Although the interest and study on ocean waves date back to antiquity, the first systematic research effort to study the characteristics of ocean waves were started during the World War II. Since then considerable attention has been paid to the study of spectral and statistical characteristics of wind generated waves. A review of the relevant literature is undertaken in the following sections.

2.1. WAVE SPECTRUM

The term 'wave energy spectrum' is derived from the concept that a random wave field is characterised by the superposition of a large number of linear progressive waves with different heights and periods. Generally the wave spectrum is presented as a plot of the component wave energies against wave frequencies.

The actual water surface profile of wind-generated ocean waves varies widely in time as well as in space, and hence the instantaneous surface elevation above the still water level, \( \eta \), at position \( x \) and time \( t \) is expressed as

\[
\eta(x,t) = \sum a_i \cos(k_i x - \omega_i t + \theta_i) \quad \ldots \ldots (2.1)
\]
where \( k \) and \( \omega \) are the wave number and angular frequency respectively. \( \phi \) is the phase angle and is assumed to be uniformly distributed over the interval \((0, 2\pi)\).

Mathematically, the spectral density function is defined as the Fourier transform of the autocorrelation function and is given by

\[
S(f) = \int_{-\infty}^{\infty} R(\tau) \exp \left[ -i2\pi f \tau \right] d\tau \quad \ldots \ldots (2.2)
\]

where \( R(\tau) \) is the autocorrelation function which is the average lagged product of the neighbouring values. For a stationary ergodic process with lag \( \tau \) it is defined as

\[
R(\tau) = E[x(t) x(t+\tau)] = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} x(t)x(t+\tau) \, dt \quad \ldots \ldots (2.3)
\]

Since \( S(f) \) and \( R(\tau) \) form a Fourier Transform pair,

\[
R(\tau) = \int_{-\infty}^{\infty} S(f) \exp \left[ -i2\pi f \tau \right] df \quad \ldots \ldots (2.4)
\]

The \( S(f) \) defined above is two-sided, but \( f \) can never be negative. Hence, physically realisable one-sided spectral density function, keeping the area under the spectrum the same, is obtained from

\[
S(f) = 2 \int_{0}^{\infty} R(\tau) \exp \left[ -i2\pi f \tau \right] d\tau \quad \text{for } 0 \leq f \leq \infty
\]
\[
= 0 \quad \text{for } f < 0 \quad \ldots \ldots (2.5)
\]
$S(f)$ can be determined directly as well as from the observed time series. From Eqs. (2.3-2.5) we obtain

$$R(0) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x^2(t) \, dt = \int_0^\infty S(f) \, df \quad \ldots (2.6)$$

If $x(t)$ is the instantaneous sea surface elevation with zero mean it follows that

$$R(0) = \sigma_x^2 \quad \ldots (2.7)$$

is the variance of the sea surface elevation and is equal to the area under the spectral curve.

For a deterministic linear wave the total energy is given by

$$E = \rho g a^2/2 \quad \ldots (2.8)$$

and hence the total energy of a random wave

$$E \propto (1/2) \sum a_i^2 \quad \ldots (2.9)$$

Kinsman (1965) has shown that

$$\sum (1/2) a_i^2 \propto S(f) \Delta f \quad \ldots (2.10)$$

That is, energy of irregular waves is proportional to the spectral density function. Hence, the total energy

$$E \propto \int_0^{\infty} S(f) \, df \quad \ldots (2.11)$$
From Eqs. (2.6), (2.7) and (2.11) it follows that the area under the wave spectrum gives the variance of the surface elevations and the total energy of the irregular wave system. The integration of the wave spectrum with different powers of frequencies yield different height and period statistics. Also, it is possible to reconstitute the time series of the sea surface fluctuations from the spectrum and this method is usually adopted to generate random waves in the laboratory.

2.2. DEEP WATER WAVE SPECTRAL MODELS

It is impossible to model wave spectrum using the basic mathematical-physical laws owing to the complexity of the wind-wave generation processes. As a result, the void due to the lack of a spectral function has been filled in by various empirical or semi-empirical models.

2.2.1. Phillips Spectrum

Most of the recent spectral models can be traced back to the spectral function proposed by Phillips (1958). He used the dimensional analysis to derive the upper limit of the equilibrium or saturation range of the spectral form for the deep water conditions and established an \( f^{-5} \) dependence. The functional form is given by

\[
S(f) = \alpha g^2 (2\pi)^{-4} f^{-5} \quad ; \quad f \geq f_m \quad \ldots \ldots \quad (2.12)
\]
This form accounts for the frequencies higher than $f_m$ only. For practical applications, rather than the high frequency range, the energy content of the spectrum is more important. However, this served as a stepping stone to the studies on the ocean wave spectrum.

2.2.2. Pierson-Moskowitz Spectrum

Based on Phillips' equilibrium theory and some additional similarity analysis by Kitaigorodskii (1962), Pierson and Moskowitz (1964) proposed a continuous functional form for fully developed sea spectrum. The spectral density is given by

$$S(f) = \alpha g^2 (2\pi)^{-4} f^{-5} \exp\left[-(5/4)\left(f/f_m\right)^{-4}\right] \quad \ldots \quad (2.13)$$

Although a large number of field and laboratory observations did provide data to support this model, there are still some difficulties in its practical application. This model simulates the high frequency range better than the portion near the spectral peak where the energy is concentrated (Huang et al., 1981). It is based on the physical conditions of equilibrium (saturated or fully developed sea state) which is an ideal condition rather than the actual in the field. As a result, the use of this model is limited in the real field conditions.
2.2.3. JONSWAP Spectrum

The efforts to arrive at a generalized spectral function for an unsaturated sea culminated in the Joint North Sea Wave Project experiments and Hasselmann et al. (1973, 1976) proposed the now well known JONSWAP spectral model for the fetch-limited (unsaturated) sea conditions. The spectral density of this form is given by

\[ S(f) = \alpha f g^2 (2\pi)^{-4} f^{-5} \exp \left[ -\left(\frac{5}{4}\frac{f}{f_m}\right)^4 \right] \gamma q \ldots (2.14) \]

where, \( q = \exp \left[ -\left(\frac{f-f_m}{2}\right)^2 / (\delta f_m)^2 \right] \ldots (2.15) \)

and \( \sigma = \begin{cases} \sigma_a & ; f \leq f_m \\ \sigma_b & ; f > f_m \end{cases} \ldots (2.16) \)

This model is basically the P-M (Pierson-Moskowitz) model (Eq.2.13) with a peak enhancement factor, \( \gamma q \). \( \gamma \) is the ratio of the maximum spectral density to the corresponding maximum derived from the P-M spectrum. When \( \gamma = 1 \) Eq.(2.14) reduces to Eq.(2.13). The mean value of \( \gamma \) for all the JONSWAP data is around 3.3. The average values of \( \sigma_a \) and \( \sigma_b \) are given as 0.07 and 0.09 respectively.

In order to use Eq.(2.14) one will have to determine the five free parameters, all of which are given empirically as functions of the non-dimensional peak frequency which cannot be determined 'apriori'. Further, it is developed for
the fetch limited developing sea state cases only. As opined in Huang et al. (1981) whether it also fits in un-saturated decaying sea is questionable.

