CHAPTER III

3. Calculations, Results and discussions.

3.1 Josephson tunnel effect.

Consider a Josephson tunnel junction consisting of two impure anisotropic superconductors which are separated by a thin oxide layer (insulator). When the superconductors are different the Josephson current can be calculated only numerically. However, for a junction made up of two identical superconductors, one can do analytic calculations. Further, all important properties of the Josephson effect are still manifested in this case. For such a junction, the amplitude of the Josephson current at a temperature $T$ can be written as

$$J_s(C, T, \lambda) = R_N^{-1} (2\pi T) \sum_{n=0}^{\infty} \frac{d\Omega}{4\pi} [1 + U_n^2(\omega)]^{-1} \ldots (3.1)$$

$R_N$ is the normal-state resistance of the junction and

$$U_n(\omega) = \omega(\omega)/\Delta(\omega)$$

$\omega(\omega)$ and $\Delta(\omega)$ are given in equations (2.71) and (2.72). Near $T_c$ $\Delta(\omega)$ is small and $U_n(\omega)$ is large. Hence equation (3.1) can be rewritten as

$$J_s(C, T, \lambda) = R_N^{-1} (2\pi T) \sum_{n=0}^{\infty} \frac{d\Omega}{4\pi} [1/U_n^2(\omega)] \ldots (3.2)$$

We consider terms containing $\Delta^2$ only. We have
shown in ref [14] that

\[
\frac{1}{U(\Omega)} = L + a(\Omega) M \tag{3.3}
\]

where

\[
L = \left[ \frac{\Delta}{(\omega + \alpha)} \right] + \left( \frac{\Delta^3}{2} \right) \left[ \frac{\alpha (2 \delta^2 - 1)}{(\omega + \alpha)^4} \right] \tag{3.4}
\]

and

\[
M = \left[ \frac{\Delta}{(\omega + \beta)} \right] + \left( \frac{\Delta^3}{2} \right) \left[ \frac{(2 \delta^2 - 1)}{(\omega + \alpha)^2} \right] \tag{3.5}
\]

Substituting equation (3.3), (3.4) and (3.5) in equation (3.1) we obtain

\[
J_\delta(\tilde{C}, T, \Omega) = \left[ \frac{\Delta^2}{(2\pi TR_N)} \right] \sum_{n=0}^{\infty} \left[ \frac{\Delta^2}{(\omega + \alpha)^2} \right] + \left< \frac{a^2}{\psi} \frac{\Delta^2}{(\omega + \beta)^2} \right>
\]

\[
= \left[ \frac{\Delta^2}{(2\pi TR_N)} \right] \left[ \psi\left(\frac{1}{2} + \frac{1}{\alpha(2\pi T)}\right) \right] + \left< \frac{a^2}{\psi} \psi\left(\frac{1}{2} + \frac{1}{\beta(2\pi T)}\right) \right> \tag{3.6}
\]

Hence

\[
J_\delta(\tilde{C}, T, \Omega) = \Delta^2(\tilde{C}, T, ) \frac{B_2(\tilde{C}, T)}{(2\pi TR_N)} \tag{3.7}
\]

Where

\[
B_2(\tilde{C}, T) = \psi\left(\frac{1}{2} + \frac{1}{\alpha(2\pi T)}\right) + \left< \frac{a^2}{\psi} \psi\left(\frac{1}{2} + \frac{1}{\beta(2\pi T)}\right) \right>
\]

Also we know that [14] that

\[-\ln\left(\frac{T}{T_{c0}}\right) = B_0(\tilde{C}, T) + B_1(\tilde{C}, T) \Delta^2(\tilde{C}, T, \Omega)/(8\kappa^2T^2) \tag{3.8}\]
Near $T_c$ equation (3.8) can be expanded to give

$$
\Delta^2(\bar{c},T,\omega) = \left[8\pi^2 T_c^2 / B(\bar{c},T) \right] \left[ 1 + \frac{\Delta}{\omega - \omega_0(\bar{c},T)} \right] \left[ 1 - T/T_c \right] \frac{T_c}{T_c} \tag{3.9}
$$

where

$$
B_0(\bar{c},T) = [1+\langle a^2 \rangle]^{-1} \left\{ \left[ \psi^0((1/2)+\rho) - \psi^0(1/2) \right] + \langle a^2 \rangle \left[ \psi^0((1/2)+\sigma) - \psi^0(1/2) \right] \right\} \tag{3.10}
$$

$$
B_1(\bar{c},T) = [1+\langle a^2 \rangle]^{-1} \left[ b_0(\bar{c},T) + \langle a^2 \rangle b_1(\bar{c},T) \right] \tag{3.11}
$$

$$
b_0(\bar{c},T) = -(1/2) \frac{\rho}{2} \frac{3}{(2\pi)^2} \frac{2}{\omega - \omega_1(1/2) + \rho} \tag{3.12}
$$

$$
b_1(\bar{c},T) = \frac{3}{(\sigma-\rho)} \left[ \psi^1((1/2)+\rho) - \psi^1((1/2)+\sigma) \right] - \frac{c_1}{4\pi^2 N(0) T} \left[ \epsilon_1^2 + (1-\epsilon)(2\epsilon^2 - 1) \right] \times \left[ \frac{1}{(\sigma-\rho)} \right]^2 \left[ \psi^1((1/2)+\rho) + \psi^1((1/2)+\sigma) \right] - 2 \left[ \frac{1}{(\sigma-\rho)} \right]^3 \left[ \psi^0((1/2)+\sigma) - \psi^0((1/2)+\rho) \right] \tag{3.13}
$$

$$
\psi_n(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}}
$$

$$
\rho = a / 2\pi T = \bar{c} \left( T_{co} / T \right) \left( 1 - \epsilon^2 \right)
$$

$$
\sigma = \rho / 2\pi T = \bar{c} \left( T_{co} / T \right) \left( 1 - \epsilon + \epsilon_1^2 \right)
$$

For small impurity concentration and $T \approx T_c$

$$
\Delta^2(\bar{c},T,\omega) = \left[8\pi^2 T_c^2 / B(\bar{c},T) \right] \left( 1 - T/T_c \right) \times \left[ 1 - \left( a / 2\pi T_c \right) \psi^1((1/2)+\rho) \right] \tag{3.14}
$$
for a pure superconductor the Josephson current is obtained as

\[ J_s(0,T,\Omega) = \Delta^2(0,T) \psi^1(1/2)[1 + \langle a^2 \rangle]/R_N(2\pi T) \] (3.15)

The normalized Josephson tunneling current can be given by

\[ \frac{J_s(\tilde{c},T,\Omega)}{J_s(0,T,\Omega)} = \left[ \frac{\Delta(\tilde{c},T)}{\Delta(0,T)} \right]^2 \left[ 1/ \psi^1(1/2)[1 + \langle a^2 \rangle] \right] \times \left[ \psi^1((1/2) + \rho) + \langle a^2 \rangle \psi^1((1/2) + \sigma) - \psi^1((1/2) + \rho) \right] \] (3.16)

where

\[ \left[ \frac{\Delta(\tilde{c},T)}{\Delta(0,T)} \right]^2 = \left[ \frac{T_c}{T_{co}} \right]^{2} \frac{B_1(0,T)/B_1(\tilde{c},T)}{1-\frac{T}{T_c}} \times \left[ 1-4.9356160/T_c \left[ 1 - \frac{T}{T_c} \right] \right] \times \left[ \epsilon^2 + \langle a^2 \rangle \right] \times \left[ \epsilon^2 + \frac{\tilde{c}}{\epsilon_i_1} \right] / \left[ 1 - \frac{T}{T_{co}} \right] \] (3.17)

For small impurity concentration and for \( T \ll T_c \)

\[ B_1(\tilde{c},T_c) = 8.414 - 32.47 \tilde{c} \left( \frac{T_{co}}{T_c} \right) \left( 1 - \frac{T}{T_c} \right) \]

\[ + \langle a^2 \rangle \left[ 42.07 - 32.47 \left( \frac{T_{co}}{T_c} \right) \right] 16.235 \left[ 15 - 8 \epsilon - 5 \epsilon^2 \right. \]

\[ - 2 \epsilon^3 + 10 \tilde{c}^2 \] (3.18)

