Chapter 4

Electrothermal Convection in a Rotating Dielectric Fluid Layer: Effect of Velocity and Temperature Boundary Conditions

4.1 Introduction

Thermal convection in a rotating fluid layer heated from below is one of the classical problems and it is considered to be a simple model system for a wide variety of geophysical and astrophysical flows. A striking example of rotation-affected convection is the deep convective chimneys in the polar region, where surface cooling increases the density of the near surface water to initiate convection. Atmospheric flow is another major application area and such flows are also found on celestial bodies other than our own. Examples include the outer convective layer of our Sun and the giant planets in our solar system. Hence, rotating convection is a relevant topic for meteorologists, climatologists, oceanographers and astrophysicists alike. Besides, the problem is also of importance in many engineering applications. Copious literature is available on thermal convection in a rotating fluid layer (Chandrasekhar, 1961; Galdi and Straughan, 1985; Kloosterziel and Carnevale, 2003 and references therein). Recently, a study of the thermal instability in an initially quiescent liquid, placed between two horizontal plates, irradiated by a volumetric heat source, and subject to rotation about the vertical axis has been carried out by Chatterjee et al. (2008).

It is observable that many convective instability problems of practical importance involve electrically conducting fluids. In such cases, the effect of external fields like magnetic and electric fields become important. In particular, the magnetic field effects become dominant if the fluid is highly electrically conducting. To the contrary, if the fluid is dielectric with low electrical conductivity then the electric forces play a major role in driving the motion. Electrohydrodynamics, coupling the electric field and fluid motion, has led to many complex and interesting instability phenomena in microchannels. It has been realized that utilizing electric force to manipulate fluids is the most efficient method in the achievement of a variety of purposes and functions in the applications of microfluidic
devices. A brief discussion on the applications of electrohydrodynamic (EHD) instability has been recently presented by Lin (2009). Here, we concentrate on EHD motion which is known to occur in liquids when spatial gradients in the electrical properties are present.

Several studies have been carried out to assess the effect of AC and/or DC electric fields on natural convection due to the fact that many problems of practical importance involve dielectric fluids. In these fluids, an applied temperature gradient produces non-uniformities in the electrical conductivity and/or the dielectric permittivity. The variation of electrical conductivity of the fluid with temperature produces free charges in the bulk of the fluid. These free charges interacting with applied or induced electric field produce a force that eventually causes fluid motion. On the other hand, when there is variation in dielectric permittivity and the electric field is intense then the polarization force which is induced by the non-uniformity of the dielectric constant causes fluid motion. In either case, convection can occur in a dielectric fluid layer even if the temperature gradient is stabilizing and such an instability produced by an electric field is called electroconvection, which is analogous to Rayleigh-Bénard convection. In addition, if the applied temperature gradient is also destabilizing then such an instability problem is called electrothermal convection. Natural convection problem under an AC and/or DC electric field has been studied extensively (Turnbull, 1968; Robert, 1969; Takashima and Aldridge, 1976; Martin and Richardson, 1984; Maekawa et al, 1992; Pontiga and Castellanos, 1994). An exhaustive review on this topic has been given by Jones (1978) and Saville (1997). The linear stability of a poorly conducting fluid in a constant electric field of a horizontal capacitor is investigated under a modulated thermal field by Smorodin (2001). The combined effects of DC electric field and volumetric heat source on the onset of convection in a dielectric fluid layer heated from below is investigated by Shivakumara et al. (2007b), while the influences of a vertical AC electric field as well as internal heat generation on the onset of electrothermooconvection in a horizontal dielectric fluid layer is analyzed by Shivakumara et al. (2009). A more detailed analysis on EHD instability in a horizontal fluid layer with electrical conductivity gradient subject to a weak shear flow is presented by Chang et al. (2009).

Studies have also been undertaken in the past to understand the effect of rotation on electroconvection but it is still in its infancy. Takashima (1976) was the first to consider the effect of rotation on the onset of instability in a dielectric fluid layer under the action of a vertical AC electric field and a vertical temperature gradient for stress-free isothermal
boundaries. The influences of an AC electric field and rotation on Benard-Marangoni instability in a layer of an incompressible fluid with small electrical conductivity are investigated by Douiebe et al. (2001). Othman (2004) has studied the stability of a rotating layer of viscoelastic dielectric liquid heated from below, while Ruo et al. (2010) have considered the EHD instability of a horizontal rotating fluid layer with a vertical electrical conductivity gradient for different kinds of velocity boundary conditions.