2.2.4. Neumann Spectrum

The above well known models involve a number of parameters which differ from model to model and are not known for all seas. For general applicability the theoretical spectrum must be expressed in terms of some common parameters which are readily available. Attempts were made in this direction and a few empirical/semi-empirical models were proposed. They are expressed mainly in terms of the total variance \( (m_0) \) and the frequency corresponding to the maximum spectral density \( (f_m) \). Working in the above direction Neumann (1953) proposed a spectral model with functional form:

\[
S(f) = 24\left(\frac{m_0}{f_m}\right)\left(\frac{f}{f_m}\right)^{-6}\left(\frac{3}{\pi}\right)^{1/2} \exp \left[-3\left(\frac{f}{f_m}\right)^{-2}\right]
\]

\[
.....(2.17)
\]

As this was the first analytically expressed spectral form, it was widely used till 1964. This model assumes an \( f^{-6} \) dependence in the high frequency region, unlike the subsequent models most of which assume an \( f^{-5} \) dependence in this part of the spectrum. However, some of the later studies (Hasselmann et al., 1973; Dattatri, 1978; Narasimhan and Deo, 1979a,b; Goda, 1983; Baba and Harish, 1986; etc.) show that higher values are possible in the deep water.
2.2.5. Darbyshire Spectrum

Based on the wind-wave data collected from the Atlantic Ocean, Darbyshire (1959) proposed an empirical spectral model, the functional form of which may be written as

\[ S(f) = 23.9 m_0 \exp \left[ -\frac{(f-f_m)^2}{0.0085(f-f_m+0.042)} \right]^{1/2} \tag{2.18} \]

This form was derived from the plot of energy densities of 64 wave records selected in such a way that the effect of extraneous swell was insignificant. In a later study, this was modified by Darbyshire (1963) to incorporate the effect of small fetch, by replacing \((f-f_m)\) with \(y(f-f_m)\), where

\[ y = \frac{X^3+3X^2+65X}{X^3+12X^2+260X+80} \tag{2.19} \]

\(X\) being the fetch in nautical miles. But, Burling (1963) observed that this modification is unnecessary since the spectral densities at high frequencies usually decrease with fetch.

2.2.6. Bretschneider Spectrum

Another model, as a modification of the Neumann spectrum was put forward by Bretschneider (1963). The spectral density of this model is given by

\[ S(f) = 5\left(\frac{m_0}{f_m}\right)\left(\frac{f}{f_m}\right)^{-5} \exp \left[ -1.25\left(\frac{f}{f_m}\right)^{-4} \right] \tag{2.20} \]
In this form the frequency dependence is assumed to be analogous to Phillips' (1958) theory. This model is identical to the P-M spectrum when found from the measured values of \( m_0 \) and \( f_m \).

2.2.7. Scott and Scott-Weigel Spectra

Based on the analysis of the wave data from a number of sources covering the Irish Sea and Atlantic Ocean, Scott (1965) modified the spectrum proposed by Darbyshire (1959) as

\[
S(f) = 21.51 m_0 \exp \left[ -\frac{[96.66(f-f_m)^2]}{(f-f_m+0.042)^2} \right]^{1/2}
\]

for \(-0.042 < (f-f_m) < 0.26\) \((2.21)\)

Weigel (1980) points out that this is not the spectrum for purely locally generated waves as the data set used for the calibration of this model included swells also. Thus for application in other seas it is necessary to compare the model with the measured spectra and calibrate it.

From the studies of a large number of energy spectra available from wave measurements in the North Atlantic Ocean Wiegel (1980) concluded that the empirical constant in Scott's spectral model is not constant, but is a function of the variance, the value increasing with the increase in variance. With this modification the form is known as Scott-Weigel spectrum and the spectral density is given by
\[ S(f) = A_m \exp \left[ -\left( \frac{(f-f_m)^2}{B(f-f_m+0.042)} \right) \right]^{1/2}. \quad (2.22) \]

When the normalized energy spectrum \( S(f)/H_s^2 \) is integrated with respect to \( (f-f_m) \) the dimensionless number \( 1/16 \) will be obtained. Based on this, Weigel has given the values of \( A \) and \( B \) for a range of \( H_s \) values.

The studies on the above spectral models carried out at different places along the southwest coast of India show that the Scott and the Scott-Weigel models fit the observed spectra in a number of cases. Dattatri et al. (1977), Dattatri (1978), Deo (1979), Prasad (1985), Sunder (1986) and Kurian (1987) obtained the best fit with the Scott spectrum for their data. Saji (1987) concludes that both the forms represents the data fairly well. Bhat (1986) found that the high frequency part is well represented by the Scott spectrum but it over-estimated the spectral peak. Baba and Harish (1986) observed that the Scott's model simulated the spectral peak closely in the cases of low energy conditions and it over-estimated the high energy conditions. From the above studies it is seen that though the Scott and Scott-Weigel spectral forms are derived for wind waves of the deep water, it could explain the observed spectra in some shallow water cases also. Further studies are required to validate the range of validity of these models in shallow water conditions.
2.2.8. Toba's Model

In the studies on the balance in the air-sea boundary processes Toba (1973) observed that the high frequency part of the spectrum is $f^{-4}$ dependent as against the $f^{-5}$ dependence assumed in other models. Based on this observation a new model is proposed, the spectral density of which is given by

$$S(f) = (2\pi)^{-3} g_* C_1 u_* f^{-4} \quad \cdots (2.23)$$

where $g_* = g(1+sk^2/\rho g)$ with $s$ as surface tension. $C_1$ is a constant and $u_*$ is the wind friction velocity at the sea surface. Later works by Goda (1974), Forristall (1981), Kahma (1981), Huang et al., (1983b), Kitaigorodskii (1983) and Battjes et al. (1987) give evidences for the existence of a negative 4th power dependence also, in the high frequency side of the spectrum.

Joseph et al. (1981) modified this model assuming a symmetrical form for the low frequency side and suggested the continuous form

$$S(f) = \begin{cases} (2\pi)^{-3} g C_1 u_* f^{-4} & ; f > f_m \\ (2\pi)^{-3} g C_1 u_* f_m^{-8} f^{-4} & ; f \leq f_m \end{cases} \quad \cdots (2.24)$$

Toba (1973) suggested 0.062 for the constant $C_1$. Based on many subsequent works Joseph et al. (1981) recommended 0.096 for $C_1$. The value of this constant is not
known for all seas. Moreover, it is not an easy task to compute the wind friction velocity at the sea surface correctly since it depends on the drag coefficient, which again depends on the wind velocity at the sea surface, and is highly variable. In a comparison of the above model with the observed spectra from the southwest coast of India Baba and Harish (1986) found that this form does not fit to the data unless the value of $C_1$ and $u_*$ are suitably adjusted.

2.3. SHALLOW WATER WAVE SPECTRAL MODELS

As against the deep water ones, the waves in the shallow water behave entirely differently. The factors that modify the wave characteristics in the shallow waters are many and are complex. The shoaling, refraction, breaking and other shallow water processes like percolation, friction, etc. play their roles and the spectral form is modified accordingly. In shallow waters the slope of the high frequency portion of the spectrum is found to be lower (Goda, 1974; Kitaigorodskii et al., 1975; Ou, 1977, 1980; Thornton, 1977, 1979; Dattatri, 1978; Vincent, 1982a,b; Vincent et al., 1982; Baba and Harish, 1986; etc.) and values as low as 1.6 are reported. As the shallow water waves are almost always unrelated to the local wind conditions, the applicability of the deep water spectral models to the shallow waters becomes restricted. The usual
practice is to adopt a deep water model and another model to propagate the wave to the shallow waters. Following this approach a few spectral models are proposed for the shallow water conditions.

