2.1 The double sinh-Gordon equation

Skyrme [140] proposed a non-linear field theory which, for the scalar case and in 1+1 dimensions, reduces to a non-linear extension of the Lagrangian density corresponding to the linear KG equation. The equation considered by him has subsequently become known as the sG equation which is characterised by a sine function in the equation of motion. In 1962, Perring and Skyrme [27] found by a computer analysis that the solitary wave solutions of the sG equation are collisionally stable and thereby paved the way for the introduction of the soliton concept. Later this was recognised as an important model in solid state physics [142] and high energy particle physics [49,141]. An equation of motion with two sine functions was subsequently introduced [142-147] and named the double sine-Gordon (DsG) equation which has led to several applications in non-linear optics [143,145] such as the study of the B-phase of liquid helium [145,146] and the treatment of quasi-one-dimensional charge-density wave condensates of organic linear conductors like TTF-TCNQ [147]. In 1+1 dimensions the sG field system undergoes a second
order phase transition [148]. The hyperbolic version of
the sG family of equations has been discussed recently.
Unlike the sG equation, the sinh-Gordon (shG) equation has
no soliton solutions [149] although like the sG equation,
this has got an ABT and an infinite number of conservation
laws.

A new member called the double sinh-Gordon (DshG)
model has recently been added to the KG family of equations
by Behera and Khare [150]. They found a kink solution for
this model and demonstrated the possibility of calculating
the exact free energy associated with the second order phase
transition that the system undergoes. Minami [151] has
recently studied this model and established its relation
to the Toda lattice model [20].

Morse [152] introduced an anharmonic potential of
the exponential type:

\[
V(\phi) = a/b \exp(-b\phi) - 2a/b \exp(-b/2\phi) \tag{2.1}
\]

which was later called the Morse potential. A more general
form of an anharmonic exponential type potential is

\[
V(\phi) = a_1/b \exp(-b\phi) + 2a_2/b \exp(-b\phi/2) + a_3/b \exp(b\phi) \\
+ 2a_4/b \exp(b\phi/2) \tag{2.2}
\]
When
\[ a_1 = a_2 = a_4 = 0 , \]  
the potential yields the classical Liouville equation \[ 142 \],
\[ \phi_{xx} - \phi_{tt} = a_3 \exp(b\phi) . \]  
This is a well studied field-theoretic model and widely used in fluid mechanics and differential geometry \[ 41,154-157 \].

For
\[ a_1 = a_3 = \alpha \]
\[ a_2 = a_4 = 0 \]
\[ b = 1 , \]
we obtain the equation of motion
\[ \phi_{xx} - \phi_{tt} = \alpha \sinh \phi , \]
which is generally known as the sinh-Gordon equation.

For the choice
\[ b_4 = 4 \]
\[ a_1 = a_3 = \eta^2/2 \]
\[ a_2 = a_4 = -2\eta , \]
(2.2) gives

\[ V(\phi) = \frac{\eta^2}{8} \cosh 4\phi - \eta \cosh 2\phi, \]  

where \( \eta \) is a real parameter. To ensure the vanishing of the potential as \( \phi \to 0 \), we may modify this trivially into the form

\[ V(\phi) = \frac{\eta^2}{8} \cosh 4\phi - \eta \cosh 2\phi - \frac{1}{8} \eta^2 + \eta. \]  

This represents the potential corresponding to the DshG equation and has minima at

\[ \phi = 0 \quad \text{for} \quad \eta > 2, \]  

and

\[ \cosh 2\phi = \frac{2}{\eta} \quad \text{for} \quad \eta < 2. \]

For the second condition (2.11) there are two degenerate minima. The values of the potential at the minima are

\[ V_{\min}(\phi = 0) = 0 \]  

and

\[ V_{\min}(\cosh 2\phi = \frac{2}{\eta}) = \frac{-1}{4}(\eta^2 - 4\eta + 4). \]

