6.1. Classical solutions of SU(2) Yang-Mills-Higgs system

According to current views the fundamental physical interactions are described by unified gauge theories. A great deal of progress has currently been made towards understanding of the classical YM field equations [57]. These properties have turned out to be of interest not only in their own right but also in connection with confinement [58-62]. In four dimensional spontaneously broken nonabelian gauge theories, solitons appear as monopoles corresponding to a suitable generalization of the Dirac quantization condition [168]. They predict the existence of heavy monopoles.

A physically significant prototype of a spontaneously broken nonabelian gauge theory is the SU(2) Yang-Mills-Higgs theory. Static monopole solutions were discovered by Prasad and Sommerfield in the limit of vanishing Higgs coupling ($\lambda \rightarrow 0$), this limit being nowadays known as the Prasad-Sommerfield (PS) limit [66]. Various workers have since obtained time-dependent, singular solutions of this system [65,188-192]. No systematic procedure was adopted in arriving at any of these solutions.
with the result that both the static and time-dependent solutions emerged as products of ingenious guesswork.

Herein it is proposed to carry out a similarity analysis of the time-dependent non-linear coupled differential equations associated with a spontaneously broken SU(2) YM theory [60-66]. This gives the invariance group of the system of equations, and we use this information to generate some of the previously known time-dependent solutions in the PS limit. We also present two new time-dependent solutions possessing surface singularities.

6.II. Similarity group of SU(2) YM-Higgs system

The SU(2) YM field coupled to an SU(2) Higgs field is given by the Lagrangian density:

\[ L = -\frac{1}{4} F_{\mu \nu}^a F_{\mu \nu}^a - \frac{1}{2} \nabla^a \nabla^a + \frac{1}{2} \phi^2 - \frac{\lambda}{4} \phi^4 , \]

where the symbols have been defined in chapter 1. At the PS limit, the time-dependent spherically symmetric Ansatz [65] reduces the equations of motion to the form
\[ r^2(H_{rr} - H_{tt}) = 2HK^2 \quad (6.2) \]

\[ r^2(K_{rr} - K_{tt}) = K(K^2 - 1) + KH^2, \quad (6.3) \]

where

\[ H_{xx} = \frac{\partial^2 H}{\partial x^2} \quad \text{and} \quad K_{xx} = \frac{\partial^2 K}{\partial x^2}, \quad (x = r, t). \quad (6.4) \]

To study the similarity group of this system, we define a generic dependent variable \( u^\alpha (\alpha = 1, 2) \) such that \( u^1 = K \) and \( u^2 = H \), and consider a one parameter family of infinitesimal transformations defined by

\[
    r^* = r + \epsilon R(r, t, u^1, u^2) + O(\epsilon^2)
\]

\[
    t^* = t + \epsilon T(r, t, u^1, u^2) + O(\epsilon^2)
\]

\[
    u^\alpha^* = u^\alpha + \epsilon U^\alpha(r, t, u^1, u^2) + O(\epsilon^2). \quad (6.5)
\]

The infinitesimals \( R, T \) and \( U^\alpha \) must ensure the invariance of (6.2) and (6.3) under the transformations (6.5). The derivatives \( \frac{\partial^2 u^\alpha}{\partial r^2} \) and \( \frac{\partial^2 u^\alpha}{\partial t^2} \) transform according to

\[
    u^\alpha_{,r^*r^*} = u^\alpha_{,rr} + \epsilon [U^\alpha_{,rr}] + O(\epsilon^2) \quad (6.6)
\]

\[
    u^\alpha_{,t^*t^*} = u^\alpha_{,tt} + \epsilon [U^\alpha_{,tt}] + O(\epsilon^2), \quad (6.7)
\]
where $u_{xx}^{\alpha} = \frac{\partial^2 u^{\alpha}}{\partial x^2}$ etc. and $[u_{xx}^{\alpha}]$ denotes an 'extension' 

$(x = r, t)$:

$$[u_{xx}^{\alpha}] = \frac{\partial^2 u^{\alpha}}{\partial x^2} + 2 \frac{\partial^2 u^{\alpha}}{\partial x \partial u^\beta} u_\beta, x - \frac{\partial^2 x^i}{\partial x^2} u^{\alpha, i} + \frac{\partial u^{\alpha}}{\partial u^\beta} u_\beta, xx$$

$$- 2 \frac{\partial x^i}{\partial x} u_\alpha, x_i + \frac{\partial^2 u^{\alpha}}{\partial u^\beta \partial u_\gamma} u_\beta, x u_\gamma, x - 2 \frac{\partial^2 x^i}{\partial x \partial u^\beta} u_\beta u_\alpha, x, i$$

$$- \frac{\partial x^i}{\partial u_\beta} (u_\alpha, x, xx + 2 u_\beta, x, x_i)$$

$$- \frac{\partial^2 x^i}{\partial u^\beta \partial u_\gamma} u_\beta, x u_\gamma, x u_\alpha, i$$