2.3.1. Kitaigorodskii et al. Spectrum

Kitaigorodskii et al. (1975) extended Phillips' (1958) argument to the shallow waters by applying a finite depth dispersion relationship. They proposed a spectral model which is a modification of the Phillips' spectrum (Eq.2.12), the functional form of which may be written as

$$S(f) = \alpha \left( \frac{2\pi}{h} \right)^{-4} \frac{1}{f} \Phi(\omega_h)$$

.....(2.25)

where $\Phi$ is a non-dimensional function of the quantity

$$\omega_h = 2\pi f (h/g)^{1/2}$$

.....(2.26)

The function $\Phi$ varies monotonously from 1 in deep water to 0 in depth $h = 0$. When $\omega_h < 1$,

$$\Phi = (\omega_h)^{2/3}$$

.....(2.27)

Then for shallow waters Eq.(2.25) becomes,

$$S(f) = 0.5 \alpha gh (2\pi)^{-4} f^{-3}$$

.....(2.28)

Observational evidence to this form has been reported by many researchers (Thornton, 1977; Ou, 1977,1980; Iwata,
1980; Vincent, 1982a,b; Vincent et al., 1982; Baba and Harish, 1986; Kurian, 1987; etc.). Although this model simulates the high frequency side of the shallow water wave spectrum it shows the same limitations in practical applications as seen in the case of the Phillips' spectrum. However, this model offers ample scope to serve as a base for the development of spectral models applicable to finite depths.

2.3.2. Thornton's Model

The functional form of Eq.(2.28) was later derived independently by Thornton (1977) based on quite different arguments. He started from the first principles and postulated reasonably that breaking occurs when particle velocity approaches the phase velocity of the wave. Consequently, the parameters controlling breaking should be the phase velocity (C) and the frequency (f). Then by dimensional analysis he obtained

\[ S(f) = \alpha C^2 (2\pi)^{-2} f^{-3} \quad \cdots \quad (2.29) \]

By applying the shallow water approximation for the phase velocity, \( C^2 = gh \), Eq.(2.29) becomes

\[ S(f) = \alpha gh (2\pi)^{-2} f^{-3} \quad \cdots \quad (2.30) \]
Although this form is similar to that proposed by Kitaigorodskii et al. (Eq. 2.28), the difference of a factor of 2 is by no means negligible.

2.3.3. Jensen's Modification

Considering the importance of the total spectrum in practical applications, Jensen (1984) modified Eq. (2.28) to account for the low frequency side of the spectrum also. The forward face (the low frequency side) is assumed to be represented by the relation

\[ S(f) = 0.5 \alpha gh(2\pi)^{-2}(f_m)^{-3} \exp \left[1-(f/f_m)^{-4}\right] \] .... (2.31)

The study areas were restricted to semi-enclosed bodies of water and the model was tested using the data obtained from Saginaw Bay, Michigan. Agreement within \( \pm 0.15 \) m significant wave height and \( \pm 1.0 \) s peak period is reported (Jensen, 1984).

2.3.4. Shadrin's Model

Assuming the equilibrium range proposed by Phillips (1958) and the deviations from it in the shallow waters due to the effects of small depths Shadrin (1982) derived a spectral model for the shallow water waves. The spectral density is given by the equation

\[ S(f) = \alpha g^2 (2\pi)^{-4} f^{-5} f_r \] .... (2.32)
where $f_r$ is a dimensionless frequency. It may be noted that this form is similar to the relation obtained by Kitaigorodskii et al. (Eq.2.25) and the procedures are exactly similar. It is further assumed that in the coastal regime as the waves propagate over uniformly decreasing depths, from a certain time when the depths become comparable to the wave height, the wave crests undergo strong deformation. Hence for very small depths relative wave height ($H/h$) is considered as the most representative parameter. Based on the above argument and dimensional considerations $f_r$ is derived as

$$f_r = \frac{g}{2 \pi f (H/g) C_2 / 2}$$  \hspace{1cm} (2.33)

where $C_2 = b r' / (1 + C_3 r')$ ; $r' = H/h$ \hspace{1cm} (2.34)

On the basis of the data from the coastal regions of the Black and Baltic Seas the values of $b$ and $C_3$ are obtained as 20 and 4 respectively (Shadrin, 1982). Verification/calibration of this model elsewhere is not seen in the literature.

2.3.5. TMA Spectrum

The postulation of Kitaigorodskii et al. (1975) that the saturation level of wind wave energy spectrum in wave number space would be independent of water depth is extended to the entire spectrum (beyond the saturation range also) by
Bouws et al. (1985). By assuming a JONSWAP spectrum in deep water and the finite depth dispersion relation of Kitaigorodskii et al. to propagate it into shallow waters a new spectral model was formulated. The functional form of this model is given as

\[ S(f) = S_J(f) \Phi(\omega_h) \]  \hspace{1cm} \text{(2.35)}

where \( S_J(f) \) is the JONSWAP spectrum defined by Eq.(2.14) and \( \Phi(\omega_h) \) is given by Eqs.(2.26 & 2.27).

The validity of the model was verified using the data collected from the so-called TEXEL storm in the North Sea and from the projects MARSEN and ARSLOE and hence named it 'TMA' spectrum.

On an evaluation of this model one may find that there are some difficulties in its use. As this is an extension of the JONSWAP spectrum the limitations of that model (discussed elsewhere) will be transmitted to this new form also. Hence, this model may find limited applications. In a recent study, Vincent (1984) shows that the scale parameter is linked to the wave steepness and derived the relation

\[ \alpha_v = 16 \pi T^2 SS^2 \]  \hspace{1cm} \text{(2.36)}

where the parameter SS is given by

\[ SS = (m_0)^{1/2}/\ell_m \]  \hspace{1cm} \text{(2.37)}
$L_m$ is the wave length corresponding to the frequency at the spectral peak. Data collected from 2 average depths (17 m and 2 m) at CERC's Field Research Facility indicated excellent fit with TMA model at high steepness and some divergence at low steepness (Vincent, 1984).

2.3.6. Wallops Spectrum

Based on the assumption that the sea surface can be represented by a linear superposition of many countable, independent Stokean wave components, Huang et al. (1981) proposed a unified 2-parameter spectral model, which is termed as 'WALLOPS' spectrum, as an alternative to the many-parameter JONSWAP spectrum. The spectral density is given by

$$S(f) = \frac{(2\pi)^{-4}}{\gamma} g^2(\tilde{f}_m)^{-5}(\tilde{f}_m/f)^m \exp \left\{-(m/4)(\tilde{f}_m/f)^4 \right\} \ldots (2.38)$$

where

$$\beta = \frac{(2\pi \cdot \text{SS})^{2(m-1)/4}}{[4^{(m-5)/4} \Gamma [(m-1)/4]]} \ldots (2.39)$$

and

$$m = \left| \log \left( \frac{\sqrt{\pi \cdot \text{SS}}}{2} \right) \right| \ldots (2.40)$$

$\gamma$ is the gamma function and SS is the significant slope of the wave field defined by Eq. (2.37).

The justification for adopting the assumption of superposition of Stokean wave components in this model is the weakness of the non-linear wave-wave interaction proposed by many researchers (Phillips, 1977; Huang and Long, 1980; Huang et al., 1981, 1983b; etc.).
Basically this model is a generalization of the Phillips' saturation range concept by relaxing the fixed negative fifth power law of the high frequency portion of the spectrum. When the slope $m$ equals 5, this model approaches to the P-M spectrum (Eq.2.13) allowing variability to the constant ($\alpha$ to $\beta$). Since the range of validity of the spectrum slope emphasized is from $f_m$ to $2f_m$ this model could provide a better representation of the energy containing range than the high frequency range alone. Mc Clain et al. (1982) have reported excellent agreement in a comparison between their data and this model for developing seas.