The zero temperature order parameter for a pure superconductor \( \Delta(0,0) \) can be obtained as \[ \text{[8]} \]

\[ 1 = N(O)V\langle 1 + a(\Omega) \rangle^2 \int d\epsilon_p \tan \left( \beta \epsilon_p/2 \right) \]

\[ = N(O)V\langle 1 + a(\Omega) \rangle^2 \int_{0}^{\omega_0} d\epsilon_p \frac{2\epsilon_p}{E_p} \]

\[ E_p = \sqrt{\epsilon_p^2 + \Delta_p^2} \]
\[
\frac{1}{N(0)V} = [1 + \langle a^2 \rangle] \ln[2 \omega_D/\Delta(0,0)] - (3/2)\langle a^2 \rangle \ldots (3.21)
\]

\[
\Delta(0,0) = 2 \omega_D [1 + \langle a^2 \rangle \left( (1/N(0)V) - (3/2) \right)] \exp\left[-1/(N(0)V) \right] \ldots (3.22)
\]

From equation (2.10) of Clem [8] we obtain

\[
\Delta(0,0) = 3.528(T_{co}/2)[1 - 1.5 \langle a^2 \rangle] \ldots (3.23)
\]

Combining equations (3.14) and (3.23) one gets

\[
\frac{\Delta^2(\bar{C},T,\Omega)}{\Delta^2(0,0)} = \frac{32\pi^2 T_c^2 \left( 1/T_c \right) [1 - 4.935\lambda_c]}{B_1(\bar{C},T) T_{co}^2 (3.528)^2 (1 - 3 \langle a^2 \rangle)} \ldots (3.24)
\]

For pure superconductor one finds

\[
\frac{\Delta^2(0,T,\Omega)}{\Delta^2(0,0)} = 3.016 \left[ 1 - 2 \langle a^2 \rangle \right] (1 - T/T_c) \ldots (3.25)
\]

Equation (3.25) is in agreement with equation (2.14) of Clem [8] in the limit of \( T \ll T_c \). Figure 1. shows the variation of \( \Delta(\bar{C},T,\Omega)/\Delta(0,T) \) with \( T/T_c \) near \( T_c \).

Significant deviation of the normalized order parameter from that without anisotropy is observed.

From equation (3.8) and (3.14) we find the slope of \( J_s(\bar{C},T,\Omega) \) against temperature at \( T_c \). This slope depends on the magnetic moment of the impurity. We define
the slope of \( J_s(\vec{c}, T, \omega) \) against temperature as

\[
K^* = \lim_{\vec{c} \to 0} \frac{[(\partial/\partial T)J_s(\vec{c}, T, \omega) - (\partial/\partial T)J_s(O, T, \omega)]/\vec{c}(\partial/\partial T)J_s(O, T, \omega)}{(T_c - T_{co})/CT_{co}}
\]

(3.26)

while writing the above equation we have used

\[
B_0(\vec{c}, T_c) = \ln(T_{co}/T_c)
\]

Following the same procedure as given in Ref. [42] we obtain the slope of \( J_s(\vec{c}, T, \omega) \) against temperature at \( T_c \) as

\[
K^* = 1.691 - 0.781[1 + \xi^2 + <a^2>/(2(1+\xi)) \times (5 - \xi - 10 \xi^2 - 10 \xi^3)]
\]

(3.27)

In the isotropic limit \(<a^2> = 0\) our equation (3.27) is in agreement with equation (27) of Nagi and Upadhyaya [42]. The effect of anisotropy on \( K^* \) is represented in Figure 2.

Here we note that when \(<a^2> = 0.01\) the effect of anisotropy is to reduce \( K^* \) for \( \xi \lesssim 0.54 \) and increases \( K^* \) for \( \xi \gtrsim 0.54 \). Our result shows a significant deviation from previous calculations.
3.2 Thermodynamic Critical Field.

In this section the thermodynamic critical field for an anisotropic superconductor with local states within the gap will be derived. The critical field density for a bulk specimen is calculated from the relation

\[ H_c^2(\vec{c}, T, \Omega) = 8\pi F_{N-S}(\vec{c}, T, \Omega) \quad \ldots (3.28) \]

where \( F_{N-S}(\vec{c}, T, \Omega) \) is the difference in the Helmholtz free energy density of the alloy in the normal and superconducting phase. We have \[43\] \[ F_{N-S}(\vec{c}, T, \Omega) = -\int_0^\infty \delta(1/|\vec{g}|) \Delta^2(\vec{c}, T) \ldots (3.29) \]

The critical field is obtained as

\[ H_c^2(\vec{c}, T, \Omega) = [N(0)B_1(\vec{c}, T, \Omega) \Delta^4(\vec{c}, T)]/(2\pi T_c^2) \quad (3.30) \]

and

\[ \left(\frac{H_c(\vec{c}, T, \Omega)}{H_c(0, T, \Omega)}\right)^2 = \left(\frac{B_1(\vec{c}, T, \Omega)}{B_1(0, T, \Omega)}\right)^4 \left(\frac{T_{co}}{T_c}\right)^2 \quad (3.31) \]

Near \( T_c \) and for low impurity concentration equation (3.30) can be simplified further. In the limit of \( \vec{c} \rightarrow 0 \) and \( \xi = 1 \) (AG approximation) we have

\[ \frac{H_c^2(\vec{c}, T, \Omega)}{8\pi y T_c} = 0.71306[1.4a^2][1 - (T/T_c)^2]^2 \quad (3.32) \]
where $\gamma = \frac{2N(0)}{(3\pi^2)}$ (in energy units)

Near $T_c$, $(1 - (T/T_c))^2$ is very small. Hence equation (3.32) is in agreement with equation (2.25) of J.R. Clem [8] in the appropriate limit. Figure 3 represents the normalized thermodynamic critical field versus $T/T_c$. 
3.3 Density of States.

It is customary to work with the density of states in $K$-space, but here it is much more convenient to consider the density of states for single particle excitations, $N(\omega)$ in $\omega$-space. The reason is this: because of collision broadening, the energy of a given momentum state is spread throughout a region in $K$-space whose width in $K$-space is $\Gamma_1/v_F$. On the other hand, the energy $\omega$ is conserved in each collision so that there is no broadening. We therefore calculate the density of states $N(\omega)$ as

$$
N(\omega) = - \int \frac{d^3k}{(2\pi)^3} \frac{1}{\pi} \text{Im} G_{11}(k, \omega + i\eta)
$$

$$
= - \int \frac{d^3k}{(2\pi)^3} \frac{1}{\pi} \text{Im} \frac{\tilde{\omega} + \xi}{\omega^2 - \xi^2 - \Delta^2}
$$

$$
= N(0) \text{Re} \frac{\tilde{\omega}}{\sqrt{\omega^2 - \Delta^2}}
$$

$$
= N(0) \text{Re} \frac{\tilde{\omega}}{\sqrt{\omega^2 - \Delta^2}}
$$

(3.33)

where $\text{Im}$ stands for the imaginary part and $\text{Re}$ for real part. Here $N(0)$ is the density of states at the Fermi surface. The path of integration can be deformed so that one encircles the positive real axis in a negative sense. $\tilde{\omega}$ and $\Delta$ are given by equations (2.71) and (1.74). Anisotropy can be introduced into the problem by assuming

$$
U = A + a(\Omega).B
$$

(3.34)
Hence

\[ \omega = \omega + \Gamma_1 \int \frac{dn}{4\pi} \frac{(A + a(\lambda), B)\sqrt{1 - (A + a(\lambda), B)^2}}{\xi^2 - (A + a(\lambda), B)^2} \]  \hspace{1cm} (3.35)\

\[ \tilde{\Delta} = \Delta + \Gamma_2 \int \frac{dn}{4\pi} \frac{\sqrt{1 - (A + a(\lambda), B)^2}}{\xi^2 - (A + a(\lambda), B)^2} \]  \hspace{1cm} (3.36)\

expanding in terms of \( a(\lambda) \) we get, by equating the isotropic part

\[ \Delta_0^A = \omega + \Gamma_1 \left( \frac{1}{\xi^2 - A^2} \right) \frac{A \sqrt{A^2 - 1}}{A} - \Gamma_2 \left( \frac{1}{\xi^2 - A^2} \right) \]  \hspace{1cm} (3.37)\

