4.1. Relationships between KG type equations

Equations of the KG family featuring scalar fields possess Lorentz invariance as a common property. This suggests the possibility of mapping a particular solution of one such equation into that of another, of the same family. For selected pairs of KG type equations this procedure has been successfully implemented before [88,169], constituting applications of the base equation technique [77-89]. The recent identification of a large variety of NDEs in the KG family possessing kink or soliton solutions with interesting properties and physical applications calls for a more detailed exploration of the base equation method. Lorentz invariance being a common trait of such equations, there can exist relationships between different members of the KG family, realizable in the form of maps between particular solutions of classes of KG type equations. Naturally, for a specified class of such equations, this would imply a multiplicity of maps or a composition of maps which generates particular solutions.

In the present chapter extensive use of the technique of composite maps is made to produce kinks and solitons in the KG family. Specifically, this approach is applied to equations...
such as $sG$, $D_sG$, $\varphi^4$, Liouville and $\varphi^6$. The non-linear correspondence between particular solutions of some of the members of the KG family through a transformation in terms of the arc sine [88] or arc tangent [169] function has been studied in the literature. We combine both of these transformations and develop a non-linear composite map that takes solutions of one KG equation to those of two other equations.

A solution of the original KG equation is found by solving the transformed equations simultaneously. The results obtained expose several 'family relationships' existing within the KG family.

4.II. Solution of $\varphi^4$ equation by bilinear operator method

Since some solutions of the $\varphi^4$ equation play an important role in our programme of composite maps, we obtain them here by the bilinear operator method [90-98]. We write the scalar field $\varphi$ satisfying the massive $\varphi^4$ equation,

$$\partial_\mu \partial^\mu \varphi + a \varphi + \beta \varphi^3 = 0, \quad \mu = 0, 1, 2, \ldots n. \quad (4.1)$$

as the quotient of two functions

$$\varphi = \frac{g}{f}. \quad (4.2)$$

This leads to the bilinear differential equation

$$f D_\mu D^\mu g \cdot f + \alpha g f^2 - g D_\mu D^\mu f \cdot f + \beta g^3 = 0, \quad (4.3)$$
where

$$D_\mu D_\nu^H = D_\nu^2 - \sum_{i=1}^{n} D_{\xi_i}^2,$$

(4.4)

and $D_\omega^2$ is a Hirota-type bilinear operator.

On splitting (4.3), we have

$$D_\mu D_\nu^H g \cdot f + \alpha g f = 0,$$

(4.5)

$$D_\mu D_\nu^H f \cdot f + \beta g f = 0.$$

(4.6)

Expanding $g$ and $f$ as power series,

$$g = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \ldots .$$

(4.7)

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \ldots .$$

(4.8)

yields a set of equations for the $g_i$ and $f_i$, with the conditions

$$g_j = 0; \quad f_j = 0, \quad \text{for all } j \geq 2.$$  

(4.9)

Choosing

$$g_0 = f_0 = 0,$$

(4.10)
the solution

\[ g_1 = \pm \sqrt{(2\alpha/\beta)} \]  

\[ f_1 = \frac{2\exp(\Theta)}{(\exp(2\Theta) - 1)}, \]  

follows, where \( \Theta = k_0 t - \sum_{j=1}^{n} k_j x_j + \delta \). Assuming a dispersion relation

\[ \sum_{j=1}^{n} k_j^2 - k_0^2 = \alpha , \]  

we find the solution

\[ \phi = \pm \sqrt{(2\alpha/\beta)} \frac{2 \exp(\Theta)}{[\exp(2\Theta) - 1]} . \]  

This is a new solution which turns out to be singular at \( \Theta = 0 \). However, as discussed later, this singularity gets eliminated under suitable transformation to other non-linear equations like the DsG and \( \phi^2 \).