In the present chapter, it is intended to study the effect of various types of velocity and temperature boundary conditions on the criterion for the onset of thermal convection in a horizontal rotating dielectric fluid layer under the influence of a vertical AC electric field. The stability characteristics of the system are discussed for three kinds of velocity boundary conditions namely, (i) free-free, (ii) rigid-rigid and (iii) lower rigid and upper free which are considered to be either isothermal or insulated to temperature perturbations. The eigenvalue problem is solved exactly for free-free isothermal boundaries, while numerically using the Galerkin method for the other combinations of velocity and temperature boundary conditions. The coupling effects of electrical body force, buoyancy force and the Coriolis force on electrothermal convection are explored in detail.

4.2 Mathematical Formulation

The physical configuration is as shown in Fig.4.1 consists of a dielectric fluid layer of thickness $d$ with a uniform vertical AC electric field applied across the layer; the lower surface is grounded and the upper surface is kept at an alternating (60 Hz) potential whose root mean square value is $V_i$ (Takashima, 1976). The dielectric fluid layer is rotating about the vertical axis with constant angular velocity $\hat{\Omega} = (0, 0, \Omega)$. The lower and upper boundaries of the layer are maintained at uniform, but different temperatures $T_L$ and $T_U$ ($< T_L$) respectively, and thus a constant temperature difference $\Delta T = T_L - T_U$ is maintained between the boundaries. A Cartesian coordinate system $(x, y, z)$ is chosen with the origin at the bottom of the fluid layer and the z- axis normal to the fluid layer in the gravitational field.

The relevant basic equations under the Oberbeck-Boussinesq approximation are:

$$\nabla \cdot \vec{q} = 0$$

(4.1)
\[ \rho_0 \left[ \frac{\partial \vec{u}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} + 2(\vec{\Omega} \times \vec{q}) \right] = -\nabla p + \rho \vec{g} + \eta \nabla^2 \vec{q} + \vec{f}_e \]  

(4.2)

\[ \frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla) T = \kappa \nabla^2 T \]  

(4.3)

\[ \rho = \rho_0 \left[ 1 - \alpha (T - T_i) \right] \]  

(4.4)

where the quantities have their pre-defined meaning and \( \vec{f}_e \) is the force of electrical origin which can be expressed as (Landau, 1960)

\[ \vec{f}_e = \rho_e \vec{E} - \frac{1}{2}(\vec{E} \cdot \vec{E}) \nabla \varepsilon + \frac{1}{2} \nabla \left( \rho \frac{\partial \varepsilon}{\partial \rho} \vec{E} \cdot \vec{E} \right) \]  

(4.5)

Here \( \vec{E} \) is the root mean square value of the electric field, \( \rho_e \) is the charge density and \( \varepsilon \) is the dielectric constant. The electrical force \( \vec{f}_e \) will have no effect on the bulk of the dielectric fluid if the dielectric constant \( \varepsilon \) and the electrical conductivity \( \sigma \) are homogeneous. Since \( \varepsilon \) and \( \sigma \) are functions of temperature, a temperature gradient applied to a dielectric fluid produces a gradient in \( \varepsilon \) and \( \sigma \). The application of a DC electric field then results in the accumulation of free charge in the liquid. The free charge increases exponentially in time with a time constant \( \varepsilon / \sigma \), which is known as the electrical relaxation time. If an AC electric field is applied at a frequency much higher than the reciprocal of the electrical relaxation time, the free charge does not have time to accumulate. Moreover, the electrical relaxation times of most dielectric liquids appear to be sufficiently long to prevent the buildup of free charge at standard power line frequencies. At the same time, dielectric loss at these frequencies is so low that it makes no significant contribution to the temperature field. Furthermore, since the second term in the above equation depends on \( (\vec{E} \cdot \vec{E}) \) rather than \( \vec{E} \) and the variation of \( \vec{E} \) is very rapid, the root mean square value of \( \vec{E} \) can be assumed as the effective value. In other words, we can treat the AC electric field as the DC electric field whose strength is equal to the root mean square value of the AC electric field (Takashima, 1976).