The equation of motion corresponding to the potential (eq.(28)) is
The first known solution for this model is a kink-like solution [150]:

\[ \phi(x,t) = \text{arc tanh} \left\{ \frac{(1-\eta/2)}{\sqrt{(1-\eta^2/2)}} \tanh \left[ \frac{\sqrt{2} \left( x-ut \right)}{\sqrt{(1-\eta^2/4)} \sqrt{m(c^2-u^2)}} \right] \right\}, \]

(2.15)

which is defined for the values \(|\eta| < 2\). As \(x \to \infty\), this behaves according to

\[ \tanh [\phi(\pm \infty)] = \mp \sqrt{\left( (2-\eta)/(2+\eta) \right)}, \]

(2.16)

which are the values of the field \(\phi\) corresponding to the two degenerate minima characterising the kink solution (2.15).

In this chapter we first show that the DshG field system possesses other types of solution besides the large amplitude kink-like solutions. For this purpose we use the bilinear operator method as well as the base equation technique. We also examine the asymptotic behaviour of multisolitary wave solutions and carry out a linear stability analysis of the single solitary wave solution. This system is shown to possess stable solitary wave solutions in 1+1 dimensions. Characterised as they are by a vanishing topological charge, these new solutions can be considered non-topological objects [30].
2.11. Solitary waves by the bilinear operator method

Define a transformation

$$\phi = \text{arc tanh} \left( \frac{g}{f} \right) \quad (2.17)$$

so that the equation of motion (2.14) yields the bilinear differential equation,

$$2 + g^2) \left( D_x^2 - D_t^2 \right) f \cdot g - f \cdot g \left( D_x^2 - D_t^2 \right) (f \cdot f + g \cdot g)$$

$$= 2\eta^2 f \cdot g (f^2 + g^2) - 4\eta (f^2 - g^2) f \cdot g \quad (2.18)$$

where $D_x^2$ is the bilinear differential operator. On splitting (2.18), so that one is linear and the other may be nonlinear in $f$ and $g$, we find:

$$\left( D_x^2 - D_t^2 \right) f \cdot g = 2\eta (\eta - 2) f \cdot g \quad (2.19)$$

$$\left( D_x^2 - D_t^2 \right) (f \cdot f + g \cdot g) = -8\eta g \cdot g \quad (2.20)$$

We introduce power series expansions for $f$ and $g$ in a parameter $\epsilon$ which is very close to unity:

$$f = 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \ldots \quad (2.21)$$

$$g = \epsilon g_1 + \epsilon^2 g_2 + \ldots \quad (2.22)$$
On equating the coefficients of same powers of \( \varepsilon \) we obtain a set of differential equations:

\[
(D_x^2 - D_t^2)g_1 = 2\eta(\eta-2)g_1 \quad (2.23)
\]
\[
(D_x^2 - D_t^2)(2f_2 + g_1g_1) = -8\eta g_1^2 \quad (2.24)
\]
\[
(D_x^2 - D_t^2)(2f_2 + g_1g_1) = 2\eta(\eta-2)g_2 \quad (2.25)
\]
\[
(D_x^2 - D_t^2)(g_3 + f_2g_1) = 2\eta(\eta-2)g_3 \quad (2.26)
\]
\[
(D_x^2 - D_t^2)(g_4 + f_2g_2) = 2\eta(\eta-2)g_4 \quad (2.27)
\]

Equation (2.23) implies,

\[
\frac{\partial^2 g_1}{\partial x^2} - \frac{\partial^2 g_1}{\partial t^2} = 2\eta(\eta-2)g_1. \quad (2.25)
\]

A simple solution for this equation (2.25) is

\[
g_1 = \exp(\Theta) \quad (2.26)
\]

where, \( \Theta = kx - \omega t + \delta \) and

\[
k^2 - \omega^2 = 2\eta(\eta-2). \quad (2.27)
\]