A known multi-dimensional solution reported earlier [169] can also be obtained by this procedure. Setting

\[ g_0 = 0; \quad f_1 = 1 . \]  

(4.15)
Equation (4.5) yields

$$\delta \mu \delta^H g_1 + \alpha g_1 = 0. \tag{4.16}$$

A simple solution for this equation is

$$g_1 = \exp(\theta). \tag{4.17}$$

When we set

$$g_j = 0; f_j = 0 \quad \text{for all } j \geq 2, \tag{4.18}$$

(4.6) gives

$$f_1 = (-\beta/8\alpha)\exp(2\theta).$$

Then an exact solution of $\phi^4$ equation is,

$$\phi = \pm \exp(\theta)/[1 - (\beta/8\alpha)\exp(2\theta)]. \tag{4.19}$$

If we change $\alpha$ to $-\alpha$ in (4.1), a non-singular soliton-like solution (i.e., asymptotically vanishing),

$$\phi = \pm \exp(\theta)/[1 + (\beta/8\alpha)\exp(2\theta)], \tag{4.20}$$

is obtained.
Setting $a = 0$ in (4.1) gives a solution of massless $\phi^4$ equation:

$$\phi = \pm A/(B + \theta),$$  \hspace{1cm} (4.21)

(A and B are arbitrary constants) which coincides with (3.45), obtained by the bilinear method.

4.III. Maps from $\phi^4$ to sG and Liouville's equations

As a first example of the composite mapping method, we shall consider maps from $\phi^4$ to sG and Liouville equations. The $\phi^4$ and sG systems are two model field theories widely employed in different branches of theoretical physics and adequate references to these models have been given in the preceding chapters. The Liouville equation was introduced by Liouville [163] in 1853 and has a variety of applications such as in the Lagrange stream function model for two dimensional steady vortex motion of an incompressible fluid [170], the theory of thermionic emission and the problem of the isothermal gas sphere [171]. It has been noticed that several coupled equations in modern gauge theories reduce to the one or two dimensional Liouville equation [41,172].

The sG equation in arbitrary dimensions may be written in the form

$$\partial_\mu \partial^\mu \psi + \alpha \sin \psi = 0.$$  \hspace{1cm} (4.22)
The non-linear map

\[ \psi = 2 \text{arc sin } \phi \]  \hspace{1cm} (4.23)

transforms (4.22) into

\[ [1-\phi^2] \partial_\mu \delta^{\mu\nu} \phi + \alpha \partial_\mu \phi \delta^{\mu\nu} \phi + \alpha \phi [1-\phi^2]^2 = 0. \]  \hspace{1cm} (4.24)

A suitable splitting of (4.24) gives \( \phi^4 \),

\[ \partial_\mu \delta^{\mu\nu} + \alpha \phi - 2\alpha \phi^3 = 0, \]  \hspace{1cm} (4.25)

and the constraint:

\[ \partial_\mu \phi \delta^{\mu\nu} + \alpha \phi^2 - \alpha \phi^4 = 0. \]  \hspace{1cm} (4.26)

Choosing \( \phi \) as in (4.14) where \( \beta \) is replaced by \(-2\alpha\) gives a complex solution of \( S_G \) equation:

\[ \psi = 2 \text{arc sin}[\pm 2i \exp(\theta)/(1+ \exp(2\theta)]]. \]  \hspace{1cm} (4.27)

Choosing \( \phi \) as in (4.19) with the replacement \( \beta \rightarrow -2\alpha \), gives an analytic solution of \( S_G \) equation:

\[ \psi = 2 \text{arc sin}[\pm \exp(\theta)/(1+ \exp(\theta)/4)]. \]  \hspace{1cm} (4.29)
The Liouville equation in arbitrary dimensions may be written,

$$\partial_\mu \partial^\mu \psi + \beta e^{2\psi} = 0. \quad (4.29)$$

Under the transformation

$$\psi = \log \phi, \quad (4.30)$$

we obtain the massive $\phi^4$, and the constraint equations:

$$\partial_\mu \partial^\mu \phi + \alpha \phi + 2\beta \phi^3 = 0 \quad (4.31)$$

$$\partial_\mu \phi \partial^\mu \phi + \alpha \phi^2 + \beta \phi^4 = 0, \quad (4.32)$$

where $\alpha$ can be zero, or a real constant. Using (4.14) we obtain an exact solution of the Liouville equation for $\alpha \neq 0$:

$$\psi = \log \left[2\gamma(\pi/\beta) \exp(\theta)/(\exp(2\theta)-1)\right]. \quad (4.33)$$

Inserting (4.20) and replacing $\alpha$ by $-\alpha$ gives another solution of (4.29):

$$\psi = \log \left[\exp(\theta)/(1+(\beta/4\alpha)\exp(2\theta))\right], \text{ when } \alpha \neq 0. \quad (4.34)$$
Setting $\alpha = 0$ in (4.31), the resulting equation is analogous to massless $\phi^4$. From (4.21) we can find the third independent solution of the Liouville equation:

$$\psi = \log \left[ \pm A/(B + \theta) \right], \quad (4.35)$$

provided $k^2 = -\beta \Lambda^2$.