Or

\[ A = \frac{\omega}{\Delta_0} - \left[ \frac{\Gamma_2 - \Gamma_1}{\Delta_0} \right] \frac{A \sqrt{1 - A^2}}{A} \]  \hspace{1cm} (3.38)\

Equating the anisotropic part we obtain

\[ \Delta_0^{A+B} + \Gamma_2 \left( \frac{B \sqrt{1 - A^2}}{\xi^2 - A^2} \right) = C \]  \hspace{1cm} (3.39)\

Using equation (3.38) we obtain,

\[ B = - \Phi A / [1 + \xi((\omega/\Delta_0) - A)/((\Gamma_2 - \Gamma_1)A)] \]  \hspace{1cm} (3.40)\

\( A \) and \( B \) can be determined from the above equations (3.38) and (3.40). Hence using equation (3.33) \( N(\omega)/N(0) \) is calculated. The results of these numerical computations are shown in the figure (7.8), where \( N(\omega)/N(0) \) is plotted against \( \omega/\Delta(\tilde{c}, T) \). The asymptotic curve is \( N(\omega)/N(0) = 1 \), achieved when \( \omega/\Delta(\tilde{c}, T) \rightarrow \infty \). We have taken \( \xi = 0.6 \) and
the different curves correspond to different impurity concentrations. When a single impurity is added, a level is introduced at $\omega/\Delta(C,T) = 0.6$ (i.e., inside the BCS energy gap) but for $\omega/\Delta(C,T) > 1$, $N(\omega)$ consists of two separate parts, one inside and one outside the BCS energy gap. We may note that the present $N(\omega)$ is very much different from the one obtained in the AG theory ($E \to 1$). We observed that the density of states of single particle excitation is great for $(\omega/\Delta(C,T)) < 1$ and small for $(\omega/\Delta(C,T)) > 1$ for anisotropic superconductor.
3.4 Ultrasonic attenuation coefficient.

As an extension to the theory of Shukla and Nagi [44], the anisotropic effect on the ultrasonic attenuation coefficient for a magnetically doped superconductor is derived in this section.

The normalized value of the longitudinal ultrasonic attenuation coefficient \( \alpha_S/\alpha_N \) near \( T_c \) is obtained as

\[
[\alpha_S/\alpha_N] = 1 - \frac{\Delta^2(C,T,\Lambda)}{(8\pi^2 T^2)} \left[ (2\pi T/\alpha) \psi^1((1/2) + (\alpha/2\pi T)) - \psi^2((1/2) + (\alpha/2\pi T)) \right]
\]  (3.41)

where \( \Delta^2(C,T,\Lambda) \) is given by equation (3.9) when \( (\alpha/2\pi T) >> 1 \) (high impurity concentration)

\[
\psi^1((1/2) + (\alpha/2\pi T)) = \frac{2\pi T}{\alpha}
\]  (3.42)

and

\[
\psi^2((1/2) + (\alpha/2\pi T)) = - \left[ \frac{2\pi T}{\alpha} \right]^2
\]  (3.43)

Using equations (3.42) and (3.43) equation (3.41) can be written as

\[
[\alpha_S/\alpha_N] = 1 - \frac{\Delta^2(C,T,\Lambda)/\alpha^2}{(8\pi^2 T^2)}
\]  (3.44)

The order parameter \( \Delta(C,T,\Lambda) \) can be rewritten by considering various quantities near \( T_c \).
The transition temperature for an impure anisotropic superconductor can be written as [14]

\[ T_c^2 = \left( 6a^2/\pi^2 \right) \left[ 1 + \langle a^2 \rangle \left( \alpha^2/\beta^2 \right) \right]^{-1} \left[ \ln(\pi T_{co}/2\gamma) \right. \\
\left. + \langle a^2 \rangle \ln(\pi T_{co}/2\gamma) \right] \quad (3.45) \]

Also

\[ \ln(T/T_{co}) = - \left[ (\Delta^2(\tilde{c},T,\mu)B_1(\tilde{c},T)/(8\pi^2T^2)) \right] - B_0(\tilde{c},T) \quad (3.46) \]

where

\[ B_0(\tilde{c},T) = \left[ 1 + \langle a^2 \rangle \right]^{-1} \left[ \ln(4\gamma / \rho) + (\pi^2T^2/6a^2) \right. \\
\left. + \langle a^2 \rangle \left[ \ln(4\gamma / \rho) + (\pi^2T^2/6\beta^2) \right] \right] \quad (3.47) \]

\( B_1(\tilde{c},T) \) is given by equation (3.12). \( b_o(\tilde{c},T) \) for large impurity concentration can be given by

\[ b_o(\tilde{c},T) = -(1/2)[2\pi T_c / \alpha]^2 - (\alpha/6\pi T)(2 \xi^2 - 1)(2\pi T_c / \alpha)^3 = (5 - 4 \xi^2)(2\pi T_c)^2 / 6a^2 \quad (3.48) \]

Near \( T_c \) polygamma function can be expanded as

\[ \psi^1((1/2)+\rho) = \psi^1((1/2)+\rho_c) + (T - T_c) \psi^2((1/2)+\rho_c) \frac{d\rho}{dT} \quad (3.49) \]

\[ \frac{d\rho}{dT} = - \rho_c / T_c \]
Hence

\[ \psi^1((1/2)+\rho) = \psi^1((1/2)+\rho_c)+(1 - T/T_c) \rho_c \psi^2((1/2)+\rho_c) \]  

(3.50)

\[ \ln(T_{co}/T) = \ln(T_{co}/T_c) + (1 - T/T_c) \]  

(3.51)

\[ B_o(\tilde{c},T_c) = \psi((1/2)+\rho_c) - \psi(1/2) + \langle a^2 \rangle [\psi((1/2)+\sigma_c) - \psi((1/2)+\rho_c)] \]  

(3.52)

\[ b_1(\tilde{c},T_c) = (3/\sigma_c \rho_c) - [\tilde{c}(1-\varepsilon) (2\varepsilon^2-1)(T_{co}/T_c)] \]

\[ [[1/(\sigma-\rho)]^2][\sigma_c + \rho_c/\sigma_c \rho_c][T/T_c]] \]

\[ -[2/(\sigma_c-\rho_c^3)][\ln\sigma_c - \ln \rho_c] \]  

(3.53)

Using equations (3.45) and (3.46) it can be shown that

\[ \ln(T/T_{co}) = - \left[ \Delta^2(\tilde{c},T,\rho) B_1(\tilde{c},T)/8\pi T^2 \right] + \left[ 1+\langle a^2 \rangle \right]^{-1} x \]

\[ [(\pi^2 T^2/6\alpha^2)[1+\langle a^2 \rangle (\alpha^2/\beta^2)] - (\pi^2 T^2/6\alpha^2) - \langle a^2 \rangle (\pi^2 T^2/6\beta^2)] - \ln(T_{co}/T_c) \]

\[ - \langle a^2 \rangle \ln(T_{co}/T_c) \]  

(3.54)

Equations (3.46) - (3.53) give

\[ \Delta^2(\tilde{c},T,\rho) = (8\pi^2 T^2/[b_o(\tilde{c},T)+\langle a^2 \rangle b_1(\tilde{c},T)]) x \]

\[ [(\pi^2 T^2/6\alpha^2) - (\pi^2 T^2/6\alpha^2) + \langle a^2 \rangle [(\pi^2 T^2/6\beta^2) - (\pi^2 T^2/6\beta^2)]] \]  

(3.55)
For isotropic case \((\langle a^2 \rangle = 0)\) equation (3.55) will be reduced to

\[
\Delta^2(\tilde{c}, T, \Omega) = \left(8\pi^2 T^2 \alpha^2 / (5 - 4 \xi^2)(2\pi T)^2 \right) \times \\
\left[ \left( \pi^2 T_c^2 / 6 \alpha^2 \right) - \left( \pi^2 T^2 / 6 \alpha^2 \right) \right] \\
= \left(2\pi^2 T_c^2 / (5 - 4 \xi^2) \right) \left(1 - (T/T_c)^2 \right)
\]

Equation (3.56) is in agreement with equation (3.11) of Shukla and Nagi [44].