Equation (2.24) yields,

\[
2\left[ \frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial t^2} \right] + 2[g_1 \frac{\partial^2 g_1}{\partial x^2} - g_1 \frac{\partial^2 g_1}{\partial t^2} - \left( \frac{\partial g_1}{\partial x} \right)^2 + \left( \frac{\partial g_1}{\partial t} \right)^2]
\]

\[
= 8\eta g_1^2. \quad (2.28)
\]
Inserting (2.26) in (2.28),

\[ f_2 = -\exp(2\theta)/(2\eta-4). \tag{2.29} \]

We can consistently set [97]

\[ g_n = 0 \quad \text{for all } n \geq 2 \]

\[ f_n = 0 \quad \text{for all } n \geq 4 \]

\[ \epsilon = 1 \tag{2.30} \]

Combining (2.17), (2.21), (2.22), (2.26), (2.29) and (2.30) an exact solitary wave solution of DshG equation (2.14) is obtained:

\[ \phi(x,t) = \arctanh \left( \frac{\pm \exp(\theta)}{1-\exp(2\theta)/(2\eta-4)} \right). \tag{2.31} \]

for \( \eta < 2 \)

This solution behaves qualitatively as sketched in fig.(1.1).

In contrast with the dark solution (eq.(2.19)), this new solution is defined for \( \eta < 2 \) and can readily be extended to arbitrary dimensions. We might expect to obtain the multi-

solitary wave solution by setting,

\[ g = \sum_{j=1}^{N} \exp(\theta_j). \tag{2.32} \]
However, the corresponding power series of the form (2.23) and (2.24) do not terminate, exposing the failure of the bilinear operator method to provide any such solutions.

2.III. Multisolitary wave solutions

The base equation technique is found useful for the construction of multisolitary or N-solitary wave solutions of the DshG equation in arbitrary dimensions.

Let us accordingly rewrite equation (2.14) as

\[
\partial_\mu \partial^\mu \phi = \frac{\eta^2}{2} \sinh 4\phi - 2\eta \sinh 2\phi , \quad (2.33)
\]

where \(\mu = 0, 1, 2, \ldots (n-1)\) and the n dimensional D'Alembertian,

\[
\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} . \quad (2.34)
\]

The transformation

\[
\phi = \arcsinh \psi \quad (2.35)
\]

converts (2.33) into the form

\[
[1 + \psi^2]^{\frac{1}{2}} \partial_\mu \partial^\mu \psi - [1 + \psi^2]^{\frac{3}{2}} \partial_\mu \partial^\mu \psi - 2\eta^2 \psi [1 + \psi^2] \frac{1}{2} [1 + 2\psi^2] [1 + 2\psi^2] \\
+ 4\eta \psi [1 + \psi^2]^{\frac{1}{2}} = 0 , \quad (2.36)
\]
where
\[ \frac{\partial}{\partial \mu} \frac{\partial^4 \psi}{\partial ^4} = \frac{\partial}{\partial t} \frac{\partial^4 \psi}{\partial t^4} - \sum_{i=1}^{n-1} \frac{\partial^4 \psi}{\partial x_i^4} \cdot \frac{\partial^4 \psi}{\partial x_i^4}. \] (2.37)

We may take the equation
\[ \frac{\partial}{\partial \mu} \frac{\partial^4 \psi}{\partial ^4} = [1+\psi^2](4\eta-6\eta^2-D)\psi^2 - [1+\psi^2](4\eta^2+B)\psi^4 \] (2.38)
as the base equation. The function \( \psi \) can then be expressed as \([89]\),
\[ \psi = u A^{--\frac{1}{2}}, \] (2.39)
where
\[ A = (1 - Bu^2/8m) - Cu^4/12D^2, \] (2.40)
\[ B = (8\eta - 8\eta^2) \]
\[ C = -6\eta^2 \]
\[ M = (\eta - \eta^2/2) \]
\[ D^2 = (4\eta - 2\eta^2), \] (2.41)
and \( u \) satisfies the equations
\[ \frac{\partial}{\partial \mu} \frac{\partial^4 u}{\partial ^4} + D^2 u = 0 \] (2.42)
\[ \frac{\partial}{\partial \mu} \frac{\partial^4 u}{\partial ^4} + D^2 u^2 = 0. \] (2.43)
Henceforth, the last two equations (2.42) and (2.43) can be employed as base equations for solving (2.38). These equations admit a simple exponential type solution:

\[ u = a \exp(ax), \]  