The relevant Maxwell equations are then

\[ \nabla \times \vec{E} = 0 \]  

(4.6)

\[ \nabla \cdot (\varepsilon \vec{E}) = 0. \]  

(4.7)

In view of (4.6), \( \vec{E} \) can be expressed as

\[ \vec{E} = -\nabla V \]  

(4.8)
where $V$ is the root mean square value of the electric potential. The dielectric constant is assumed to be a linear function of temperature in the form

$$\varepsilon = \varepsilon_0 [1 - \gamma (T - T_0)]$$  \hspace{1cm} (4.9)

where, $\gamma$ (>0) is the thermal expansion coefficient of dielectric constant and is assumed to be small. The basic state is quiescent and is given by

$$\ddot{q} = \ddot{q}_b = 0, \quad T = T_b(z), \quad p = p_b(z), \quad \tilde{E} = \tilde{E}_b(z), \quad \varepsilon = \varepsilon_b(z)$$  \hspace{1cm} (4.10)

where the subscript $b$ denotes the basic state. It is found that

$$T_b - T_L = -\Delta T z / d$$  \hspace{1cm} (4.11)

$$\tilde{E}_b(z) = \frac{E_0}{(1 + \gamma \Delta T z / d)}$$  \hspace{1cm} (4.12)

$$p_b = p_0 - \rho_0 g z - \frac{\rho_0 g \alpha \Delta T z^2}{2 d} + \frac{\varepsilon_0 E_0^2}{(1 + \gamma \Delta T z / d)}$$  \hspace{1cm} (4.13)

and hence

$$V_b(z) = -\frac{E_0 d}{\gamma \Delta T} \log \left(1 + \gamma \Delta T z / d \right)$$  \hspace{1cm} (4.14)

where,

$$E_0 = -\frac{V_i \gamma \Delta T / d}{\log (1 + \gamma \Delta T)}$$  \hspace{1cm} (4.15)

is the root mean square value of the electric field at $z = 0$.

To study the stability of the basic state, we superimpose infinitesimally small perturbations $(\ddot{q}', \ p', \ \tilde{E}', \ T', \ \rho', \ \varepsilon')$ on the basic state in the form

$$\ddot{q} = \ddot{q}', \quad p = p_b + p', \quad \tilde{E} = \tilde{E}_b + \tilde{E}', \quad T = T_b + T', \quad \rho = \rho_b + \rho', \quad \varepsilon = \varepsilon_b + \varepsilon'.$$  \hspace{1cm} (4.16)

Substituting quantities from (4.16) into (4.1) - (4.9), linearizing by neglecting the products of primed quantities, eliminating the pressure from the momentum equation by operating curl twice and retaining the vertical component and non-dimensionalizing the resulting equations by scaling $(x, y, z)$ by $d$, $t$ by $d^2 / \kappa$, $\ddot{q}'$ by $\kappa / d$, $\tilde{E}'$ by $\kappa / d^2$, $T'$ by $\Delta T$ and $V'$ by $\gamma E_0 \Delta T d$, we obtain the stability equations (after neglecting the primes for simplicity) in the form
\[
\left( \frac{1}{Pr} \frac{\partial}{\partial t} - \nabla^2 \right) \nabla^2 w = R_t \nabla_h^2 T - T d^{1/2} \frac{\partial \xi}{\partial z} + R_c \nabla_h^2 \left( T - \frac{\partial V}{\partial z} \right)
\]
(4.17)
\[
\left( \frac{1}{Pr} \frac{\partial}{\partial t} - \nabla^2 \right) \xi = T d^{1/2} \frac{\partial w}{\partial z}
\]
(4.18)
\[
\left( \frac{\partial}{\partial t} - \nabla^2 \right) T = w
\]
(4.19)
\[
\nabla^2 V = \frac{\partial T}{\partial z}
\]
(4.20)

where, \( R_t = \alpha g \Delta T d^3 / \nu \kappa \) is the thermal Rayleigh number, \( R_c = \gamma^2 \varepsilon_0 E_0^2 (\Delta T)^2 d^2 / \eta \kappa \) is the AC electric Rayleigh number, \( Ta = 4 \Omega d^4 / \nu^2 \) is the Taylor number, \( Pr = \nu / \kappa \) is the Prandtl number and \( \zeta = \partial v / \partial x - \partial u / \partial y \) is the \( z \)-component of vorticity.

The boundaries of the fluid layer are either stress-free or rigid which are considered to be either isothermal or insulated to temperature perturbations. We have thus considered the following boundary conditions:
\[
w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial \xi}{\partial z} = \frac{\partial V}{\partial z} = 0, \quad T = 0 \text{ or } DT = 0.
\]
(4.21)
on the stress-free boundary and
\[
w = \frac{\partial w}{\partial z} = \xi = V = 0, \quad T = 0 \text{ or } DT = 0.
\]
(4.22)
on the rigid boundary. It may be noted that only one type of boundary condition on \( V \) is considered on the rigid and stress-free boundaries in investigating the problem although either of the conditions can be imposed on these boundaries (Turnbull, 1968; Maekawa et al. 1992).

