation methods due to Derrida et al and Helleman, we calculate the bifurcation ratio $\delta$ and the scaling factor $\alpha$ of the map. Their values suggest that the map should be included in the universality class of Feigenbaum. A detailed analysis of the 3-dimensional bifurcation structure of the map in the region $\mu > \mu_\infty$ gives numerical evidence for the fact that the map has infinite number of periodic windows which are arranged along the parameter axis exactly as in the case of the logistic map. The importance of this result lies in the fact that this ordering — known as Sarkovskii ordering — is thought to be characteristici
of only one hump maps and so far no higher dimensional maps are known to show this property.

In Chapter IV, we study the chaotic region of the map making use of some important quantitative measures. We mainly concentrate on the Lyapunov exponent and the fractal dimension which are quite general and are applicable to a wide spectrum of problems. The former is the rate of exponential divergence of nearby trajectories and hence is a measure of the loss of predictability, whereas the latter is a quantitative measure of the strangeness of the attractor. We calculate the two Lyapunov exponents of the map for a number of parameter values and show that the Lyapunov exponent along the $X$-direction is always negative. It is then shown that the map can generate uncountable number of strange attractors in the unit square. We present numerical plot of the attractors for various values of $\mu$ and the self-similar structure is explicitly shown in one case. The fractal dimension $D_0$ and the information dimension $D_1$ of a number of strange attractors corresponding to various values of $\mu$ have been calculated and in most of the cases, found to lie between 1 and 2 confirming the fractal structure.
The last Chapter provides summary and evaluation of investigations presented in the preceding chapters. An important conclusion that we derive is that the 'modulated' logistic map can be considered as an analogue of the logistic map in two dimensions. Other important results are also specified and possibilities for future investigations are indicated.

Part of the investigations presented in the thesis has provided materials for the following publications.