4.IV. The composite map $\phi^4 \rightarrow \text{Dsg} \rightarrow \phi^6$

It has been shown that [173] the $\phi^6$ model has soliton-like solutions in 1+1 dimensions. The Dsg equation involving arbitrary parameters $\alpha$ and $\beta$,

$$\delta_\mu \partial^n \psi + \alpha \sin \psi + \beta \sin(\psi/2) = 0, \quad (4.36)$$

transforms under the map,

$$\psi = 4 \arctan \phi \quad (4.37)$$

into:

$$2[1+\phi^2] \partial_\mu \partial^\mu \phi - 4 \partial_\mu \phi \partial^\mu \partial^2\phi + (2\alpha + \beta)\phi + (-2\alpha + \beta)\phi^3 = 0. \quad (4.38)$$

On splitting this equation, there emerge the $\phi^4$ equation

$$\delta_\mu \partial^n \phi + (\alpha + \beta/2)\phi + \beta \phi^3 = 0, \quad (4.39)$$
and a constraint:

\[ \partial_\mu \phi \partial^\mu \phi + (\alpha + \beta/2)\phi^2 + \beta/2 \phi^4 = 0 \]  \hspace{1cm} (4.40)

Inserting the form given in (4.14) with the replacement \( \alpha \rightarrow (\alpha + \beta/2) \), a 'On-pulse' like solution of the DsG equation is obtained:

\[ \psi = 4 \text{ arc tan} \left[ \pm \frac{2\sqrt{(2\alpha + \beta)/\beta}\exp(\theta)/(\exp(2\theta) - 1)}{1 - [\beta/(8\alpha + 4\beta)]\exp(2\theta)} \right] \]  \hspace{1cm} (4.41)

Also corresponding to (4.19) we get another solution:

\[ \psi = 4 \text{ arc tan} \left( \frac{\pm \exp(\theta)}{1 - [\beta/(8\alpha + 4\beta)]\exp(2\theta)} \right). \]  \hspace{1cm} (4.42)

The dispersion relation associated with both these solutions is

\[ \sum_{j=1}^{n} k_j^2 - k_o^2 = \alpha + \beta/2. \]  \hspace{1cm} (4.43)

For \( \alpha = -\beta/2 \), the massless \( \phi^4 \) equation follows from (4.30). The corresponding DsG equation is

\[ \partial_\mu \partial^\mu \psi - \beta/2 \sin \psi + \beta \sin(\psi/2) = 0. \]  \hspace{1cm} (4.44)
Using (4.21) we obtain two pairs of 'On—pulse' like solutions of DsG equation (4.44):

$$\psi = 4 \arctan \left[ \pm \frac{A}{B \pm \theta} \right].$$  \hspace{1cm} (4.45)

By the non-linear transformation

$$\psi = 4 \arcsin \phi,$$  \hspace{1cm} (4.46)

one can pass from the DsG equation (4.36) to the equation

$$2(1-\phi^2) \partial_\mu \partial^\mu \phi + 2\phi \partial_\mu \phi \partial^\mu \phi + 2\alpha \phi(1-\phi^2)^2 (1-2\phi^2) + \beta \phi(1-\phi^2)^2 = 0. \hspace{1cm} (4.47)$$

This yields the $\phi^6$ model defined by

$$\partial_\mu \partial^\mu \phi + (\alpha+\beta/2)\phi - (4\alpha+\beta)\phi^3 + 3\alpha \phi^5 = 0 \hspace{1cm} (4.48)$$

plus the constraint:

$$\partial_\mu \partial^\mu \phi + (\alpha+\beta/2)\phi - (2\alpha+\beta/2)\phi^4 + \alpha \phi^6 = 0. \hspace{1cm} (4.49)$$

The DsG solution (4.41) now yields an exact solution of $\phi^6$:

$$\phi = \pm \frac{2 \sqrt{(2\alpha+\beta) \exp(\theta)}}{\sqrt{[\beta+(8\alpha+2\beta) \exp(2\theta) + \exp(4\theta)]^{yz}}}.$$  \hspace{1cm} (4.50)
The associated dispersion relation is

$$\sum_{j=1}^{n} k_j^2 - k_0^2 = \frac{(2\alpha + \beta)}{2}. \quad (4.51)$$

The solution (4.50) can be interpreted as a nontopological soliton-like configuration.