### 4.3 Linear Stability Analysis

To carry out the linear stability analysis, we employ the normal mode analysis procedure in which we look for the solution of the form
\[
(w, T, V, \zeta) = (W, \Theta, \Phi, Z)(z) \exp(i \ell x + im y + i \omega t)
\]
(4.23)
where \( \ell \) and \( m \) are the horizontal wave numbers in the \( x \) and \( y \) directions respectively and \( \omega (= \omega_r + i \omega_i) \) is the growth rate. Substituting (4.23) into (4.17) - (4.20), we obtain

\[
\begin{align*}
\left\{ \frac{\omega}{Pr} - (D^2 - a^2) \right\} \left( D^2 - a^2 \right) W &= -R_e a^2 \Theta - Ta^{1/2} DZ - R_c a^2 \left( \Theta - D\Phi \right) \\
\left\{ \frac{\omega}{Pr} - (D^2 - a^2) \right\} Z &= Ta^{1/2} DW \\
\left\{ \omega - (D^2 - a^2) \right\} \Theta &= W \\
\left( D^2 - a^2 \right) \Phi &= D\Theta
\end{align*}
\] (4.24)

where \( D = \frac{d}{dz} \) and \( a = \sqrt{\ell^2 + m^2} \) is the overall horizontal wave number.

On using (4.23) in the boundary conditions (4.21) and (4.22), we get respectively

\[
W = D^2 W = DZ = D\Phi = 0, \quad \Theta = 0 \quad \text{or} \quad D\Theta = 0.
\] (4.28)

and

\[
W = DW = Z = \Phi = 0, \quad \Theta = 0 \quad \text{or} \quad D\Theta = 0.
\] (4.29)

The above set of equations is a double eigenvalue problem for \( R_c \) or \( R_e \) and \( \omega \), to be solved with respect to the chosen boundary conditions.

**4.3.1 Exact solution for free-free isothermal boundaries**

For both boundaries considered, let us assume the solution in the following form such that they satisfy the respective boundary conditions:

\[
W = A_1 \sin \pi z, \quad \Theta = A_2 \sin \pi z, \quad Z = A_3 \cos \pi z, \quad \Phi = A_4 \cos \pi z
\] (4.30)

where \( A_1 - A_4 \) are constants. Substituting (4.30) into (4.24) - (4.27), we find the condition for the existence of a non-trivial eigenvalue is
\[ \begin{vmatrix} \frac{\omega}{Pr} + \delta^2 & -(R_t + R_e)a^2 & \pi Ta^{1/2} & -R_e a^2 \pi \\ -1 & \omega + \delta^2 & 0 & 0 \\ -\pi Ta^{1/2} & 0 & \left( \frac{\omega}{Pr} + \delta^2 \right) & 0 \\ 0 & \pi & 0 & \delta^2 \end{vmatrix} = 0 \] (4.31)

where \( \delta^2 = \pi^2 + a^2 \). Expanding the above determinant yields an expression for the thermal Rayleigh number in the form

\[ R_t = \frac{\delta^2 \left( \delta^2 + \omega \right) \left( \delta^2 + \frac{\omega}{Pr} \right)}{a^2} + \frac{\pi^2 Ta}{a^2} \left( \frac{\delta^2 + \omega}{Pr} \right) - \frac{a^2}{\delta^2} R_e. \] (4.32)

To examine the stability of the system, the real part of \( \omega \) is set to zero and take \( \omega = i\omega_l \) in (4.31). After clearing the complex quantity from the denominator, (4.32) yields

\[ R_t = \frac{\delta^2}{a^2} \left( \delta^4 - \frac{\omega_l^2}{Pr} \right) + \frac{\pi^2 Ta}{a^2} \left( \frac{\delta^4 + \frac{\omega_l^2}{Pr}}{\delta^2 + \frac{\omega_l^2}{Pr^2}} \right) - \frac{a^2}{\delta^2} R_e + i\omega_l N \] (4.33)

where

\[ N = \frac{\delta^2}{a^2 Pr} \left[ \delta^2 \left( Pr + 1 \right) + \frac{\pi^2 Ta (Pr - 1)}{\left( \delta^4 + \frac{\omega_l^2}{Pr^2} \right)} \right]. \] (4.34)

Since \( R_t \) is a physical quantity, it must be real and hence either \( \omega_l = 0 \) or \( N = 0 \) in (4.33). Accordingly, we get the condition for the occurrence of stationary and oscillatory convection. It is found that stationary convection (i.e., \( \omega_l = 0 \)) occurs at \( R_t = R_t^s \), where

\[ R_t^s = \frac{\delta^6}{a^2} + \frac{\pi^2 Ta}{a^2} - \frac{a^2}{\delta^2} R_e. \] (4.35)