Combining equations (3.44) and (3.55), (3.45) we get

\[
[\alpha_s/\alpha_N] \times \left[1 - \left(8\pi^2 T_c^2 / [b_o(\tilde{c}, T) + \langle a^2 \rangle b_1(\tilde{c}, T)] \right)^{-1} \times \\
\left[\left( \pi^2 T_c^2 / 6 \right) - \left( \pi^2 T^2 / 6 \right) + \langle a^2 \rangle \left[\left( \pi^2 T_c^2 / 6 \right) - \left( \pi^2 T^2 / 6 \right) \right] \times \\
(\alpha/\beta)^2 \right]
\]

(3.57)

In the case when \(\langle a^2 \rangle = 0\) equation (3.57) can be written as

\[
[\alpha_s/\alpha_N] = 1 - [12\left(1 - (T/T_c)^2 \right) \ln(\pi T_{co} / 2\gamma\alpha)] / (5 - 4 \xi^2)
\]

(3.58)

Equation (3.58) is in agreement with equation (3.13) of Shukla and Nagi [44]. The initial slope at \(T_c\) is given by

\[
C^N = \frac{\partial}{\partial t} \left[ \alpha_s/\alpha_N \right]_{T \to T_c} = \left[8\pi^2 T_c^2 / \alpha^2 \right] \left[ b_o(\tilde{c}, T) + \langle a^2 \rangle b_1(\tilde{c}, T) \right]^{-1} \times \\
\left[\left( \pi^2 / 6 \alpha^2 \right) + \langle a^2 \rangle \left( \pi^2 / 6 \beta^2 \right) \right]
\]

(3.59)
It can be shown that in the isotropic limit equation (3.59) will be reduced to equation (3.14) of Shukla and Nagi [44].

\[ C^*_A = 24[1 + <a^2>4[1 - (b_1(C,T_c)/b_0(C,T_c))]] x \ln[(\pi T_{co}/2\gamma a)[\pi T_{co}/2\gamma b]^{<a^2>}/[1 + <a^2>4] \quad (3.60) \]

For large impurity concentration

\[ \frac{C^*_A}{C^*_A} = \frac{5 - 4 \varepsilon^2}{1 + <a^2>[(1 + \varepsilon)^2 -(1 + \varepsilon)^2x} \]

\[ (b_1(C,T)/b_0(C,T))][1+4<a^2>][1+<a^2>(1 + \varepsilon)^2]^{-1} \]

\[ [1+4<a^2>[1 - (b_1(C,T_c)/b_0(C,T_c)]^{-1} \quad (3.61) \]

\[ \frac{C^*_A}{C^*_A} = 1 \quad \text{when} \quad \varepsilon = 1 \]

\[ = 0.283, \quad \varepsilon = 0.6 \quad (3.62) \]

\[ = 0.204, \quad \varepsilon = 0 \]

It was observed that when \( \varepsilon = 0.6 \) % change in \( C^*/C^*_A \) = 0.936 and when \( \varepsilon = 0 \), % change in \( C^*/C^*_A \) = 2.245

When \( (a/2\pi T) \ll 1 \) (low impurity concentration equation (3.41) can be shown to be
\[
\begin{align*}
\frac{[\alpha_S/\alpha_N]}{[\Delta^2(\bar{C},T)]/\Delta^2(\bar{C},T)} &= 1 - \left[ \frac{\Delta^2(\bar{C},T)}{8\pi^2T^2} \right] \left[ (2\pi T/\alpha) [\psi^{1/2}] + (\alpha/2\pi T) \psi^{2}(1/2) + (\alpha/2\pi T)^2 \psi^{3}(1/2) \right] \\
&\quad - \psi^{2}(1/2) - (\alpha/2\pi T) \psi^{3}(1/2) \\
&= \left[ \frac{\Delta^2(\bar{C},T)}{8\pi^2T^2} \right] \left[ \psi^{1/2} - (\alpha/2\pi T) \psi^{3}(1/2) \right]
\end{align*}
\]

Also
\[
\Delta^2(\bar{C},T) = -8\pi^2T^2/B_1(\bar{C},T) [B_0(\bar{C},T) + \ln(T/T_{co})]
\]

Where
\[
\begin{align*}
B_0(\bar{C},T) &= \rho \psi^{1/2} + \langle a^2 \rangle [\sigma - \rho] \psi^{1/2} \\
&= 4 \lambda(2) [\rho + \langle a^2 \rangle (\sigma - \rho)] \\
b_0(\bar{C},T) &= 8 \lambda(3) - 32 \lambda(4) (\alpha/2\pi T) (1 + \xi^2) \\
b_1(\bar{C},T) &= -\left( \frac{\rho}{6} \right) \psi^{3}(1/2) (2\xi^2 - 1) + 48 [\lambda(3) - 3\lambda(4) (\sigma - \rho)]
\end{align*}
\]

Hence \(\Delta^2(\bar{C},T)\) is obtained as
\[
\Delta^2(\bar{C},T) = -8\pi^2T^2/B_1(\bar{C},T) [4 \lambda(2) [\rho + \langle a^2 \rangle (\sigma - \rho)] + \ln(T/T_{co})]
\]

If we put
\[
A(T) = \ln(T/T_{co}) + 4 \lambda(2) [\rho + \langle a^2 \rangle (\sigma - \rho)]
\]

Using above equations the normalized ultrasonic attenuation coefficient is given as
\[ \frac{\alpha_S}{\alpha_N} = 1 + \left( \frac{1}{B_1(C,T)} \right) \left[ (2\pi T/\alpha) \psi^1(1/2) - (\alpha/2\pi T)\psi^3(1/2) \right] A(T) \]
\[ = 1 + \left( \frac{A(T)/B_1(T)}{B_1(T)} \right) \left[ (2\pi T/\alpha) 4A(2) - (\alpha/2\pi T) 96A(4) \right] \quad (3.68) \]

\[ \frac{\alpha_S}{\alpha_N} = 1 + \left[ 8\pi T A(T) A(2)/B_1(C,T) \alpha \right] \left[ 1 - (\alpha/2\pi T)^2 24A(4)/A(2) \right] \quad (3.69) \]

Since \((\alpha/2\pi T) \ll 1\) the second term in the bracket can be neglected. Hence equation (3.69) is in agreement with equation (3.16) of Shukla and Nagi[44]. But \(A(T)\) and \(B_1(C,T)\) are different because of anisotropy.