For shallow waters Huang et al. (1983b) modified the Wallops spectrum (Eqs.2.38-2.41) and derived two cases based on the non-dimensional depth ($k_ph$) with $k_p$, the wave number corresponding to the peak energy:

(i) For $0.75 \leq k_ph < 3$, Stoke's shallow water wave theory was used. This lead to the following relations for $m$ and $\beta$ in the Wallops spectrum

\[
m = \left| \log \left[ \sqrt{2} \Pi S \, \coth(k_ph) \left[ 1 + 3/(2 \sinh^2 k_ph) \right] \right] / \log \sqrt{2} \right| \quad \text{(2.41)}
\]

\[
\beta = (2 \Pi S)^2 m(m-1)/4 \tanh^2 (k_ph) / \left[ 4(m-1)/5 \Gamma[(m-1)/4] \right] \quad \text{(2.42)}
\]
(ii) For \( k_p h < 0.75 \), solitary wave theory was applied to yield,

\[
m = \left| \frac{\log(\cosh \mu)}{\log \sqrt{2}} \right| \quad \text{.....(2.43)}
\]

\[
\mu = \frac{\pi}{(3 U_r)^{1/2}} \quad \text{.....(2.44)}
\]

where \( U_r \) is the Ursell number given by

\[
U_r = \frac{2 \pi SS}{(kh)^3} \quad \text{.....(2.45)}
\]

\[
\beta = \left( 2 \pi SS \right)^2 m^{(m-1)/4} \left( \frac{C_p}{k_p} \right)^2 \left( \frac{g}{4(m-5)/4} \right) \Gamma \left[ \frac{(m-1)/4}{4} \right]
\]

.....(2.46)

where \( C_p \) is the phase velocity corresponding to the peak frequency. Huang et al. (1983) recommended the use of the phase velocity of Stoke’s wave, quoting Bona et al. (1981), to give a highly accurate answer for most studies. Then,

\[
\beta = \left( 2 \pi SS \right)^2 m^{(m-1)/4} \tanh 2k_p h \left( \frac{4(m-5)/4}{\Gamma \left[ (m-1)/4 \right]} \right)
\]

.....(2.47)

This is perhaps the first full representation of a shallow water wave spectrum developed by using Stoke’s and solitary wave theories. Liu (1985) on a comparison with field data collected from the south eastern coast of Lake Erie at depths ranging from 1.4-3.8 m found that the semi-empirical Wallops model provides fair agreement with the observed data at the deeper stations but only marginal agreement in very shallow waters. In a later study Liu
(1987) found that the value of 0.75 for the non-dimensional depth \((k_p h)\) as the division between solitary and Stoke's wave theories should be modified to 1.5 to give better results. That is, for \(k_p h\) between 0.75 and 1.5 solitary wave theory fits the spectrum better than the Stoke's theory. However, it may be noted that the expressions for the spectral parameters and coefficients used in the study (Liu, 1987) differ from those suggested in the original.

As the Wallops model depend on the internal parameters it maintains a variable band width as a function of the significant slope which measures the non-linearity of the wave field. Also, it contains the exact total energy of the true spectrum since the total energy content is a built-in feature in the definition of the coefficient \(\beta\).

2.3.7. GLERL Spectrum

In most of the spectral models, though the overall forms are basically similar, they consists of a number of empirical coefficients and exponents that vary from location to location, depending on the environmental conditions. This restricts the universal applicability of these models. With the aim to solve this problem, Liu (1983) proposed an empirical 'Generalized Spectrum' in a form similar to the Wallops spectral model. The spectral density is given by
\[ S(f) = C_4(m_0/f_m)(f/f_m)^{-C_5} \exp \left[-C_6(f/f_m)^{-C_5/C_6}\right] \]

\[ \ldots (2.48) \]

where \( C_i = 4, 5, 6 \) are dimensionless coefficients and exponents that are to be determined from the given spectral parameters. The following relations are provided to derive these coefficients iteratively:

\[ C_4 = \exp \left(C_6 S(f_m)f_m/m_0\right) \]

\[ \ldots (2.49) \]

\[ m_0/S(f_m)f_m = \exp \left[C_6+(1-C_6+C_6/C_5)\ln C_6\right] \frac{\Gamma(C_6-C_2/C_5)/C_5}{C_5} \]

\[ \ldots (2.50) \]

\[ C_5 = \exp \left[C_6+(1-C_6+3C_6/C_5)\ln C_6\right] \frac{\Gamma(C_6-3C_6/C_5)/D}{C_5} \]

\[ \ldots (2.51) \]

\[ D = \left(f_a/f_m\right)^2m_0/[S(f_m)f_m] \]

\[ \ldots (2.52) \]

\[ f_a = (m_2/m_0)^{1/2} \]

\[ \ldots (2.53) \]

The practical application of this form requires the parameters \( m_0, f_m, f_a \) and \( S(f_m) \) to be known. In other words, the spectrum has to be fully defined for its shape and energy apriori. Usually, the total energy \( (m_0) \) and the peak frequency \( (f_m) \) are obtained from design wave information. \( S(f_m) \) and \( f_a \) have to be obtained from the individual spectrum (this limits the generalization of the model) or from empirical relations. Liu (1983) derived the following empirical relations for deep water wave spectra when \( S(f) \) is in \( m^2s \) and \( f \) in Hz:
\[ f_a = 0.82 \left( f_m \right)^{0.74} \quad \ldots \quad (2.54) \]
\[ S(f_m) = 17.0 \left( m_0 \right)^{1.13} \quad \ldots \quad (2.55) \]

The applicability of Eqs. (2.54 and 2.55) has been further corroborated in Liu (1984). In the subsequent works Liu (1985, 1987) shows that this form applies equally well in shallow water and deep water. Though water depth is not a parameter in this model it seems that the effect of depth is included through the exponents and coefficients. This model requires validation for different environmental conditions.

2.4. STATISTICAL CHARACTERISTICS

The deterministic approach, which incorporates wave theories derived from the equations of classical hydrodynamics, rarely represent the observed ocean wave characteristics. Each wave theory assumes the wave as regular, i.e., having a fixed profile that repeats exactly after a certain time (wave period) and then gives fixed solutions. The wind-generated ocean waves are usually irregular and hence the solutions based on the wave theories are approximate and inadequate to describe the actual complex phenomena. A meaningful description of the irregular wave field can be obtained from the various statistical methods.
The statistical analysis of waves is mainly aimed at deriving the probability distributions of wave heights and periods. The magnitude of the different parameters like significant, average, root-mean-square wave heights and periods, zero-crossing period, etc., having a specified recurrence can be derived easily if the probability distributions of heights and periods are known.

The probability distribution of a random variable is the probability that the given random variable will be less than or equal to a specified value. If \( x_n \) denotes a specified value of the random variable \( x \), then the probability distribution is given by

\[
P(x_n) = \text{Prob} (x \leq x_n)
\]  \( .....(2.56)\)

Hence, by convention, \( P(-\infty) = 0 \) and \( P(\infty) = 1 \)  \( .....(2.57)\)

The probability density function (pdf) is basically the probability that the random variable lies in a given range. From Eq.(2.56) it follows that

\[
P(x_n < x \leq x_n + \Delta x) = \text{Prob} (x \leq x_n + \Delta x) - \text{Prob} (x \leq x_n) = P(x_n + \Delta x) - P(x_n) = \Delta P(x_n)
\]  \( .....(2.58)\)

At the limit \( x \to 0 \), this is the probability density at \( x = x_n \) and is given by

\[
p(x) \bigg|_{x=x_n} = \lim_{x \to 0} \frac{P(x_n)}{\Delta x} = \frac{d[P(x_n)]}{dx} \quad (2.59)
\]
Generalizing for all the \( x_n \) values, the probability density function is given by

\[
p(x) = \frac{d[P(x)]}{dx} \quad \text{.....(2.60)}
\]

Similarly the probability distribution is given by

\[
P(x) = \int p(x) dx \quad \text{.....(2.61)}
\]

The concepts of the statistical height and period parameters of ocean waves were made more meaningful by the studies of many researchers (Seiwell, 1948; Weigel, 1949; Rudnick, 1951; Munk and Arthur, 1951; Darbyshire, 1952; Putz, 1952; Pierson and Marks, 1952; Watters, 1953; Yoshida et al., 1953; Darlington, 1954; etc.) on the distribution of wave heights and periods about their mean values.