To find the critical value of \( R_t^s \), (4.35) is differentiated with respect to \( a^2 \) and equated to zero. A polynomial in \( (a^2) \) whose coefficients are functions of the physical parameters influencing the instability is obtained in the form
\[2 \left( a_c^2 \right)^5 + 7 \pi^2 \left( a_c^2 \right)^4 + 8 \pi^4 \left( a_c^2 \right)^3 + \left( 2 \pi^6 - \pi^2 R_c + \pi^2 Ta \right) \left( a_c^2 \right)^2 \]
\[- 2 \pi^8 \left( 1 + \frac{Ta}{\pi^4} \right) \left( a_c^2 \right)^3 - \pi^{10} \left( 1 + \frac{Ta}{\pi^4} \right) = 0. \] (4.36)

The above equation is solved numerically for various values of \( R_c \) as well as \( Ta \) and the minimum value of \( a_c^2 \) is obtained each time and using this \( a_c^2 \) in (4.35), the critical Rayleigh number above which the electrothermal convection sets-in is determined. It is interesting to check (4.35) for existing results in the literature under some limiting cases. When \( R_c = 0 \), (4.35) reduces to

\[ R_i = \delta^6 \frac{\pi^2 Ta}{a^2} \] (4.37)

and coincides with Chandrasekhar (1961). When \( Ta = 0 \), (4.35) reduces to

\[ R_i = \delta^6 - \frac{R_c a^2}{\delta^2} \] (4.38)

and coincides with Roberts (1969).

The oscillatory onset corresponds to \( N = 0 (\omega_j \neq 0) \) in (4.33) and this gives an expression for the oscillation frequency \( \omega_i^2 \) as

\[ \omega_i^2 = -\delta^4 Pr^2 + \frac{(1 - Pr) Pr^2 \pi^2 Ta}{\delta^2 (1 + Pr)} \] (4.39)

Now the condition that \( \omega_i^2 > 0 \) provides the fact that oscillatory instability can occur only if

\[ 0 < Pr < 1 \quad \text{and} \quad Ta > \frac{\delta^6 \left( 1 + Pr \right)}{\pi^2 \left( 1 - Pr \right)}. \] (4.40)

It is interesting to note that these conditions are exactly in the same form as that of the ordinary viscous fluid (Chandrasekhar, 1961) suggesting that the vertical AC electric field does not influence the necessary conditions for the existence of oscillatory convection. Eliminating \( \omega_i^2 \) from (4.33) using (4.39) and noting \( N = 0 \), we note that the oscillatory convection occurs at \( R_i = R_i^0 \), where
When $R_e = 0$, the above equation coincides with the expression given by Chandrasekhar (1961). Moreover, it is interesting to note that $\omega_i^2$ can be written in the form

$$\omega_i^2 = \frac{Pr^2 a^2}{\delta^2 (1+2Pr)} \left( R_i^e - R_i^0 \right).$$

(4.42)

From (4.42) it is evident that if the instability sets in as oscillatory motions, it always occurs at a thermal Rayleigh number less than that of stationary onset.

### 4.4 Numerical Solution

It has been observed that oscillatory convection occurs only if the Prandtl number $Pr$ is less than unity and the Taylor number exceeds a threshold (see (4.39)). But for dielectric fluids, Prandtl number is much greater than unity (for example, for corn oil $Pr = 480$, silicone oil $Pr = 100$ and for caster oil $Pr = 10000$) and hence the oscillatory convection is ruled out as the preferred mode of instability for dielectric fluids. Under the circumstances, we restrict ourselves to the case of steady onset and put $\omega = 0$ in (4.24) - (4.27). As in the case of stress-free isothermal boundaries, an exact solution is not possible for the other types of velocity and temperature boundary conditions and then one has to resort to numerical methods to extract the critical stability parameters. For this, the Galerkin method is adopted to solve the resulting eigenvalue problem. Accordingly, the variables are written in a series of basis functions as

$$W = \sum A_i W_i, \quad \Theta = \sum B_i \Theta_i, \quad Z = \sum C_i Z_i, \quad \Phi = \sum D_i \Phi_i$$