(6.8)

with

$$x^1 = R, \quad x^2 = T.$$

(6.9)

When equations (6.5) are substituted into the transformed system corresponding to (6.6) and (6.7) and coefficients of first order in $\epsilon$ are equated to zero, we find:

$$r^2 [u_{xx}^1] - r^2 [u_{tt}^1] + 2r x^4 (K_{rr} - K_{tt})$$

$$+ U^1 (1 - 3K^2 - H^2) - 2 U^2 KH = 0,$$

(6.10)
\[ r^2[U^2_{rr}] - r^2[U^2_{tt}] + 2r X^4(H_{rr} - H_{tt}) \\
- 2 K^2 U^2 - 4HKU^2 = 0. \quad (6.11) \]

Solving (6.10) and (6.11) for the infinitesimals (see Appendix),

\[ R = 2 \lambda r t + \kappa r \]  
\[ T = \lambda (r^2 + t^2) + \kappa t + \sigma \]  
\[ U^\alpha = 0, \]  
\[ (6.12) \]
\[ (6.13) \]
\[ (6.14) \]

where \( \lambda, \kappa \) and \( \sigma \) are constants.

Equation (6.14) states the invariance of any solution of the SU(2) YM-Higgs system reduced by the Ansatz (1.62-1.64) under the similarity transformation (6.5).

The occurrence of three independent parameters \( \lambda, \kappa \) and \( \sigma \) above permits us to define the generators \( G_a, a = 1, 2, 3. \)

\[ G_1 = 2rt \frac{\partial}{\partial r} + (r^2 + t^2) \frac{\partial}{\partial t}, \]  
\[ (6.15) \]
\[ G_2 = r \frac{\partial}{\partial r} + t \frac{\partial}{\partial t}, \]  
\[ G_3 = \frac{\partial}{\partial t}, \]  
\[ (6.16) \]
which satisfy the Lie algebra,

\[ [G_1, G_2] = -G_1 \]  \hspace{1cm} (6.17)

\[ [G_2, G_3] = -G_3 \]  \hspace{1cm} (6.18)

\[ [G_3, G_1] = 2G_2. \]  \hspace{1cm} (6.19)

We identify this as the Lie algebra associated with the similarity group \( \mathcal{G} \) of equations (6.2) and (6.3).

6.III. Time-dependent solutions

In the preceding chapter we developed a method of constructing particular solutions of non-linear KG equations under various subgroups of the similarity group. That procedure may be extended to the SU(2) YM-Higgs system reduced by the Ansatz (1.62-1.64). The idea is to consider different subgroups of the similarity group \( \mathcal{G} \), define a similarity variable for each subgroup, set up the corresponding similarity-reduced equations and solve them. Solutions are obtained in cases where the reduced equations are of the PS type. The different cases are discussed below.
A. Full group $G: \lambda \neq 0, \chi \neq 0, \sigma = \chi^2/4\lambda$

Equations (6.12-6.14) yield the similarity variable

$$\chi = r/(t^2 - r^2 + \frac{\chi t}{\lambda} + \frac{\chi^2}{4\lambda}).$$ \hspace{1cm} (6.20)

The similarity-reduced system of equations are

$$\chi^2 \frac{d^2\chi}{d\chi^2} = K(K^2 - 1) + KH^2,$$ \hspace{1cm} (6.21)

$$\chi^2 \frac{d^2H}{d\chi^2} = 2HK^2.$$ \hspace{1cm} (6.22)

This is of the same form as the equation considered by Prasad and Sommerfield for the static case ((1.82) and (1.83)). A solution of (6.21) and (6.22) is ((1.84) and (1.86)):

$$K(\chi) = \frac{C}{\sinh (C\chi)},$$ \hspace{1cm} (6.23)

$$H(\chi) = \frac{C}{\chi} \coth (C\chi) - 1,$$ \hspace{1cm} (6.24)

where $\chi$ has been defined in (6.20). This solution coincides with that reported in Ref [65].
However, a new solution can be obtained by replacing $r$ by $\chi$ in the static solution reported in Ref [172]. Thus we are led to the solution

$$K(\chi) = \chi/(A + \chi) \quad (6.25)$$

$$H(\chi) = A/(A + \chi), \quad (6.26)$$

where $A$ is a nonzero arbitrary constant. Both $K(\chi)$ and $H(\chi)$ are singular on the surface $(A + \chi) = 0$.