The slope of the normalized ultrasonic attenuation coefficient is obtained as

\[ C^* = \left[ 4A(2)/(\alpha/2\pi T_c)B_1(C,T) \right] \frac{\delta A(T)}{\delta T} \quad \text{as} T \to T_c \quad (3.70) \]

\[ \frac{\delta A(T)}{\delta T} = 1 - \left[ 4A(2)/2\pi T_c \right] [\alpha + <a^2>(\beta - \alpha)] \quad (3.71) \]

\[ C^* = \left[ 8\pi T_c A(2) [1 - (4A(2)/2\pi T_c) [\alpha + <a^2>(\beta - \alpha)]] \times \left[ 1 + <a^2>(1/\alpha) [8A(3) - 32A(4)(\alpha/2\pi T_c)(1 + \varepsilon^2)] + <a^2>[\alpha/6] (2\varepsilon^2 - 1) \psi^3(1/2)/(1 + \varepsilon) + 48[\lambda(3) - 3\lambda(4)(\alpha + \rho)]^{-1} \right] \quad (3.72) \]

It can be shown that in the isotropic limit equation (3.72) reduces to equation (3.20) of Shukla and Nagi[44].
\[ C^*_A = [1 + \langle a^2 \rangle] (4 \lambda(2) / \rho)[1 - 4 \lambda(2)[\rho + \langle a^2 \rangle (\sigma - \rho)]] x \]

\[ [8 \lambda(3) - 64 \lambda(4) \rho + \langle a^2 \rangle [48[\lambda(3) - 3 \lambda(4)(\sigma + \rho)]]]^{-1} \]

(3.73)

Hence from equation (3.72) and (3.73) \([C^*/C^*_A]\) can be derived.

\[ [C^*/C^*_A] = [8.416 - 64.96 \rho_c + \langle a^2 \rangle [50.49 - 227.27 \rho_c]] x \]

\[ [8.416 - 32.48 \rho_c (1 + \rho_c) + \langle a^2 \rangle [50.49 - 146.1 \rho_c [(2+\varepsilon)/(1+\varepsilon)] - 16.24 \rho_c (2\varepsilon^2 - 1)/(1+\varepsilon)]^{-1} \]

(3.74)
3.5 Nuclear Spin Relaxation.

In this section the behaviour of the normalized value of unclear spin relaxation rate \( \frac{R_S}{R_N} \) near \( T_c \) will be discussed.

In this limit one obtains

\[
\frac{R_S}{R_N} = 1 + \left[ \frac{\Delta^2(C,T)}{8\pi^2T^2} \right] \left[ \frac{(2\pi T/\alpha)\psi^1((1/2) + (\alpha/2\pi T))}{\psi^2((1/2) + (\alpha/2\pi T))} \right] + 3\psi^2((1/2) + (\alpha/2\pi T)) \tag{3.75}
\]

For high impurity concentration \( (\alpha/2\pi T) >> 1 \), the above equation gives

\[
\frac{R_S}{R_N} \approx 1 + \left[ \frac{\Delta^2(C,T)}{8\pi^2} \right] \left[ \psi^1((1/2) + (\alpha/2\pi T)) \right] \tag{3.76}
\]

The initial slope of \( \frac{R_S}{R_N} \) can be shown to be of the same form as \( C^* \) in equation (3.72)

For \( (\alpha/2\pi T) << 1 \) (low impurity concentration) we get

\[
\frac{R_S}{R_N} = 1 - \left[ \frac{4C(2)}{\lambda(2)} \right] \frac{A(T)}{B_1(C,T)} \left[ 1 - \frac{16\lambda(3)}{\lambda(2)} \right] \tag{3.77}
\]

The initial slope of \( \frac{R_S}{R_N} \) against \( T/T_c \) near \( T_c \) is

\[
S^* = \left[ \frac{d}{dt} \left( \frac{R_S}{R_N} \right) \right] \quad \text{as} \quad T \to T_c
\]

\[
S^* \approx -\left[ \frac{4\lambda(2)}{B_1(C,T_c)} \right] \left[ 1 - \frac{16\lambda(3)}{\lambda(2)} \right] \left[ 1 - 4\lambda(2)\left( \ddot{\beta} + \frac{a^2}{\beta - \alpha} \right) \right] \tag{3.78}
\]
The isotropic limit result can be obtained from equation (3.78) and found to agree with Shukla and Nagi's result [45].

\[
\left[ \frac{S^*}{S_{AG}^*} \right] = 1 - 3.86 (1 - \xi^2) + \langle a^2 \rangle \ 72D(\bar{\rho} - \bar{\xi})
\]

(3.79)

where

\[
D = - \left[ \left( \frac{C}{6} \right) \left( \frac{T_{co}}{T_c} \right) (1 - \xi)(2\xi^2 - 1) \psi^3(1/2) + 48[\lambda(3) - 3\lambda(4)(\sigma + \bar{\rho})] \right]
\]
3.6 Upper Critical Field. $H_{c2}(T)$

The Upper critical field for a superconductor with anisotropic coupling will be discussed in this section. If the magnetic impurities have randomly oriented spins, we have for an anisotropic superconductor [45]

$$-\ln\left[\frac{T}{T_{co}}\right] = \Psi\left[\left\{\frac{1}{2}\right\} + \left\{\frac{1}{2}\right\} + \left(\frac{D H_{c2}(T)}{2\pi T}\right)\right] - \Psi\left(\frac{1}{2}\right) + \left\langle a^2 \right\rangle \left[\Psi\left[\left\{\frac{1}{2}\right\} + \left(\frac{D H_{c2}(T)}{2\pi T}\right)\right] - \frac{1}{2} - \frac{1}{1 + \xi}\right]\left(\frac{1}{2}\right)$$

where $D$ is the diffusion constant given by

$$D = \frac{\tau_F}{\text{tr} V^2}$$

Then we can write

$$\rho_o = \rho_i + \rho_H$$

$$\rho_i = \left(\frac{1}{2\pi T}\right)\left(\frac{1}{T_2}\right)$$

$$\rho_H = \left(\frac{1}{2\pi T}\right)D H_{c2}(T)$$

we know when $T = T_c$, $H_{c2}(T) = 0$. Hence equation (3.80) reduces to

$$-\ln\left[\frac{T_c}{T_{co}}\right] = \Psi\left[\left\{\frac{1}{2}\right\} + \left(\frac{1}{2}\right) + \left(\frac{D H_{c2}(T)}{2\pi T}\right)\right] - \Psi\left(\frac{1}{2}\right) + \left\langle a^2 \right\rangle \left[\Psi\left[\left\{\frac{1}{2}\right\} + \left(\frac{1}{2}\right) + \left(\frac{D H_{c2}(T)}{2\pi T}\right)\right] - \frac{1}{1 + \xi}\right]$$

(3.81)
since near $T_c$, $\Delta$ is small and $\Delta^2$ can be neglected.

For small impurity concentration ($\langle a/2\pi T \rangle \ll 1$)

$$-\ln t_c = \left(\frac{a}{2\pi T_c}\right) \psi^1(1/2) + \langle a^2 \rangle \left[\left(\frac{a}{2\pi T_c(1+\xi)}\right) \psi^1(1/2) \right]$$

$$-\left(\frac{a}{2\pi T_c}\right) \psi^1(1/2))$$

$$\rho = \left(\frac{a}{2\pi T}\right) = C(1-\xi^2)/t_c$$

$$t_c = T_c/T_{co}$$

$$-\ln t_c = \left(\frac{C(1-\xi^2)/t_c}{(1+\xi)}\right) \psi^1(1/2) + \langle a^2 \rangle \left[\left(\frac{C(1-\xi^2)}{t_c(1+\xi)}\right) \psi^1(1/2) \right]$$

$$\psi^1(1/2) = \left(\frac{C(1-\xi^2)}{t_c}\right) \psi^1(1/2)]$$

(3.83)

that is

$$-t_c \ln t_c = 4.935 \frac{C}{1-\xi} \left[1 + \xi (1 - \langle a^2 \rangle)\right]$$

(3.84)

considering the BCS case ($\langle a/2\pi T \rangle = 0$) as $T \to 0$,

$$\rho_H \gg 1 \Rightarrow$$

$$\ln[T/T_{co}] = \ln[4\gamma DeH_c^2(T)/2\pi T] + \langle a^2 \rangle [\ln((1/(1+\xi))]$$

(3.85)

that is

$$4\gamma DeH_c^2(0)/2\pi T_{co} = (1+\xi)^{\langle a^2 \rangle}$$

$$\rho_{H_{red}} = \frac{4\gamma t}{DeH_c^2(T)/2\pi T} = \frac{(1+\xi)^\langle a^2 \rangle}{\rho_{H_{red}}{^{H_{c2}(T)}}}$$

$$\rho_{H_{red}}{^{H_{c2}(0)}}$$
Equation (3.80) will be reduced to

\[-\ln[T/T_{co}] = \psi((1/2) + \rho_i + \rho_{H_{\text{red}}}) - \psi(1/2)\]

\[+ \langle a^2 \rangle \left[ \psi((1/2) + (\rho_i/(1+\xi)) + \rho_{H_{\text{red}}}) \right.\]

\[\left. - \psi((1/2) + \rho_i + \rho_{H_{\text{red}}}) \right] \tag{3.86}\]