(2.44)

where \( a \) is an arbitrary parameter and

\[ k = (k_0, k_1, \ldots, k_{n-1}) \]  

(2.45)

\[ x = (t, x_1, x_2, \ldots, x_{n-1}) \]  

(2.46)

\[ \alpha = \left[ -\frac{4\eta-2\eta^2}{k^2} \right]^{1/2}, \]  

(2.47)

so that \((4\eta-2\eta^2) < 0\) or \(\eta \notin (0,2)\).

Equations (2.39)-(2.46) imply an exact solution of the DshG system (2.33):

\[ \phi = \text{arc sinh} \left( \frac{u}{1 - (1-2\eta)u^2/(8-4\eta) + u^4/2(4-2\eta)^2} \right)^{1/2}. \]  

(2.48)

All the solutions of (2.42) and (2.43) are automatically the solutions of the DshG equation. Since (2.42) is a linear equation, the linear superposition
\[ u = \sum_{j=1}^{N} a_j \exp(\alpha_j k_j x) \quad (2.49) \]

is also a solution of (2.42) and (2.43). On substituting this form in (2.48), a multisolitary wave solution of the DshG equation emerges with the additional set of conditions:

\[ \alpha_j \alpha_i k_j k_i + (4\eta - 2\eta^2) = 0 , \quad (2.50) \]

where

\[ k_i = (k_{i0}, k_{i1}, k_{i2}, \ldots, k_{in-1}) = (k_{i0}, k_i) \quad (2.51) \]

and

\[ k_i \neq k_j , \quad k_i \cdot k_j \neq 0 , \quad (2.52) \]

for any \( i \) and \( j \).

For a multisolitary wave solution in an \( n \) dimensional space-time, the number of independent components of the wave vector \( \alpha_j k_j x \) is \( N(n-1) \), whereas the number of constraints in equation (2.50) is \( N(N-1)/2 \), for \( i \neq j \). Hence for the system not to be overdetermined,

\[ N \leq (2n-1) . \quad (2.53) \]
Thus the solitary wave index $N$ is restricted by the dimensional of the space-time $n$. Nevertheless, in $1+1$ dimensions only one solitary wave can be found as there is only one independent wave vector and any other wave vector is necessarily parallel to it, as can be verified in the following way [89].

For any two vectors $k_i$ and $k_j$, if $\Theta_{ij}$ be the angle between them, then [89]

$$\cos \Theta_{ij} = \frac{k_{i0} k_{j0}}{|k_i| |k_j|} + \frac{(k_i^2 k_j^2)^{1/2}}{|k_i^2| |k_j^2|}$$

$$= v_i v_j + \sqrt{[(v_i^2 - 1)(v_j^2 - 1)]}, \quad (2.54)$$

where $v_i^2 = k_{i0}^2 / k_i^2$ and $k_i^2 = k_{i0}^2 - K_i^2$.