B. Subgroup $G_1$: $\chi = \sigma = 0$

Under the subgroup $G_1 \subset G$ specified by $\chi = \sigma = 0$, the infinitesimals read

$$R = 2\lambda rt \quad (6.27)$$

$$T = \lambda (r^2 + t^2). \quad (6.28)$$

With a similarity variable

$$\eta = r/(t^2 - r^2) \quad (6.29)$$

the reduced system assumes the form of (6.21) and (6.22) on the replacement $\chi \rightarrow \eta$. We note that there exist two
families of solutions just as in the case of the full group \( G \) mentioned above. They are

\[
K(\eta) = C \eta / \sinh(C\eta) \quad (6.30)
\]

\[
H(\eta) = C \eta \coth(C\eta) - 1 \quad (6.31)
\]
as found in Ref [65], and

\[
K(\eta) = \eta / (A + \eta) \quad (6.32)
\]

\[
H(\eta) = A / (A + \eta), A \neq 0 \quad (6.33)
\]

which constitutes a new time-dependent solution. The later is singular on the surface \((A + \eta) = 0\).

C. Subgroup \( G_2: \lambda = 0, \chi \neq 0, \sigma \neq 0 \)

For the subgroup \( G_2 \subseteq G \), \( \lambda = 0 \), and the infinite-simials are

\[
R = \chi r \quad (6.34)
\]

\[
T = \chi t + \sigma \quad (6.35)
\]
The corresponding similarity variable is

$$\xi = \frac{r}{(t - a)}$$  \hspace{1cm} (6.36)

where \(a\) is a nonzero arbitrary constant. The similarity equations read

$$\left(\xi^2 - \xi^4\right) K'' - 2\xi^3 K' = K(K^2 - 1 + H^2)$$  \hspace{1cm} (6.37)

$$\left(\xi^2 - \xi^4\right) H'' - 2\xi^3 H' = 2 HK^2$$  \hspace{1cm} (6.38)

where a prime denotes differentiation with respect to \(\xi\). It has not been possible for us to find an exact solution for this system.

6.IV. Discussion

The similarity method of analysis of the non-linear coupled differential equations equivalent to the classical SU(2) YM-Higgs system gives the similarity group \(\mathcal{G}\), which evidently depends on the Ansatz employed. There is an explicit time-dependent similarity variable for each subgroup of \(\mathcal{G}\). Under the full group \(\mathcal{G}\) as well as under one of its subgroups \(\mathcal{G}_1\), time-dependent solutions arise as generalizations of
the well known static solutions of Refs[66] and [172]. This indicates the possibility of transforming any static solution of (6.2) and (6.3) into a non-trivial time-dependent form. The two new solutions herein obtained as well as those reported earlier in the literature, can be continued to the Euclidean space.

APPENDIX

The equations (6.12)-(6.14) are obtained by solving nearly 100 constraint relations obtained from (6.10) and (6.11) by equating coefficients of different orders of derivatives of K and H. These constraints can be divided into primary and secondary types such that the latter arise as combinations of the former. We list only the primary constraints.

\[
\begin{align*}
\mathbf{r}^2 U^{\prime}_{rr} - \mathbf{r}^2 U^{\prime}_{tt} + (1-3K-H^2)U^1 - 2KH^2 &= 0 \\
2\mathbf{r}^2 U^{\prime}_{tH} - \mathbf{r}^2 R_{rr} - \mathbf{r}^2 R_{tt} &= 0 \\
2\mathbf{r}^2 U^{\prime}_{tK} - \mathbf{r}^2 R_{rr} - \mathbf{r}^2 R_{tt} &= 0 \\
\mathbf{r}^2 U^{\prime}_{tr} - \mathbf{r}^2 U^{\prime}_{tt} - 2K^2U^2 - 4KH^1 &= 0 \\
\mathbf{r} U^{\prime}_K - 2\mathbf{r} R_t + 2K &= 0, \quad \mathbf{r} U^{\prime}_H - 2\mathbf{r} T_t + 2H &= 0 \\
T_t + \mathbf{r} T_{rt} - R_t &= 0, \quad \mathbf{r} R_t - R &= 0, \quad \mathbf{r} T_t - K &= 0 \\
T_{rr} - T_{tt} &= 0, \quad T_{rt} = T_K = T_H = 0 \\
U^1 = U^2 = 0, \quad R_K = R_H = 0. 
\end{align*}
\]