Corresponding to (4.42) we get another solution of the $\phi^6$ equation (4.48):

$$\phi = \frac{\pm \exp(\theta)}{\sqrt{\left[1 - \beta \exp(2\theta)/(\alpha + \beta/2)\right]^2 + \exp(2\theta)}} \quad (4.52)$$

The associated dispersion relation is the same as (4.51).

4.V. The composite map: $\phi^2 \rightarrow \text{sG} \rightarrow \phi^4$

Starting from solutions of the linear KG or $\phi^2$ equation, a pair of maps can be constructed yielding solutions of sG and $\phi^4$ equations. A solution of the $\phi^2$ equation:

$$\phi = e^{\theta}, \quad (4.53)$$

where $\theta$ is defined as in (4.12), can be used to obtain a solution of sG equation in the following manner. The
non-linear map (4.37) converts sG equation (4.22) into:

\[ [1+\phi^2] \partial_{\mu} \partial^{\mu} \phi - 2\phi \partial_{\mu} \partial^{\mu} \phi + \alpha \phi [1-\phi^2] = 0. \] (4.54)

On splitting this gives

\[ \partial_{\mu} \partial^{\mu} \phi + \alpha \phi = 0 \] (4.55)

\[ \partial_{\mu} \phi \partial^{\mu} \phi + \alpha \phi^2 = 0. \] (4.56)

Inserting (4.53), which is a solution of \(\phi^2\), the well known solution of the sG, namely:

\[ \psi = 4 \text{ arc tan} [\exp(\theta)] \] (4.57)

follows. Using (4.23) we obtain a well behaved soliton-like solution of the \(\phi^4\) equation (4.25):

\[ \phi = 2 \exp(\theta)/[1+\exp(2\theta)]. \] (4.58)

4.VI. Multisolitary wave solutions

A fact of some importance is that \(N\) solitary wave solutions or multisolitary wave solutions can easily be constructed for cases where \(e^{\Theta}\) appears, by the replacement

\[ e^{\Theta} \rightarrow \sum_{j=1}^{N} e^{\Theta_j}, \] (4.59)
where \( \Theta_j = k_{oj} t - \sum_{i=1}^{n} k_{ij} x_i + \Theta_j \), for \( j = 1, 2, \ldots, N \), and by

imposing the additional constraints

\[
(k_{ij} - k_{sl})^2 - (k_{oj} - k_{ol})^2 = 0, \tag{4.60}
\]

where \( i, s = 1, 2, \ldots, n, j, l = 1, 2, \ldots, N \). As discussed in Chapter 2, the relation (4.60) restricts the number of solitary wave solutions according to \( N \leq (2n-1) \), where \( n \) is the space-time dimensionality. However, in 1+1 dimensions there will exist only one solitary wave as there is only one independent wave vector and any other vector is necessarily parallel to it.

4.VII. Discussion

The composite mapping method has been shown to be a powerful tool for exposing family relationships among non-linear differential equations of importance to physics and also for generating kink and soliton-like solutions. Of the several new solutions herein reported, the solution (4.45) for DsG and solutions (4.50) and (4.52) for \( \phi^3 \), are of special interest. The other known solutions for the DsG collapse to a single kink or anti-kink in 1+1 dimensions [89]. Our solution (4.45) is an exception to this behaviour. Since all the four distinct solutions specified by (4.45)
can be simultaneously constructed for given values of the parameter $\beta$, it should be possible to study their interactions.

Another interesting feature is that when the parameters $\alpha$ and $\beta$ in (4.36) are varied, the solution (4.41) disappears at $\alpha = -\beta/2$, but the solution (4.45) arises precisely at this point in such a manner that corresponding to a kink or antikink given by (4.41), there are now two pairs of 'On-pulse' like solutions given by (4.45) of the DsG equation. Similar phenomena have been studied for the KdV system [174,175], in which the varied parameter is the depth.

The $\phi^6$ model, apart from being a classical field theory in its own right, is a model of the first order ferroelectric phase transition discussed in condensed-matter physics [176], whose finite-temperature behaviour has recently been studied [177]. The only other known (time-dependent) soliton-like solution of the $\phi^6$ equation is that reported in Ref [89].

When the composite mapping method is applied to the hyperbolic counterparts of the KG models discussed here it is found that resulting solutions are all singular.