For small impurity concentration and upper critical field, equation (3.80) can be written for large \(T\)

\[-\ln t = \left[ \left( (C(1-\xi^2)/t) + \rho_{H_{\text{red}}} \right) + \langle a^2 \rangle \left[ \left( (1/(1+\xi)) \right) \right. \right.\]

\[\left. \left. \left[ (C(1-\xi^2)/t) + \rho_{H_{\text{red}}} \right] - \left[ (C(1-\xi^2)/t) \right. \right. \right.\]

\[\left. \left. + \rho_{H_{\text{red}}} \right] \right\} 4.935 \tag{3.87}\]

For small \(T\) we obtain

\[-\ln t = \ln[4\gamma(\rho_i + \rho_{H_{\text{red}}})] + \langle a^2 \rangle [\ln((1/(1+\xi))] \tag{3.88}\]

that is we get the reduced upper critical field as

\[h_{c2}(T) = 1 - 4\gamma C(1-\xi)(1+\xi) \tag{3.89}\]

Differentiate equation (3.80) with respect to \(T\) we obtain

\[-(1/T) = \psi^1((1/2) + \rho_i + \rho_{H_{\text{red}}}) \frac{d[\rho_i + \rho_H]}{dT} + \langle a^2 \rangle \left[ \psi^1((1/2) \right.\]

\[\left. \left( (\rho_i + \rho_H)/(1+\xi) \right) - \psi^1((1/2) + \rho_i + \rho_H) \right] dT \left[ \rho_i + \rho_H \right] \tag{3.90}\]
where

\[ \frac{d}{dT} \left( \rho_1 + \rho_H \right) = -\left( \frac{a}{2\pi T} \right) + \frac{\Delta H_{c2}(T)}{2\pi T} \]

\[ -\frac{\Delta H_{c2}(T)}{2\pi T^2} \quad (3.91) \]

where \( \Delta H_{c2}(T) = \frac{d}{dT} H_{c2}(T) \)

For \( T \to T_c \), \( H_{c2}(T) \to 0 \), then equation (3.91) will be reduced to

\[ -\left[ \frac{1}{T_c} \right] = \psi^1((1/2) + \rho_c') \left[ \Delta H_{c2}'(T_c)/2\pi T_c - \rho_c/T_c \right] \]

\[ + \left< a^2 \right> \left[ \psi^1((1/2) + \rho_c/(1+\xi)) \left[ \Delta H_{c2}'(T)/2\pi T_c \right] \right] \]

\[ - \frac{\rho_c}{T_c} - \psi^1((1/2) + \rho_c') \left[ \Delta H_{c2}'(T)/2\pi T_c \right] \]

\[ - \rho_c/T_c \right] \quad (3.92) \]

For pure superconductors \( C \to 0 \), \( T_c \to T_{co} \), equation (3.93) reduces to

\[ -\left[ \frac{1}{T_{co}} \right] = \left[ \Delta H_{c2}'(T_{co})/2\pi T_{co} \right] \psi^1(1/2) \quad (3.93) \]

from which we obtain

\[ \Delta H_{c2}'(T_{co}) = -1.27 \quad (3.94) \]

Using equation (3.94) equation (3.92) can be given by

\[ -1 = -\frac{\rho_c}{T_c} \psi^1((1/2) + \rho_c') - \left< a^2 \right> \psi^1((1/2) + \rho_c/(1+\xi)) \rho_c \]

\[ + \left< a^2 \right> \psi^1((1/2) + \rho_c') c - 0.2026 \tilde{H}_{c2}(T_c) \]

\[ \left[ \psi^1((1/2) + \rho_c') \right] + \left< a^2 \right> \left[ \psi^1((1/2) + \rho_c/(1+\xi)) - \psi^1((1/2) + \rho_c) \right] \quad (3.95) \]
where

$$\tilde{h}_{c2}(T_c) = \frac{H_{c2}(T_c)}{H_{c2}(T_{co})}$$

For small impurity concentration \((\alpha/2\pi T) \ll 1\) we get

$$\tilde{h}_{c2}(T_c) = [1 - \rho_c 4.935]/P \quad (3.96)$$

where

$$P = 1 - 3.4094 \rho_c [1 + \langle a^2 \rangle \xi (1 + \xi)]$$

and for large impurity concentration \((\alpha/2\pi T) \gg 1\) we obtain

$$\tilde{h}_{c2}(T_c) = -4.935 \rho_c \xi /[1 + \xi \langle a^2 \rangle] \quad (3.97)$$

The upper critical field \(H_{c2}(T)\) of \(PbMo_6S_8\) was measured in a temperature region from 1.3K to superconducting transition temperature \(T_c\) by Kiichi Okuda et al [47].

The broad transition observed in the poly crystalline sample may partly be attributed to the anisotropic \(H_{c2}(T)\) in this compound. Recent measurements of \(H_{c2}(T)\) on single crystal by Decroux et al [48] showed that this compound has an anisotropy of about 20% with the maximum \(H_{c2}(T)\) where the field is perpendicular to the ternary axis.
W. Biberacher et al. [49] measured the upper critical field $H_{c2}(T)$ in Nb$_3$S$_4$. The observed large anisotropy of $H_{c2}(T)$ in Nb$_3$S$_4$. Our graph connecting $H_{c2}(T)$ versus $T_c/T_{co}$ is comparable with the corresponding graphs of Jun Takeuchi et al. [50] and Guertin et al [51]. $H_{c2}(T)$ versus $T/T_{co}$ can be compared with the graph of Nagaoka [52].
3.7 Response Function.

In this section we derive the response function to a weak transverse field for an isotropic superconductor described by SR model. The calculations are done in a similar line with Skalski [46]. Dick and Reif [17] have measured the real part $\sigma_1$, of the conductivity of rare earth impurities like Gd in Pb and transition metal impurities like Mn in Pb by measuring the infrared radiation absorbed by a film of the above alloys.

In the above measurements, it was found that the experimental results in the case of rare earth impurities agree with the theoretical ones derived from the AG theory. But in the case of transition metal impurities, the agreement is not satisfactory. The observed effects in this case are more pronounced than those predicted by the AG theory. In the case of rare earth impurities, the localized spin is due to the inner $f$ - electrons and, hence, the coupling between the electrons and the impurities may be weak and the Born approximation, as used by AG, may be valid. In the case of transition metal impurities the spin of the impurities is due to the $d$ - electrons. In this case, the coupling between the electron and magnetic impurities may not be weak, hence the Born approximation may not be justified. So here we make an attempt to derive the response to a weak
transverse magnetic field for an isotropic superconductor described by GR model, where the coupling between the electron and the impurities is strong.