2.5. DISTRIBUTION OF INDIVIDUAL WAVE HEIGHTS

The distribution of individual wave heights, especially in the shallow waters, has attracted the attention of many researchers (Longuet-Higgins, 1952; Putz, 1952; Gluhovskii, 1968; Goda, 1975; Lee and Black, 1978; Tayfun, 1980, 1981, 1983a,b; Huang et al., 1983a; Tang et al., 1985; etc.) and different mathematical/empirical models are put forward for the probability densities. The important ones are discussed in the following sections.
2.5.1. Rayleigh Distribution

Based on the works of Rice (1944, 1945) it was shown by Longuet-Higgins (1952) that the distribution of individual wave heights are Gaussian and follow the distribution function suggested by Rayleigh (1880). Similar conclusions are drawn by Putz (1952) also. The pdf is given by

\[ p(H) = \left( \frac{H}{4 \sigma^2} \right) \exp \left[ -\frac{H^2}{8 \sigma^2} \right] \quad \ldots \ldots (2.62) \]

where \( H \) is the individual wave height and \( \sigma^2 \), the variance. In terms of the significant wave height \( (H_s) \) this can be written as

\[ p(H) = \left( \frac{4H}{H_s^2} \right) \exp \left[ -2 \left( \frac{H}{H_s} \right)^2 \right] \quad \ldots \ldots (2.63) \]

and in terms of the average height \( (\bar{H}) \), this becomes

\[ p(H) = \left( \frac{\pi H}{2 \bar{H}^2} \right) \exp \left[ -\left( \frac{\pi}{4} \right) \left( \frac{H}{\bar{H}} \right)^2 \right] \quad \ldots \ldots (2.64) \]

The assumptions made in deriving the above relations are

(i) the wave spectrum contains a single narrow band of frequencies, and

(ii) the wave energy is being received from a large number of different sources whose phases are random.

This model is tested worldover by many researchers and evidences are provided for the applicability of this to
waves with broad-band spectra also under the condition of individual waves being defined by zero-crossing method (Bretschneider, 1959; Chakraborti and Snider, 1974; Longuet-Higgins, 1975; Tayfun, 1977; Dattatri et al., 1979; Goda, 1979; Huang and Long, 1980; etc.). The use of this distribution is sometimes extended to the shallow waters also. Good agreement of the shallow water data with the Rayleigh distribution is reported in some studies (Goodknight and Russe1, 1963; Koele and Bruyn, 1964; Harris, 1972; Manohar et al., 1974; Ou and Tang, 1974; Thornton and Guza, 1983; etc.). However, from detailed studies, many authors (Thompson, 1974; Dattatri, 1973; Goda, 1974; Black, 1978; Deo, 1979; Deo and Narasimhan, 1979; Baba, 1983; Baba and Harish, 1985; etc.) have reported that the waves higher than $H_s$ depart from the Rayleigh distribution in many cases to an alarming extent. Kuo and Kuo (1975) suggested that this is due to

(i) the non-linear effects of wave interactions yielding more larger waves,

(ii) the effect of bottom friction yielding reduction in the low frequency components, and

(iii) the effect of wave breaking which would truncate the distribution and transfer some of the kinetic energy to the high frequency components.
From an experimental study of the surface elevation probability distribution of wind waves in the laboratory Huang and Long (1980) derived a form for highly non-Gaussian conditions, based on Gram-Charlier expansion and a 4-term relation was suggested as a good approximation. This 4-term expansion is closely similar to the one given by Longuet-Higgins (1963). As a viable alternative to the computation of the pdf by the Gram-Charlier approximation, Huang et al. (1983a) derived equation for the probability density function of non-linear random wave field based on Stokes expansion to the 3rd order. For finite waters an additional parameter, the non-dimensional depth, is incorporated. This model is strictly for narrow band cases and is therefore more restrictive as far as the band width is concerned.

The Rayleigh distribution has a strong mathematical base and it has been widely used for quite some time in predicting the crest-to-trough heights of sea waves with apparent success (Huang and Long, 1980; Tayfun, 1983a; etc.). In spite of the deviation of the higher waves in shallow waters from this model, it is being used as a first approximation in view of its simplicity as a single-parameter model.
2.5.2. Gluhovskii's Distribution

Considering the fact that waves experience transformation in accordance with the water depth, once they enter shallow waters, Gluhovskii (1968) after analysing a large number of wave records from different coasts suggested a semi-empirical function for the distribution of wave heights by incorporating the relative water depth as a controlling parameter. The probability density function is given by

\[ p(H) = \left( \frac{\pi}{2H_\ast} \right) \left( 1 + H_\ast / (2\pi)^{1/2} (1-H_\ast) \right)^{-1} \left( H / H_\ast \right) \left( 1 + H_\ast / (1-H_\ast) \right) \exp \left[ -\pi / 4 (1 + H_\ast / (2\pi)^{1/2} (H / H_\ast)^2 / (1-H_\ast) \right] \] \hspace{1cm} (2.65)

where \( H_\ast = H / h \) \hspace{1cm} (2.66)

This model is a modification of the Rayleigh distribution with the introduction of a relative depth parameter. In the deep water conditions where \( H \ll h \) this model assumes the form of Rayleigh (Eq.2.64). This form was verified for different bottom slopes (0.1-0.001) and bottom sediments (sand and gravel) and it is claimed that this can be applied to any region from the deep water to the breaker zone (Gluhovskii, 1968). Some observational evidence to the applicability of this model to the shallow water waves are available from the southwest coast of India (Baba, 1983; Baba and Harish, 1985; Rachel, 1987; Saji, 1987; etc.).

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2.5.3. Ibrageemov's Distribution

In an analysis of Gluhovskii's function (Eq. 2.65) and field data, Ibrageemov (1973) found that the distribution of wave heights in and near the breaker zone is controlled not only by depth, but also by the periods of the individual waves. In addition to the depth, the wave period is introduced as a controlling parameter and an empirical function is suggested for the distribution of wave heights, the pdf of which is given by

\[
p(H) = \left(\frac{\pi}{2H} \frac{1}{\psi} \right) (H/H_0)^{2-\psi} / \exp \left[ (-\piT/4) (H/H_0)^2/\psi \right]
\]

...(2.67)

where \( \psi = 1 - 0.56 \exp (-4.6h/T^2) \) .....(2.68)

In deep water conditions, when \( h >> T \), this model also assumes the form of Rayleigh distribution (Eq.2.48). That is, though the dependence of the wave period was established for the surf zone conditions it is capable of predicting the deep water wave height distribution also. Based on this observation, it is argued that this model can be applied to all regions ranging from deep water to the breaker zone. In the shallow waters the depth is small and the empirical parameter assumes definite values. The validity of this factor in the shallow water conditions is yet to be verified.
2.5.4. Truncated Rayleigh Distributions