(4.43)

where $A_i, B_i, C_i$ and $D_i$ are constants and the basis functions $W_i, \Theta_i, Z_i$ and $\Phi_i$ will be represented by the power series satisfying the respective boundary conditions. Substituting (4.43) into (4.24)–(4.27) (after noting $\omega = 0$), multiplying the resulting momentum equation by $W_j(z)$, vorticity equation by $\xi_j(z)$, energy equation by $\Theta_j(z)$, electric potential equation by $\Phi_j(z)$; performing the integration by parts with respect to $z$ between $z = 0$ and $z = 1$ and using the boundary conditions, leads to the following system of linear homogeneous algebraic equations:
\[ E_{ji} A_i + F_{ji} B_j + G_{ji} C_i + H_{ji} D_i = 0 \]

\[ I_{ji} A_i + J_{ji} B_i = 0 \tag{4.44} \]

\[ K_{ji} A_i + L_{ji} C_i = 0 \]

\[ M_{ji} B_i + N_{ji} D_i = 0 \]

where,

\[ E_{ji} = < D^2 W_j D^2 W_i + a^4 W_j W_i + 2a^2 DW_j DW_i >, \quad F_{ji} = - < R_i a^2 W_j \Theta_i + R_j a^2 W_j \Theta_i > \\ G_{ji} = - < Ta^{1/2} W_j DZ_i >, \quad H_{ji} = < R_i a^2 W_j D\Phi_i >, \quad I_{ji} = < \Theta_j W_i > \]

\[ J_{ji} = - < D\Theta_i D\Theta_j + a^2 \Theta_i \Theta_j >, \quad K_{ji} = < Ta^{1/2} Z_j DW_i >, \]

\[ L_{ji} = - < DZ_j DZ_i + a^2 Z_j Z_i >, \quad M_{ji} = - < \Phi_j D\Theta_i >, \quad N_{ji} = - < D\Phi_j D\Phi_i + a^2 \Phi_j \Phi_i > \]

Here the inner product is defined as \(< f_1, f_2 > = \frac{1}{0} \int f_1 f_2 \, dz \).

The above set of homogeneous algebraic equations can have a non-trivial solution if and only if

\[
\begin{vmatrix}
E_{ji} & F_{ji} & G_{ji} & H_{ji} \\
I_{ji} & J_{ji} & 0 & 0 \\
K_{ji} & 0 & L_{ji} & 0 \\
0 & M_{ji} & 0 & N_{ji}
\end{vmatrix} = 0. \tag{4.45}
\]

We select trial functions satisfying the appropriate boundary conditions as follows:

(i) Both boundaries rigid

\[ W_i = (z^2 - 2z^3 + z^4) T_{i-1}^*, \quad \Phi_i = (z - z^2) T_{i-1}^* = Z_i \tag{4.46a} \]

(ii) Both boundaries free

\[ W_i = (z - 2z^3 + z^4) T_{i-1}^*, \quad \Phi_i = z^2 (1 - 2z / 3) T_{i-1}^* = Z_i \tag{4.46b} \]

(iii) Lower rigid and upper free boundaries

\[ W_i = (3z^3 - 5z^3 + 2z^4) T_{i-1}^*, \quad \Phi_i = (z^3 - z^2 - z) T_{i-1}^* = Z_i \tag{4.46c} \]
with

\[
\Theta_i = \left( z - z^2 \right) T_{i-1}^* \quad \text{(if the boundaries are isothermal)} \tag{4.47a}
\]

\[
\Theta_i = z^2 \left( 1 - 2z/3 \right) T_{i-1}^* \quad \text{(if the boundaries are insulated)} \tag{4.47b}
\]

where \( T_i^* (i = 1, 2, \ldots, n) \) is the modified Chebyshev polynomial of \( i \)th order. Substituting the above trial functions, depending on the boundary combinations considered, in (4.45) and expanding the determinant leads to the characteristic equation giving the thermal Rayleigh number \( R_t \) or the AC electric Rayleigh number \( R_e \) as a function of the wave number \( a \) as well as other parameters \( R_c \) or \( R_e \) as the case may be and the Taylor number \( Ta \). The inner products involved in the determinant are evaluated analytically rather than numerically in order to avoid errors in the numerical integration. Numerical computations carried out reveal that the convergence in finding \( R_e^* \) or \( R_c^* \) with respect to the wave number crucially depends on the value of \( Ta \), and for higher values of \( Ta \) more number of terms are found to be required in the Galerkin expansion. The results presented here are for \( i = j = 8 \) the order at which the convergence is achieved, in general.