The Green's function $\tilde{G}(p, \omega_n)$ of the superconducting alloy, averaged over the positions and the spin directions of the impurities, is given by [4,5]

$$\tilde{G}(p, \omega_n) = \tilde{G}^0(p, \omega_n) + c_i \tilde{G}^0(p, \omega_n) \tilde{L}(p, p', \omega_n) \tilde{G}(p, \omega_n). \tag{3.98}$$

where $\tilde{G}^0(p, \omega_n)$ is the Green's function of the pure superconductor. $c_i$ is the impurity concentration and $L(p, p', \omega_n)$ is the exact vertex part. Further

$$\tilde{G}^0(p, \omega_n) = [i \omega_n \not{p} - \not{\epsilon} + i \Delta(0, T)p \sigma_2]^{-1} \tag{3.99}$$

$$\omega_n = (2n + 1) \pi T \quad \text{and} \quad \not{\epsilon} = (p^2/2m) - E_F$$

where $E_F$ is the Fermi energy, $\sigma_2$ and $p'$ are the Pauli matrices operating on the ordinary spin states and on the space composed of electron and hole states respectively, while $\Delta(0, T)$ is the temperature dependent order parameter of the pure superconductor. The vertex part is related to the interaction $\tilde{V}(p, p')$ between the electron and the impurity by the relation [5]

$$\tilde{L}(p, p', \omega_n) = \tilde{V}(p, p') + \not{\epsilon} \tilde{V}(p, p') \tilde{G}(p_1, \omega_n) \tilde{L}(p_1, p', \omega_n)$$

where

$$\tilde{V}(p, p') = \begin{bmatrix} V_{\alpha\beta}(p, p') & 0 \\ 0 & V_{\beta\alpha}(p', p) \end{bmatrix}$$
and

\[ V(p, p') = U(p, p') + J(p, p') \sigma \cdot \gamma \]  \hspace{1cm} (3.100)

where \( U(p, p') \) refers to the potential scattering, \( J(p, p') \) is the strength of the exchange interaction, \( \sigma \) is the spin of the impurity and \( \gamma \) denotes the pauli matrices.

Using the above equations, and neglecting Kondo effect Rusinov [5] has calculated the Green's function for a superconducting alloy. According to him \( \bar{G}(p, \omega_n) \) may be written as

\[ \bar{G}(p, \omega_n) = \left[ i \tilde{\omega}_n \gamma_3 - \xi_p + i \tilde{\Delta}_n \gamma_1 \sigma_2 \right]^{-1} \]  \hspace{1cm} (3.101)

where

\[ \tilde{\omega}_n = \omega_n + \left( \frac{\pi C_1}{m_p} \right) \sum_{\ell} (2 \ell + 1) (\sin^2 \delta^+ + \sin^2 \delta^-) \times \]  
\[ U_n \gamma[U_n^2 + 1][U_n^2 + \cos^2(\delta^+ - \delta^-)]^{-1} \]

\[ \tilde{\Delta}_n = + \left( \frac{2 \pi C_1}{m_p} \right) \sum_{\ell} (2 \ell + 1) \sin \delta^+ \sin \delta^- \times \]  
\[ \cos(\delta^+ - \delta^-) \gamma[U_n^2 + 1][U_n^2 + \cos^2(\delta^+ - \delta^-)]^{-1} \]

while

\[ U_n = \tilde{\omega}_n / \tilde{\Delta}_n \]

and \( \xi_p = \cos(\delta^+ - \delta^-) \)

Various physical quantities can be calculated in a closed form when \( \ell = 0 \) (isotropic scattering). Hence we get

\[ \tilde{\omega}_n = \omega_n + \Gamma_1 \left[ U_n \gamma[1 + U_n^2][U_n^2 + \xi_p^2]^{-1} \right] \]  \hspace{1cm} (3.102)
We investigate the effect of magnetic impurities on the electrodynamic properties of superconductors in SR model. As a first problem we calculate the response function.

The response to a weak transverse field is described in terms of the wave-number and frequency-dependent kernel $Q(q,q_o)$ ($q$ - wave number and $q_o$ frequency).

\[ j(q,q_o) = Q(q,q_o) A(q,q_o) \]  

where \( A(q,q_o) \cdot q = 0 \)

We expect that most samples of the alloys will have short mean free paths so that we can assume an essentially local relation between field and current, and therefore have neglected the dependence of the response function on the wave number q. Hence

\[ Q(0,q_o) = \sigma^N_{q_o} + K(0,q_o) \]  

\[ \sigma^N = Ne^2q/m, \quad T = 1/2\Gamma_2 \]

Following the calculations of Skalski [46] the zero-temperature response function is obtained as

\[ K(0,q_o) = (Ne^2i/2mc) \int d\omega \left[ \frac{1}{\sqrt{[\Delta^2 + \omega^2]}} \right] \]

\[ + \int \frac{d^4 \varepsilon}{\pi^2} \frac{\omega^2 + \Delta^2 - \omega^2 - \Delta^2}{(\varepsilon^2 + \Delta^2 - \omega^2)(\varepsilon^2 + \Delta^2 - \omega^2)} \]  

\[ \Delta_n = \Delta + \Gamma_2 u_n \sqrt{[1 + u_n^2]} [u_n^2 + \xi^2]^{-1} \]  

\[ \xi^2 = \cos^2(\delta^+ - \delta^-) \]
The second integration can be performed and can be shown to be equal to \[\left[\gamma(\Delta_+^2 - \omega_+^2)(\Delta_-^2 - \omega_-^2)\right]^{-1}\left[\gamma(\Delta_+^2 - \omega_+^2) + \gamma(\Delta_-^2 - \omega_-^2)\right]^{-1}\]

Hence from equation (3.106) we get
\[
K(0,q_0) = \frac{Ne^2}{2mc} \int \frac{d\omega}{C} \frac{[1 - (1 + \omega_+ \omega_-)][\gamma(\omega_+^2 - 1)(\omega_-^2 - 1)]^{-1}}{[\Delta_+\gamma(\omega_+^2 - 1) + \Delta_-\gamma(\omega_-^2 - 1)\]}
\]

Near \(T_c\)
\[
\int d\omega \rightarrow 2\pi T \sum_{n=0}^{+\infty}
\]

Finite temperature response function (near \(T_c\)) is obtained as

\[
K(0,q_0) = (Ne^2/mc)(2\pi T) \sum_{n=0}^{\infty} \frac{[\omega_+\omega_-]}{\left[1 - \frac{\gamma(\omega_+^2 + 1)(\omega_-^2 + 1)}{\gamma(\omega_+^2 - 1)(\omega_-^2 - 1)}\right]} x \left[\Delta_+\gamma(\omega_+^2 + 1) + \Delta_-\gamma(\omega_-^2 + 1)\right]^{-1}
\]

where

\[
\tilde{\omega}_+ = \omega + (q/2) + \Gamma_1[\gamma(\omega_+^2 + 1)]u_+/[u_+^2 + \xi^2]
\]

and

\[
\tilde{\Delta}_+ = \Delta + \Gamma_2[\gamma(\omega_+^2 + 1)]/[u_+^2 + \xi^2]
\]

\[
\tilde{\omega}_- = \omega - (q/2) + \Gamma_1[\gamma(\omega_-^2 + 1)]/[u_-^2 + \xi^2]
\]

\[
\tilde{\Delta}_- = \Delta + \Gamma_2[\gamma(\omega_-^2 + 1)]/[u_-^2 + \xi^2]
\]

\[
\tilde{u}_+ = \tilde{\omega}_+ / \tilde{\Delta}_+
\]

\[
\tilde{u}_- = \tilde{\omega}_- / \tilde{\Delta}_-
\]

Near \(T_c\) equations (3.110) and (3.112) become
Equation (3.108) reduces to

\[
K(0, q_0) = \langle N e^2 / m c^2 \rangle^2 (2\pi T) \sum_{n=0}^{\infty} \left[ 1 - \frac{[U_+ U_- - 1]}{[(U_+^2 + 1)(U_-^2 + 1)]} \right] \times \\
\left[ [(U_+^2 + 1)/(U_+^2 + \varepsilon^2)] + [(U_-^2 + 1)/(U_-^2 + \varepsilon^2)]^{-1} \right]^{-1}
\]

(3.113)

Near \( T_c \), \( \Delta \) is small and \( U \) will be very large. So we can expand various quantities as following

\[
\frac{U_+^2 + 1}{U_+^2 + \varepsilon^2} = 1 - \frac{\varepsilon^2}{U_+^2} + \frac{1}{U_+^2}
\]

(3.114)

Equation (3.113) can be shown to be

\[
K(0, q_0) = \langle N e^2 / m c^2 \rangle^2 (2\pi T) \sum_{n=0}^{\infty} \left[ (1/U_+^2) + (1/U_-^2) \right] \\
+ (2/(U_+ U_-)) + (\varepsilon^2 - 2)[(1/U_+^2 U_-) + (1/U_+ U_-^2)] \\
+ (\varepsilon^2 - (3/2))(1/U_+^2 U_-) + (\varepsilon^2 - (5/4)) x \\
[1/U_+^4] + (1/U_-^4)]
\]