The wave attenuation due to irregular breaking was studied by Goda (1975) and a theory was formulated. Based on this theory the Rayleigh distribution was modified to explain the distribution of breaking and broken components in area of water depth shallower than about 2.5 times the equivalent deep water significant wave height. The probability density of this truncated Rayleigh distribution (Goda,1975) is given by

\[ p(x) = \mu 2 \lambda^2 x \exp \left( -\lambda^2 x^2 \right) \quad \ldots \ldots \ (2.69) \]

where

\[ 1/\mu = 1 - \left( 1 + 0.2x_1(x_1 - x_2) \right) \exp \left( -0.01x_1^2 \right) \quad \ldots \ldots \ (2.70) \]

\[ x = H/H_0, \text{ the non-dimensional wave height,} \]

\[ H_0 = K_r H_0, \text{ the deep water equivalent wave height, and} \]

\[ \lambda = 1.416/K_s \]

in which \( K_r \) and \( K_s \) are the refraction and shoaling coefficients. \( x_1 \) and \( x_2 \) are the ranges of breaker heights which can be calculated using Goda's breaker index

\[ x_b = C_7 (L_0/H_0) \left[ 1 - \exp \left( -1.5(\pi h/H_0)(H_0/L_0)(1 + K_t n \phi) \right) \right] \quad \ldots \ldots \ (2.71) \]

\( \phi \) is the angle of inclination of sea bed. For a best fit of the index curves Goda recommended the values
\[ K = 15 \]
\[ n = 4/3 \]
\[ C_7 = \begin{cases} 
0.18 \text{ for } x_1 \\
0.12 \text{ for } x_2 
\end{cases} \]

(2.72)

This model assumes the constancy of wave number and mean wave period. For deep water conditions, \( x \leq x_2 \), this assumes the form of Rayleigh distribution. The simultaneous wave observations carried out at depths of about 20, 14 and 10 m at the Port of Sakata supported Goda's theory of decrease of wave heights due to irregular breaking in very shallow waters (Irie, 1975).

In a study to investigate the effect of breaking on wave statistics, Kuo and Kuo (1975) also observed that the probability density function of wave heights with a certain intensity of breaking waves could be explained by a truncated Rayleigh distribution. In extreme cases for very shallow waters they used the limiting height of solitary wave to predict the breaking wave heights. A similar form of truncated Rayleigh distribution is recommended by Battjes (1974) for the shallow water wave heights. In a comparison with the field data collected from the Ala Moana Beach (Hawaii) Black (1978) obtained excellent fit with the truncated Rayleigh distribution in the breaker zone, but poor fit in both offshore and shoreward of this region.
2.5.5. Weibull Distribution

In search for a general distribution which fits for all positions in a reef Lee and Black (1978) suggested the Weibull distribution, the pdf of which is given by

\[ p(H) = C_8 C_9 H^{C_9 - 1} \exp \left[-C_8 H^{C_9}\right] \quad \ldots (2.73) \]

The peakedness coefficient, \( C_9 \), for a Weibull distribution is given by

\[ C_9 = 4 \int_0^\infty H^2 p(H) \, dH \quad \ldots (2.74) \]

The approximate value of \( C_9 \) for data divided into bins of width \( \Delta H \) can be computed from the relation

\[ C_9 = \left(\frac{4}{N^2 \Delta H}\right) \sum_{i=1}^{N_{\text{BINS}}} \frac{H_i^2 m_i^2}{N} \quad \ldots (2.75) \]

where \( N \) is the total number of waves, \( N_{\text{BINS}} \) is the number of bins and \( H_i \) and \( m_i \) are the heights and number of occurrences respectively in the 'i'th bin. Since the squared probability density term in the distribution width function tends to magnify small deviations from the theoretical distribution and \( C_9 \) is sensitive to the choice of the number of bins, it should be selected such that \( N_{\text{BINS}} < N/10 \).

The Weibull coefficient, \( C_8 \), may be determined using the relation

\[ C_8 = \left[\frac{1}{(1 + 1/C_9)^{C_9}}\right] \quad \ldots (2.76) \]
It should be noted that Rayleigh is a special case of the Weibull distribution with \( C_9 = 2 \) and \( C_8 = 1/(H_{rms})^2 \). Where Rayleigh distribution is a function of the variance, the Weibull distribution is a function of the higher moments about the mean.

In a comparison of the probability densities predicted by this model with the measured ones, Lee and Black (1978) obtained correlation coefficients greater than 0.98, which is unity for a perfect fit. Forristall (1978) also recommended the use of Weibull distribution for wave heights from the analysis of hurricane-generated waves in the Gulf of Mexico. As reported in Lee and Black (1978), Arhan and Ezraty (1975) also used Weibull distribution successfully to fit their wave data in the shallow waters.

2.5.6. Tayfun's Distribution

From the studies on the consequences of the fact that the crest and trough of the wave do not occur at the same time, Tayfun (1981) developed an envelope approach to explain the distribution of crest-to-trough wave heights. The equation of Rice (1945) for the joint distribution for the amplitudes of two points on the envelope separated by time \( T \) is integrated to give the distribution of zero crossing wave heights. The probability density is given by
\[ p(H') = 2 \int_{-1/\mu}^{1/\mu} \hat{p}(T) \int p(A', 2H' - A'; \bar{T}/2) dA' d\bar{T}, \quad \text{for } H' \geq 0 \].....(2.77)

where \( \hat{p}(T) \) represents the probability density of normalised zero-up-crossing periods such as that given by Longuet-Higgins (1975) and \( H' = H/\bar{H} \) is the normalised wave height.

\[ p(A', A''; \bar{T}/2) = \frac{\pi^2}{4} \frac{A' A''}{(1 - r^2)} I_0 \left[ \pi r A' A'' / 2(1 - r^2) \right] \exp \left[ -\frac{\pi}{4} \frac{A'^2 + A''^2}{(1 - r^2)} \right] \].....(2.78)

in which \( A' \) and \( A'' \) are the normalized trough and crest heights given by

\[ A' = A(t)/\bar{A} = 2A(t)/\bar{H} \].....(2.79)
\[ A'' = A(t+\bar{T}/2)/\bar{A} = 2A(t+\bar{T}/2)/\bar{H} \].....(2.80)

\( I_0 \) denotes the zero-order modified Bessel function of the first kind, and

\[ r(\bar{T}/2) = (F_1^2 + F_2^2)^{1/2} \].....(2.81)
\[ F_1(\bar{T}/2) = (m_0)^{-1} \int S(\omega) \cos(\omega - \bar{\omega})(\bar{T}/2) d\omega \].....(2.82)
\[ F_2(\bar{T}/2) = -(m_0)^{-1} \int S(\omega) \sin(\omega - \bar{\omega})(\bar{T}/2) d\omega \].....(2.83)

Under narrow band conditions, as \( \nu^2 \to 0 \), \( S(\omega) \) and \( \hat{p}(T) \) tend to behave as pseudo-delta functions centered at \( \omega = \bar{\omega} \) and \( T = \bar{T} \) respectively. With \( T = \bar{T} \) being fixed, the trigonometric terms in Eqs.(2.82 and 2.83) when expanded in Taylor series to second-order in \( (\omega - \bar{\omega}) \) gives
F_2(\bar{T}/2) \approx 0 \text{ and } r(\bar{T}/2) \approx F_1(\bar{T}/2) = 1 - ((\pi \nu)^2/2) \ldots (2.84)

On this basis Eq. (2.77) reduces to

\[ p(H') = 2 \int_0^{2H'} p(A', 2H' - A'; \bar{T}/2) \, dA' \ldots (2.85) \]

Tayfun (1983) has shown that as \( \nu^2 \to 0 \), the approximation improves considerably and, for \( \nu^2 \leq 0.04 \) the error is less than 2% relative to the exact solution for values of \( H' \) in the range 0.25-3.50, which represents at least 98% of the total probability mass in all cases.