### 4.5 Results and Discussion

The simultaneous effects of Coriolis force and a vertical AC electric field on the criterion for the onset of thermal convection in a dielectric rotating fluid layer are investigated. Attention is focused on three kinds of velocity boundary conditions namely, free-free, rigid-rigid and lower rigid upper free which are considered to be either isothermal or insulated to temperature perturbations. To solve the resulting eigenvalue problem, both analytical and numerical techniques are used depending on the choice of boundary conditions. The analytical study performed in the case of free-free isothermal boundaries reveals that the necessary conditions for the occurrence of oscillatory convection are independent of vertical AC electric field and found to be same as in those of ordinary viscous fluids. Since the Prandtl number is greater than unity for dielectric fluids, oscillatory convection is not a preferred mode of instability for the problem considered. Hence the critical stability parameters are computed numerically using the Galerkin method for stationary convection. The numerical results have been validated first by comparing the results with those of Chatterjee et al. (2008) for the case of no AC electric field \( (R_e = 0) \) and
the results are presented in Table 4.1 for rigid-rigid isothermal boundaries. The comparisons of the results show excellent agreement of the numerical results and verify the accuracy of the numerical method used.

Figures 4.2 and 4.3 respectively demonstrate the neutral curves for isothermal boundaries in the \((R'_y, a)\) plane for various values of \(R_y\) when \(Ta = 100\) and also for different values of \(Ta\) when \(R_y = 500\). Whereas, Figs. 4.4 and 4.5 exhibit the neutral curves for the same boundaries in the \((R_x, a)\) plane for different values of \(R'_y\) when \(Ta = 100\) and for various values of \(Ta\) when \(R'_y = 100\) respectively. The neutral curves exhibit single but different minimum with respect to the wave number for various values of physical parameters and also for different kinds of velocity boundary conditions. The region below each neutral curve corresponds to the stable state of the system. Thus we note that increasing \(R_y\) (Fig. 4.2) is to decrease the region of stability, while opposite is the trend with increasing \(Ta\) (Fig. 4.3). The region of stability increases for negative values of \(R'_y\) (i.e., heated from above) but the trend is reversed if the system is heated from below (i.e., \(R'_y > 0\)) (Fig. 4.4). From the figures it is also evident that, except for a quantitative change, the stability characteristics are identical for different kinds of velocity boundary conditions.

To ascertain the stability characteristics of the system, the critical thermal Rayleigh number \(R''_y\) or critical AC electric Rayleigh number \(R''_e\) and the corresponding critical wave number \(a_c\) are computed for different values of physical parameters as well as for various types of velocity and temperature boundary conditions. The variations of \(R''_y\) and \(a_c\) as a function of \(R_c\) are presented for different values of \(Ta\) in Figs. 4.6 (a) and (b) respectively for free-free, rigid-rigid and rigid-free isothermal boundaries. For any fixed value of Taylor number, increasing the value of AC electric Rayleigh number amounts to decrease in the critical thermal Raleigh number monotonically (Fig. 4.6a) indicating its effect is destabilizing. In other words, the presence of electric field is to augment heat transfer and to hasten the onset of convection in a dielectric fluid layer. This is so irrespective of the velocity boundary conditions considered. This is due to the destabilizing electrical body force induced by the gradient of dielectric constant due to variations in temperature under the action of electric field which eventually drive an upward fluid motion. From Fig 4.6(a), it is also obvious that
rigid-rigid boundaries are stabilizing compared to rigid-free boundaries and the least stable is free-free boundaries for values of $Ta = 0$ and 1000. Nonetheless, in the same figure we note that the trend is reversed when $Ta = 10000$. That is, at higher values of $Ta$ both free-free and rigid-free boundaries become more stable than the rigid-rigid boundaries. This is because a viscous boundary layer will appear near the rigid boundary which in turn arrests the fluid motion. Although the boundary layer exerts a pure stabilizing mechanism up to moderate values of $Ta$, it exhibits a dual effect at higher values of $Ta$. On the one hand, the viscosity dissipates the kinetic energy required for the onset of instability in the layer; on the other hand, the viscous force resists the fluid to attach to the vortex lines and makes the fluid to find a means for achieving cross-isobar flow through which potential energy is released. The dual mechanisms coupled with the EHD force exhibit a more instability behavior in the case of rigid boundaries. Besides, the critical wave number increases slowly in the presence of rotation with increasing $R_i$ but significantly increases with increasing $Ta$ (Fig. 4.6b). That is, increasing $R_i$ and $Ta$ is to diminish the size of convection cells.