For a time-like $k_i$, $k_i^2 > 0$ implies

$$k_{i0}^2 - K_i^2 > 0, \quad (2.55)$$

giving

$$\sqrt{(k_{i0}^2 / k_i^2)} = |v_i| > 1. \quad (2.56)$$

Similarly for a space-like $k_i$, we have

$$|v_i| < 1. \quad (2.57)$$
To ensure the consistency of the trigonometric function

\[ |\cos \theta_{ij}| \leq 1. \] (2.58)

Equation (2.54) now gives

\[ -1 - v_i v_j \leq \pm \sqrt{(v_i^2 - 1)(v_j^2 - 1)} \leq 1 - v_i v_j. \] (2.59)

For time-like \( k_i \) we have \(|v_i| > 1\), implying

\[ v_i v_j > 1 \]

or

\[ 1 - v_i v_j < 0. \] (2.60)

From (2.59) and (2.60) we find

\[ (1 + v_i v_j)^2 \geq (v_i^2 - 1)(v_j^2 - 1) \geq (1 - v_i v_j)^2, \] (2.61)

which gives

\[ (v_i - v_j)^2 \leq 0, \text{ for all } i \text{ and } j. \] (2.62)

For real values of \( v_i \) and \( v_j \) (2.62) is true only for the equality, implying

\[ v_i = v_j \text{ for all } i \text{ and } j. \] (2.63)
Equations (2.54) and (2.63) yield,\[\cos \theta_{ij} = 1, \quad (2.64)\]

and thus\[\theta_{ij} = 0. \quad (2.65)\]

Hence in 1+1 dimensions all vectors are necessarily parallel and therefore all are dependent and so the multi-solitary waves reduce to a single solitary wave. For a space-like \(k_1\), this argument can be repeated and the result will not be contradicted.

2.IV. Linear stability of solutions in 1+1 dimensions

To investigate the linear stability of a particular solution of a NDE a small perturbation is applied to the solution, and examine whether or not this small perturbation grows with time. If the perturbation remains small enough, the non-linear equation that it obeys may be approximated by a linear equation. In this section we analyse the linear stability of the single solitary wave solution (2.31).

The static form of the single solitary wave solution (2.31) is
\[ \phi_s(x) = \text{arc tanh} \left( \frac{\exp(kx)}{1 - \exp(2kx)/(2\eta - 4)} \right). \quad (2.66) \]

Let

\[ \phi(x,t) = \phi_s(x) + \phi_p(x,t), \quad (2.67) \]

where \( \phi_p(x,t) \) is a perturbation such that \(|\phi_p| \ll 1\).

On substituting (2.67) in (2.14) we obtain,

\[ \phi_p,xx - \phi_p,tt = \eta^2/2 \sinh[4(\phi_s + \phi_p)] - 2\eta \sinh[2(\phi_s + \phi_p)] \]

\[ - [\eta^2/2 \sinh 4\phi_s - 2\eta \sinh 2\phi_s]. \]

(2.68)

By the linearity assumption \(|\phi_p| \ll 1\), so

\[ \phi_p,xx - \phi_p,tt = \phi_p[2\eta^2 \cosh 4\phi_s - 4\eta \cosh 2\phi_s]. \quad (2.69) \]

This is not of the same form as (2.14); nevertheless, it is linear and therefore easier to solve. Now consider a separable form of the solution,

\[ \phi_p(x,t) = f(x) \exp(\lambda t). \quad (2.70) \]
This leads to the Schrödinger eigenvalue problem,

\[
\left[ -\frac{\partial^2}{\partial x^2} + V_0 (\varphi_s) + \left( \lambda^2 + 2\eta^2 - 4\eta \right) \right] f(x) = 0 \tag{2.71}
\]

for the potential

\[
V_0(\varphi_s) = 2\eta^2 \cosh 4\varphi_s - 4\eta \cosh 2\varphi_s - (2\eta^2 - 4\eta) \tag{2.72}
\]

\[
= V''(\phi_s) - (2\eta^2 - 4\eta).
\]

\(V_0(\varphi_s)\) is smooth and bounded and tends to zero as \(x \to \pm \infty\). Thus, there exists atmost a finite number of bound product solutions for which \(|f| \to 0\) as \(x \to \pm \infty\). But corresponding to the eigenvalue \(\lambda^2 + 2\eta^2 - 4\eta = 0\), there exists a non-zero eigenfunction \(f(x)\) given by

\[
f(x,0) = \frac{\partial \varphi_s(x)}{\partial x} \tag{2.73}
\]

The nodes of \(f(x,0)\) are infinitely separated; so \(\lambda = 0\) is the lowest eigenvalue \([158]\). This demonstrates the linear stability \([34]\) of the solution \((2.31)\).