If the spectrum is not narrow, the envelope will change during half-wave period and if the wave is high, so that the crest is near the extreme on the envelope, it is likely that the associated trough will have a small amplitude, and the wave height will be less than twice the value of the envelope at the crest. Thus different amplitudes are considered for the envelope at the time of the crest and trough. This approach seems to give a good representation of the effect of spectral width on the wave height distribution. A case study given by Tayfun (1981) showed very favourable results supporting the concepts developed in the study. Forristall (1984) compared this distribution to simulated waves with different spectral shapes as well as to field observations from the Gulf of Mexico and reports excellent agreement.
2.6. JOINT DISTRIBUTION OF WAVE HEIGHTS AND PERIODS

Studies on probability density function of period, height and wave lengths have been conducted over the last 2-3 decades. However, studies on the joint distribution of heights and periods of sea waves are not as extensive as that on the distribution of heights. Knowledge of the joint distribution of heights and periods of ocean waves is essential for any ocean/coastal engineering application. The majority of the studies reported have been mainly concerned with theoretical or deep water aspects of the problem. The few theories available in the literature are discussed here.

2.6.1. Rayleigh Distribution

On the basis of wave data from various deep and shallow water locations Krilov (1956) and later based on field and laboratory data Bretschneider (1959) recommended the use of Rayleigh distribution for the square of wave periods. The probability density of the distribution function is given by

\[ p(T) = \exp \left[ -0.675 \left( \frac{T}{\bar{T}} \right)^4 \right] \]  \hspace{1cm} \text{.....(2.86)}

Different authors opined differently regarding the suitability of this in modelling the distribution of wave periods. Chakraborti and Snider (1974) observed that the Rayleigh's form shows poor fit with the height data and better fit with \( T^2 \). According to Goda (1979) this semi-
empirical proposal of the Rayleigh distribution for $T^2$ provides a fair approximation to the wind waves although the formulation of a joint distribution in a closed form is lacking. Studies conducted with field data collected from different parts of the Indian coasts show that this model is incapable of simulating the period distributions (Narasimhan and Deo, 1979a; Dattatri et al., 1979).

Assuming that the marginal pdfs of wave heights and square of periods to be Rayleigh distributed, Bretschneider (1959) examined its joint distribution for the extreme cases of correlation (0 and +1). For the cases of total independence (zero correlation) the pdf is given by

$$p(H',T') = 1.35H' \exp \left(-\frac{H'^2}{4}T'^3\exp \left[-0.67T'^4\right]\right)$$  \hspace{1cm} (2.87)

where $T' = T/\bar{T}$ and $H' = H/\bar{H}$  \hspace{1cm} (2.88)

For the case of total dependence (correlation coefficient equals to 1) all data points on a plot of joint Rayleigh pdfs fall on a 45 degree straight line through the origin.

2.6.2. Gluhovskii's Distribution

Following the assumption of total independence between the wave height and period distribution Gluhovskii (1968)
suggested a depth controlled relation for the joint distribution of heights and periods of sea waves. The probability density is given by

\[ p(H,T) = \frac{0.4165}{\pi} \frac{\pi^2}{(1+0.4H_*)(1-H_*)} \frac{T^3}{T^4} (H' P - 1) \exp \left[ -\frac{\pi}{4} \left( \frac{T}{1+0.4H_*(H')P+0.833(T/T)^4} \right) \right] \]

\[ \text{...(2.89)} \]

where \( p = 2/(1-H_*) \). \( H_* \) and \( H' \) are given by Eqs.(2.66 & 2.88).

Although the theoretical models developed by Bretscheneider and Gluhovskii (Eqs.2.87 and 2.89) are based on the primary condition that the two variables (height and period) are mutually independent, observational evidence show considerable correlation (Chakraborti and Cooley, 1977; Goda, 1978,1979; Thornton and Schaeffer, 1978; Dattatri et al., 1979; Baba and Harish, 1985; Harish and Baba, 1986; etc.). Baba and Harish (1985) obtained correlation coefficients ranging up to 0.69. Dattatri et al. (1979) observed still higher correlation between the individual heights and periods for the monsoon data collected off Mangalore. Houmb and Overik (1976) have shown that the assumption of independency between the height and period leads to an overestimation of the heights of breaking waves lower than \( H_s \). This deviation leads to the failure of these models in predicting the joint distribution of heights and periods as observed in the field.
2.6.3. Longuet-Higgins' Distributions

The theory of the joint distribution of wave heights and periods in a closed form was provided by Longuet-Higgins (1975) under the assumption of a narrow band spectrum. It is actually a recapitulation of a previous work of the author (Longuet-Higgins, 1957) on the statistical properties of random moving surface. The joint probability density is given by

\[ p(H', T') = \frac{\pi H'^2}{4 \nu} \exp \left\{ -\frac{(\pi /4)H'^2}{4} \left[ 1 + (T' - 1)^2 / \nu^2 \right] \right\} \]

\[ \ldots \ldots (2.90) \]

where \( \nu = (m_0m_2/m_1^2 - 1)^{1/2} \)

\[ \ldots \ldots (2.91) \]

and \( H' \) and \( T' \) are given by Eq.(2.88).

The joint distribution given by Eq.(2.90) has its axis of symmetry at \( T' = 1 \) (or \( T = \bar{T} \)) and yields no correlation between wave heights and periods. Generally, ocean waves exhibit a distinctly positive correlation (as seen before) especially for the low waves, which sometimes amounts to more than 0.7 among sea waves (Goda, 1978). The results obtained from the analysis of 89 wave records from the Japanese coast (Goda, 1978) and data from the 1961 storm in the North Atlantic (Chakrabarti and Cooley, 1977) it is seen that this theory could explain the characteristics of the
joint distribution in its upper portion with high waves if the spectral width parameter is selected in such a way that it would fit the marginal distribution of wave periods. But the theory disagrees with the observed joint distribution in its lower portion with low waves. Disagreement of the theory with the observed are reported by others also (Houmb and Overik, 1976; Ezraty et al., 1977; Tayfun, 1981, 1983a,b; etc.). Lee and Black (1978) explained the poor fit of the data with the model as mainly due to the positive skewness observed in the actual shallow water distribution of periods. However, Shum and Melville (1984) observed good agreement with their data from both calm and hurricane sea states, when an integral transform method was used to obtain continuous time series of wave amplitude and period from ocean wave measurements.

More recently Longuet-Higgins (1983) remodelled his earlier equations to incorporate the effects of non-linearities and finite band width. The modified function is derived by considering the statistics of the wave envelope and in particular the joint distribution of envelope amplitude and the time derivative of the envelope phase. By relating the frequency to the period and assuming that the phase envelope is an increasing function of time, the following distribution function is derived:
\[ p(H'', T'') = C_{LH} \left( \frac{H''}{T''} \right)^2 \exp \left[ -\left( \frac{H''^2}{8} \right) \left[ 1 + \nu^{-2} \left( 1 - \frac{1}{T''} \right)^2 \right] \right] \]

\[ C_{LH} = \left( \frac{1}{8} \right) (2\pi)^{-1/2} \nu^{-1} \left[ 1 + \nu^2 \right]^{-1/2} \]

\[ H'' = \frac{H}{(m_0)^{1/2}} \text{ and } T'' = \frac{T}{(m_0/m_4)} \]

It can be seen that this distribution depends only on the spectral width parameter \( \nu \). Shum and Melville (1984) obtained good agreement of this model with their data based on wave envelope analysis (rather than waves themselves). Srokosz and Challenor (1987) report that this model did not fit their broad band data, but gave good fit with zero-crossing height and period distribution for \( \nu < 0.4 \), rather than \( \nu \leq 0.6 \) as suggested by Longuet-Higgins (1983). In a subsequent study Srokosz (1988) reports that for spectra narrower than those examined in the earlier study, while the overall shape of the predicted distribution is similar to the observed, the mode is incorrectly placed by this model.