Figures 4.7(a) and (b) respectively show the variation of $R_{cc}$ and the corresponding $a_c$ as a function of $R_i^r$ for different values of $Ta$ when the boundaries are isothermal. As expected on the physical ground, heating from above ($R_i^r < 0$) is more stabilizing than from below. Moreover, the effect of boundaries on the stability of the system under the influence of rotation is found to be dual in nature depending on the strength of buoyancy forces (Fig. 4.7a). In the absence of rotational effect, the rigid-rigid boundaries are more stable than rigid-free and free-free boundaries. However, when $Ta = 1000$ it is seen that the rigid-rigid boundaries are least stable compared to other two types of boundaries and more stable are the free-free boundaries up to certain range of values of $R_i^r$ ($\leq 250$, say) exceeding which the trend is reversed. With further increasing $Ta (=10000)$, however, the rigid-rigid boundaries found to be more unstable than the other two types of boundaries for all the values of $R_i^r$ considered. Increase in the value of $R_i^r$ is to decrease the critical wave number but rather slowly with increasing value of $Ta$ and the same is evident from Fig. (4.7b). A closer inspection of Figs. 4.6 and 4.7 further reveals that $R_{cc} > R_i^r$ under all the conditions considered.
To know distinctly the similarities as well as differences between isothermal and insulated temperature conditions on the stability characteristics of the system, the critical thermal Rayleigh number for different values of $Ta$ for these two boundaries are compared in Fig. 4.8 for a representative velocity boundary conditions namely, rigid-rigid boundaries. From the figure it is evident that the isothermal boundaries offer more stabilizing effect on the system compared to insulated ones. Besides, the results obtained for different velocity boundary conditions are tabulated in Table 4.2 for various values of $R_c$ and $Ta$. From the tabulated values, it is noted that the known results $R_c^* = 720, 320$ and $120$ documented in the literature are retrieved for rigid-rigid, rigid-free and free-free boundaries respectively when $R_c = 0 = Ta$. In contrast to isothermal case results, it is observed that (i) rigid-free boundaries offer more stabilizing effect than rigid-rigid boundaries even at $Ta = 1000$ and (ii) free-free boundaries are the least stable when $Ta = 10000$.

4.6 Conclusions

The problem of electrothermal convective instability in a dielectric fluid layer heated either from below or above subject to Coriolis force and a vertical AC electric field has been investigated for three different combinations of velocity boundary conditions namely, free-free, rigid-rigid and lower rigid upper free which are either isothermal or insulated to temperature perturbations. The results indicate that the criterion for the onset of electrothermal convection crucially depends on the nature of velocity and temperature boundary conditions and the strength of rotation. Irrespective of the type of temperature conditions, the system is more stable when both boundaries are rigid, while the free boundaries are least stable only up to moderate values of Taylor number. The range of Taylor number up to which this trend is noticed is decreased for insulated boundaries. At higher Taylor number, however, the tendency is reversed in the case of isothermal boundaries since the stability of the free-free boundaries will be enhanced rapidly than the rigid-rigid boundaries. Nonetheless, free boundaries offer least stability on the system even at higher value of Taylor number if the boundaries are insulated. Moreover, the boundaries found to exhibit a dual effect on electrothermal convection depending on the strength of rotation and buoyancy forces. The effect of increasing AC electric field is to enhance the heat transfer and to hasten the onset of convection, while increasing the strength of rotation is to inhibit the onset of electrothermal convection for a fixed type of boundary conditions. Furthermore,
increasing $R_e$ and $Ta$ is to reduce the size of convection cells and quite opposite is the trend with increasing $R_i$. Thus the foregoing results give an insight for the coupling effect of electrical body force, buoyancy force and the Coriolis force on the stability characteristics of the system for different types of velocity and temperature boundary conditions.
Figure 4.1 Physical Configuration.
Figure 4.2 Neutral curves for different values of $R_e$ with $Ta = 100$.

Figure 4.3 Neutral curves for different values of $Ta$ with $R_e = 500$. 
Figure 4.4 Neutral curves for different values of $R_e^i$ with $Ta = 100$.

Figure 4.5 Neutral curves for different values of $Ta$ with $R_e^i = 100$. 
Figure 4.6 Variation of (a) $R_{te}$ and (b) $a_c$ as a function of $R_e$ for different values of $Ta$. 
Figure 4.7 Variation of (a) $R_{ec}'$ and (b) $a_c$ as a function of $R_t$ for different values of $Ta$. 
Figure 4.8 Variation of $R'_c$ as a function of $R_e$ for different values of $Ta$ for isothermal and insulated boundaries.

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Table 4.1 Comparison of critical stability parameters with the earlier published works for rigid-rigid isothermal boundaries.
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Table 4.2 Comparison of critical stability parameters for free-free, rigid-free and rigid-rigid insulated boundaries for different values of $R_e$ and $Ta$. 