2.V. Asymptotic behaviour of multisolitary wave solutions

The multisolitary wave solutions in more than 1+1 dimensions are of the form
\[ \phi_N(x,t) = \text{arc sinh} \left( \frac{u_N}{\sqrt{1 - \left( \frac{1}{2} - \eta \right) u_N^2 + \frac{u_N^4}{2(4-2\eta)^2}}} \right) \]  

(2.74)

where

\[ u_N = \sum_{j=1}^{N} a_j \exp(\alpha_j k_j x_j). \]  

(2.75)

This can be seen to break up into \(N\) simple waves in the asymptotic regions. For as

\[ \alpha_j k_j x_j \rightarrow -\infty \]

\[ \phi_N \rightarrow \text{arc sinh } u_N \]  

(2.76)

and as \( \alpha_j k_j x_j \rightarrow +\infty \), the dominant term in the braces of (2.74) is

\[ \frac{u_N^2}{2(4-2\eta)^2}. \]  

(2.77)

Consequently,

\[ \phi_N \approx \text{arc sinh}(u_N/(4-2\eta)) \text{ as } \alpha_j k_j x_j \rightarrow +\infty. \]  

(2.78)

To calculate the phase shift we consider the \(j^{th}\) wave in the asymptotic regions,
\( \phi_j = \text{arc sinh}\left[ a_j \text{ exp}(a_j k_j x_j) \right] \text{ as } x \to -\infty \) \hspace{1cm} (2.79)

and

\( \phi_j \approx \text{arc sinh}\left[ a_j \text{ exp}(-a_j k_j x_j)/(4-2\eta) \right] \text{ as } x \to +\infty \). \hspace{1cm} (2.80)

Defining the corresponding phases [89] as,

\( \delta_j^- = \log a_j \) \hspace{1cm} (2.81)

\( \delta_j^+ = \log \left[ (4-2\eta)^{-1} \right] \) \hspace{1cm} (2.82)

the phase shift for the \( j \)th wave is given by

\( \Delta_j = \delta_j^+ - \delta_j^- \) \hspace{1cm} (2.83)

\( \approx \log\left[ (4-2\eta)/a_j^2 \right] \). \hspace{1cm} (2.84)

The multisolitary wave solutions behave as if they were simple waves both at \(-\infty\) and \(+\infty\), and each component wave nearly undergoes a phase shift given by (2.84). However, there is no loss of stability for the multisolitary wave profile as a whole.
in the asymptotic regions. Similar behaviour has been noted [29] for KdV solitons in one space dimension.

2.VI. Topological charge

The conserved topological charge \( Q \) associated with a solitary wave in \( 1+1 \) dimensions is defined as

\[
Q = \int_{-\infty}^{\infty} j^0 \, dx , \quad (2.85)
\]

where

\[
j^\mu = \epsilon^{\mu\nu} \partial_\nu \phi , \quad (2.86)
\]

and

\[
\epsilon^{01} = 1, \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} . \quad (2.87)
\]

The Behera-Khare kink (eq.(2.15)) can be shown to possess a topological charge

\[
Q = 2 \arctanh \sqrt{(2-\eta)/(2+\eta)} , \quad |\eta| < 2. \quad (2.88)
\]

However, the solitary wave solutions reported herein are associated with vanishing topological charge and are, therefore, non-topological configurations [30].
Even though the multisolitary wave solutions which are defined in more than one space dimension possess very good stability properties, Derrick's theorem [38] does not permit them to possess finite energy.