### 2.6.4. CNEXO Distribution

The asymmetric pattern of the joint distribution of wave heights and periods is incorporated in the theory developed by the group of CNEXO (Ezraty et al., 1977) which is basically for the joint distribution of the amplitudes and periods of positive maxima. The time interval between successive positive maxima is estimated by extending the
theory of Cartwright and Longuet-Higgins (1956). They further presumed that it could be applicable to the joint distribution of heights and periods of zero-up-crossing waves by replacing the amplitude of positive maximum with one-half wave height and the quasi-period of positive maximum with zero-crossing wave period. The probability density of this joint distribution has the following form:

\[
p(H'', T_1) = \epsilon'^3 H''^2 / [4(2\pi)^{1/2} \epsilon_s (1-\epsilon_s^2) T_1^5] \times 
\exp\left[\left(\frac{H''^2}{8\epsilon_s^2 T_1^4}\right) \left(\frac{(T_1^2-\epsilon'^2)^2+\epsilon'^4 \epsilon''^2}{(T_1^2-\epsilon'^2)^2+\epsilon'^4 \epsilon''^2}\right)\right] 
\]

(2.95)

where

\[
\epsilon' = \left[1+(1-\epsilon_s^2)^{1/2}\right]/2;
\]
\[
\epsilon'' = \epsilon_s / (1-\epsilon_s^2)^{1/2};
\]
\[
\epsilon_s = \left(1-m_2^2/m_0 m_4^2\right)^{1/2}
\]

(2.96)

and \(H''\) is defined in Eq.(2.94). The mean value of the non-dimensional wave period, \(\bar{T}_1\), can be obtained by numerically integrating the marginal distribution of wave period of the following form

\[
p(T_1) = \epsilon'^3 \epsilon''^2 T_1 / [(T_1^2-\epsilon'^2)^2+\epsilon'^4 \epsilon''^2]^{3/2} \]

(2.97)

Goda (1978) found that the value of \(\bar{T}_1\), obtained by numerical integration of Eq.(2.97), remained close to 1 for the range of \(0 < \epsilon_s < 0.95\). When this theory was applied for ocean waves by the group of CNEXO, the spectral width
parameter was estimated by the formula given in Eq. 3.3. In an analysis of the governing parameters of the joint distribution Goda (1978) observed that the apparent spectral width parameter is less influential and the correlation coefficient between individual wave heights and periods could be the governing parameter. The correlation coefficient is defined as:

\[ r(H, T) = \frac{1}{\sigma_H \sigma_T N_Z} \sum_{i=1}^{N_Z} (H_i - \bar{H})(T_i - \bar{T}) \]  

(2.98)

where \( \sigma_H \) and \( \sigma_T \) denotes the standard deviations of wave heights and periods respectively and \( N_Z \) is the number of zero-crossing waves.

Battjes (1977) pointed out that the theory of CNEXO is theoretically inconsistent, especially in its approximation of the mean zero-up-crossing wave period with the mean interval of positive maxima. However, Goda (1978, 1979) and Harish and Baba (1986) observed that it could provide a good approximation to the joint distribution. Goda (1978) points out that as the correlation coefficient \( r(H, T) \) increases, especially when \( r(H, T) \geq 0.4 \) the asymmetry of joint distribution becomes conspicuous, which is in accordance with the theory of CNEXO.

A shortcoming of this theory is that the asymmetry of the joint distribution with respect to the wave period
becomes too pronounced with the increase of the spectral width parameter (Goda, 1979; Harish and Baba, 1986) and the theory predicts the probability of long periods much higher than that observed (Goda, 1979). Further, Goda (1978) found that the observed density followed the theory of CNEXO only when \( H' < 0.4 \) and the rest followed closely the theory of Longuet-Higgins. That is, the portion of high waves in the joint distribution retains the symmetry around their mean value of periods irrespective of the value of spectral width parameter.

2.6.5. Tayfun's Distribution

Another theoretical expression for the joint distribution of crest-to-trough wave height and zero-up-crossing periods is developed by Tayfun (1983b) from a modified extension of presently available results relevant to wave envelopes and periods under narrow band conditions. The wave profile is viewed in time as a slowly modulated carrier wave considering the narrow band approximation. The joint probability density of this model is given by

\[
p(H', T_2) = 2 \frac{H'}{P(T_2)} \int_0^{2H'} p[A', 2H' - A'; (T/2)(1+\nu T_2)]dA'
\]

for \( H' \geq 0 \) and \( T_2 \leq \nu^{-1} \) ....(2.99)

in which \( H' \) and \( \nu \) are given in Eqs. (2.88 & 2.91) and
\[ T_2 = (T - \bar{T}) / \sqrt{T} \]
\[ \hat{p}(T_2) = (1 + \nu^2)^{1/2} / 2(1 + T_2^2)^{3/2} \quad \text{for } |T_2| \leq \nu^{-1} \]

The other parameters are same as given in section 2.5.6. As in the case of height distribution Tayfun (1983) has shown that for \( \nu^2 \leq 0.04 \), this displays an error of less than 5% relative to the exact solution.

2.7. SUMMARY

From the literature it is seen that a generalized spectral form is very much lacking. Different models are proposed for different environmental conditions like deep water, shallow water, fully developed seas, developing sea state, etc. The parameters and coefficients vary from researcher to researcher. For instance, in the deep water conditions the high frequency face of the spectrum was first intuitively set by Neumann (1953) to be proportional to \( f^{-6} \). Later Phillips (1958) deduced from dimensional considerations that it should be \( f^{-5} \) and this has been substantiated by many field and laboratory experiments. More recently some workers found that it is \( f^{-4} \) instead. Further, in the shallow water conditions observational evidences are abundant for a negative power of 3 and less.
Most of the spectral models are described by one, two or more independent wave parameters. However, a large number of these models are based on two commonly used design wave parameters, the total energy and the peak frequency. In an analysis of these spectral forms it can be seen that most of these models unify into one form with varying coefficients. Chakraborti (1986) has reported such a form in a comparison of a few two-parameter models.

The models derived for deep water conditions are found to fail in simulating the shallow water wave spectra satisfactorily. However, the empirical Scott's model and its modified form (Scott-Weigel spectrum) are found to fit the observed shallow water spectra in some cases. Most of the shallow water wave spectral models are the modification of one or other of the deep water models with a finite depth dispersion relationship. Extensive field calibration under a wide range of wave conditions is required for any model to be used for a particular location, until a generalized model with universal applicability is developed.

Most of the models to predict the distribution of individual heights of sea waves are based on the theoretically sound Rayleigh distribution derived for the deep water conditions. Owing to the simplicity as a single parameter model, the Rayleigh distribution is extensively used as a
first approximation, even in the shallow waters. Observations at different environmental conditions reveal that though the Rayleigh form is valid beyond the narrow band cases, the distribution of heights in the shallow waters are non-Gaussian. The depth-controlled forms appear to be promising for the future. The envelope approach to describe the height distribution offers scope for the future studies as this new concept seems to be more realistic. Most of the shallow water models lack sufficient field verifications.

Studies on the distribution of wave periods and its joint distribution with heights are not extensive. Most of the studies are based on the assumption that the height and period distributions are mutually independent. Recently, dependency of heights and periods are reported by many researchers and correlation coefficients of the order of 0.7 are reported. Asymmetry of the joint distribution is revealed in the recent studies and this concept seems to throw more light into the study of the joint distribution. The models developed on the basis of this concept also require sufficient field verification.