(3.115)

But \( U_+ \) and \( U_- \) are obtained from Ref. [53] as given below

\[
[1/U_+] = \left[ \Delta / (\omega_+ \alpha) \right] + [(2\varepsilon^2 - 1) \alpha \Delta^3 / (2(\omega_+ \alpha)^4)]
\]

\[
[1/U_-] = \left[ \Delta / (\omega_- \alpha) \right] + [(2\varepsilon^2 - 1) \alpha \Delta^3 / (2(\omega_- \alpha)^4)]
\]

\[
[1/U_+ U_-] = \Delta^2 [(\omega_+(q/2)+\alpha)(\omega_-(q/2)+\alpha)]^{-1} + (2\varepsilon^2 - 1) \alpha \Delta^4 \times \\
[2(\omega_+(q/2)+\alpha)^4(\omega_-(q/2)+\alpha)]^{-1} + (2\varepsilon^2 - 1) \alpha \Delta^4 \times \\
[2(\omega_+(q/2)+\alpha)(\omega_-(q/2)+\alpha)^4]^{-1}
\]
where we use

\[ \omega_+ = \omega + (q/2) \]
\[ \omega_- = \omega - (q/2) \]

\[ \frac{1}{U^2_+} = \Delta^2 [\omega_+(q/2) + \alpha]^2 - \alpha \Delta^4 (2\xi^2 - 1) [\omega_+(q/2) + \alpha]^5 \]
\[ [1/U^2_-] = \Delta^2 [\omega_-(q/2) + \alpha]^2 - \alpha \Delta^4 (2\xi^2 - 1) [\omega_-(q/2) + \alpha]^5 \]
\[ [1/U^3_+ U^-_1] = \Delta^4 [(\omega_+(q/2) + \alpha)^2 (\omega_-(q/2) + \alpha)^2]^{-1} \]
\[ [1/U^3_-] = \Delta^4 [(\omega_+(q/2) + \alpha)^3 (\omega_-(q/2) + \alpha)]^{-1} \]
\[ [1/U_+ U^-_3] = \Delta^4 [(\omega_+(q/2) + \alpha)(\omega_-(q/2) + \alpha)^3]^{-1}\]

After doing partial fraction and converting sum into polygamma function equation (3.115) becomes

\[ K(0,q_0) = [\sigma^N(nT/2C)(\Delta^2/(2\pi T)^2)] \left[ \psi^{1}((1/2)+\hat{p}+q/4\pi T) \right. \]
\[ + \psi^{1}((1/2)+\hat{p}-q/4\pi T)] - [(2\xi^2 - 1)\hat{p} \Delta^4/(24(2\pi T)^4) \]
\[ \left[ \psi^{4}((1/2)+\hat{p}+q/4\pi T) + \psi^{4}((1/2)+\hat{p}-q/4\pi T) \right] \]
\[ + (\Delta^2/q\pi T)[\psi((1/2)+\hat{p}+q/4\pi T) - \psi((1/2)+\hat{p}-q/4\pi T)] \]
\[ + (2\xi^2 - 1)\hat{p} \Delta^4 [(1/(q^3 2\pi T))\psi^{1}((1/2)+\hat{p}-q/4\pi T)] \]
\[ - \psi^{1}((1/2)+\hat{p}+q/4\pi T)] + (1/2q^2(2\pi T)^2) \times \]
\[ \left[ \psi^{2}((1/2)+\hat{p}+q/4\pi T) + \psi^{2}((1/2)+\hat{p}-q/4\pi T) \right] \]

contd.
\[+(1/6q(2\pi T)^3)[\psi^3((1/2)+\rho -q/4\pi T)-\psi^3((1/2)+\rho +q/4\pi T)]\]
\[+(\varepsilon^2-2)\Delta^4[(1/q^3\pi T)[\psi^3((1/2)+\rho +q/4\pi T) - \psi^3((1/2)+\rho -q/4\pi T)] - (1/q^2(2\pi T)^2)[\psi^1((1/2)+\rho +q/4\pi T) + \psi^1((1/2)+\rho -q/4\pi T)] - (1/2q(2\pi T)^3) x \]
\[\psi^2((1/2)+\rho -q/4\pi T) - \psi^2((1/2)+\rho +q/4\pi T)]\]
\[+(\varepsilon^2-(3/2))[(2\Delta^4/q^3(2\pi T))[\psi^3((1/2)+\rho -q/4\pi T) - \psi^3((1/2)+\rho +q/4\pi T) + (\Delta^4/q^2(2\pi T)^2)[\psi^1((1/2)+\rho +q/4\pi T) + \psi^1((1/2)+\rho -q/4\pi T)] + (\varepsilon^2-(5/4)\Delta^4/6(2\pi T)^4) x \]
\[\psi^3((1/2)+\rho +q/4\pi T) + \psi^3((1/2)+\rho -q/4\pi T)]\]

(3.116)

where
\[\sigma^N = Ne^2/2m\Gamma_2\]

For small \(q_0\) we get
\[K(0,q_0) = \frac{\sigma^N}{C_n T} [\Delta^2\psi^1((1/2)+\rho) + (\Delta^2/12)(q/2\pi T)^2\psi^3((1/2)+\rho) - (\Delta^4/(2\pi T)^2)((2-\varepsilon^2)/6)\psi^3((1/2)+\rho) + (2\varepsilon^2-1)\psi^4((1/2)+\rho)/24]\]

(3.117)

as \(q_0 \to 0\) equation (3.117) agrees with equation (9) of K. Maki[54] when \(\omega >> \alpha\).

\[Q(0,q) = \sigma^N q + K(0,q)\]

and from Skalski [46] the conductivity is given by
\[\text{Re } \sigma(0,q) + \text{Im } \sigma(0,q) = (C/\omega) K(0,q)\]

(3.118)
3.8 Kondo Effect in the Problem

In the Shiba–Husinov model, the Kondo effect is neglected. The effect is essentially arises while considering the scattering of a conduction electron from a magnetic impurity exactly if, (i) the exchange coupling between the electron and the impurity spins is antiferromagnetic and (ii) the non-commutativity of the spin operators is taken into account. The role of the Kondo effect in superconductors is considerably clarified by the work of Muller–Hartmann and Zittartz [3]. Within certain approximations one can include the Kondo effect in the SR model by assuming that depends on \( \ln \left( \frac{T_K}{T} \right) \), where \( T_K \) is the Kondo temperature. These approximations are (i) the electron energy is near the Fermi energy (ii) the temperature \( T \) is near the transition temperature \( T_c \) and (iii) the impurity concentration is very low. The actual relation between and is more complicated but several authors have used the simpler relation

\[
\xi^2 = \frac{\nu^2}{\nu^2 + \pi^2 S(S+1)} \quad \text{and} \quad \nu = \ln \left( \frac{T_K}{T_K} \right)
\]

where \( S \) is the impurity spin. We take \( \nu = \ln \left( \frac{T_K}{T_c} \right) \)

\[
\nu_c = \ln \left( \frac{T_K}{T_{c0}} \right) + \ln \left( \frac{T_{c0}}{T_c} \right)
\]

Taking \( S = 1/2 \) we get

\[
\xi^2 = \frac{0.1351 \ln^2 \left( \frac{T_c}{T_K} \right)}{\left[ 0.1351 \ln^2 \left( \frac{T_c}{T_K} \right) + 1 \right]}
\]
We have calculated the slope $K^*$ of the Josephson current versus $T/T_c$, $K^*$ of the Josephson current versus $T_C/T_{co}$ for various values of $T_K/T_{co}$ and are tabulated in the tables.

We have also observed that $\xi$ is anisotropic. We have tabulated the different values of the local states within the gap $\xi$ for $\langle a^2 \rangle = 0$ and $\langle a^2 \rangle = 0.05$. We observed significant change in $\xi$ due to anisotropy.