Chapter 5

SOME RESULTS ON NONLINEAR DIFFERENTIAL POLYNOMIALS SHARING A SMALL FUNCTION
5.1 NONLINEAR DIFFERENTIAL POLYNOMIALS SHARING SMALL FUNCTION

5.1.1 INTRODUCTION, DEFINITIONS AND RESULTS

Let $f$ and $g$ be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. A meromorphic function $\alpha$ is said to be a small function of $f$ provided that $T(r, \alpha) = S(r, f)$, that is $T(r, \alpha) = o(T(r, f))$ as $r \to +\infty$, outside of a possible exceptional set of finite linear measure. Clearly if $f$ is rational then $\alpha$ is a constant and if $f$ is transcendental then $\alpha$ is a nonconstant meromorphic function. We denote by $S(f)$ the set of all small function of $f$.

If for some $\alpha \in S(f) \cap S(g)$, $f - \alpha$ and $g - \alpha$ have the same set of zeros with the same multiplicities, we say that $f$ and $g$ share $\alpha$ CM(counting multiplicities), and if we do not consider the multiplicities then $f$ and $g$ are said share $\alpha$ IM(ignoring multiplicities).

We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to +\infty$, outside of a possible exceptional set of finite linear measure.

Let $N_E(r, \alpha; f, g)$ ($\overline{N}_E(r, \alpha; f, g)$) be the counting function (reduced counting function) of all common zeros of $f - \alpha$ and $g - \alpha$ with same multiplicities and $N_0(r, \alpha; f, g)$ ($\overline{N}_0(r, \alpha; f, g)$) be the counting function (reduced counting function) of all common zeros of $f - \alpha$ and $g - \alpha$ ignoring multiplicities.

If

$$\overline{N}(r, \alpha; f) + \overline{N}(r, \alpha; g) - 2\overline{N}_E(r, \alpha; f, g) = S(r, f) + S(r, g)$$

then we say that $f$ and $g$ share $\alpha$ "CM".

On other hand if

$$\overline{N}(r, \alpha; f) + \overline{N}(r, \alpha; g) - 2\overline{N}_0(r, \alpha; f, g) = S(r, f) + S(r, g)$$
then we say that $f$ and $g$ share $\alpha$ “IM”.

We use $I$ to denote any set of infinite linear measure of $0 < r < \infty$.

In [31] Lahiri studied the problem of uniqueness of meromorphic function when two linear differential polynomials share the same 1-points. In the same paper [31] regarding the nonlinear differential polynomials Lahiri asked the following question. What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM?

Naturally several authors investigate the possible answer to the above question and continuous efforts are being carried out to relax the hypothesis of the results. (cf. [3], [16], [10], [36], [37], [39], [40], [41]).

In 2002 Fang and Fang [16] and in 2004 Lin-Yi [40] independently proved the following result.

**Theorem 5.1.A.** Let $f$ and $g$ be two nonconstant meromorphic functions and $n(\geq 13)$ be an integer. If $f^n(f - 1)^2 f'$ and $g^n(g - 1)^2 g'$ share 1 CM, then $f \equiv g$.

In 2004 Lin-Yi [41] improved Theorem 5.1.A by generalizing it in view of fixed point. Lin-Yi [41] proved the following result.

**Theorem 5.1.B.** Let $f$ and $g$ be two transcendental meromorphic functions and $n(\geq 13)$ be an integer. If $f^n(f - 1)^2 f'$ and $g^n(g - 1)^2 g'$ share z CM, then $f \equiv g$.

In the same paper Lin-Yi [41] mentioned that in Theorem 5.1.B $z$ can be replaced by $\alpha(z)$.

In 2001 an idea of gradation of sharing of values was introduced in ([33], [34]) which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

**Definition 5.1.1 ([33], [34]).** Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f$, $g$ share the value $a$ with weight $k$.

The definition implies that if $f$, $g$ share a value $a$ with weight $k$ then $z_0$ is an $a$-point...
of $f$ with multiplicity $m(\leq k)$ if and only if it is an a-point of $g$ with multiplicity $m(\leq k)$ and $z_0$ is an a-point of $f$ with multiplicity $m(> k)$ if and only if it is an a-point of $g$ with multiplicity $n(> k)$, where $m$ is not necessarily equal to $n$.

We write $f, g$ share $(a,k)$ to mean that share $f$, $g$ value $a$ with weight $k$. Clearly if $f, g$ share $(a,k)$, then $f$, $g$ share $(a,p)$ for any integer $p$, $0 \leq p < k$. Also we note that $f, g$ share a value $a$ IM or CM if and only if $f, g$ share $(a,0)$ or $(a,\infty)$ respectively.

With the notion of weighted sharing of value recently the first author [3] improved Theorem 5.1.A as follows.

**Theorem 5.1.C.** Let $f$ and $g$ be two nonconstant meromorphic functions and

$$n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\Theta(\infty; f), \Theta(\infty; g)],$$

is an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share $(1,2)$ then $f \equiv g$.

In the mean time Lahiri and Sarkar [37] also studied the uniqueness of meromorphic functions corresponding to nonlinear differential polynomials which are different from that of previously mentioned and proved the following.

**Theorem 5.1.D.** Let $f$ and $g$ be two nonconstant meromorphic functions such that

$f^n(f^2 - 1)f'$ and $g^n(g^2 - 1)g'$ share (1,2), where $n(\geq 13)$ is an integer then either $f \equiv g$

or $f \equiv -g$. If $n$ is an even integer then the possibility of $f \equiv -g$ does not arise.

From the above discussion it will be a natural query to investigate the uniqueness of meromorphic functions when two nonlinear differential polynomials of more general form namely $f^n(a(f^2 + bf + c)f'$ and $g^n(ag^2 + bg + c)g'$ where $a \neq 0$ and $|b| + |c| \neq 0$ share a small function.

In the paper [2] we will study the above problem with the notion of weakly weighted sharing which has recently been introduced by Lin and Lin [38] generalizing the idea of weighted sharing of values. We are now giving the definition.

**Definition 5.1.2 ([38]).** Let $f, g$ share $\alpha$ “IM” for $\alpha \in S(f) \cap S(g)$ and $k$ is a positive integer or $\infty$.

(i) $\overline{N}^E(r, \alpha; f, g \mid \leq k)$ denotes the reduced counting function of those $\alpha$-points of $f$
whose multiplicities are equal to the corresponding $\alpha$-points of $g$, both of their multiplicities are not greater than $k$.

(ii) $\overline{N}_0^0(r, \alpha; f, g \mid > k)$ denotes the reduced counting function of those $\alpha$-points of $f$ which are $\alpha$-points of $g$, both of their multiplicities are not less than $k$.

**Definition 5.1.3 ([38]).** For $\alpha \in S(f) \cap S(g)$, if $k$ is a positive integer or $\infty$ and

$$\overline{N}(r, \alpha; f \mid \leq k) - \overline{N}_E^0(r, \alpha; f, g \mid \leq k) = S(r, f),$$

$$\overline{N}(r, \alpha; g \mid \leq k) - \overline{N}_E^0(r, \alpha; f, g \mid \leq k) = S(r, g),$$

$$\overline{N}(r, \alpha; f \mid \geq k + 1) - \overline{N}_0^0(r, \alpha; f, g \mid \geq k + 1) = S(r, f),$$

$$\overline{N}(r, \alpha; g \mid \geq k + 1) - \overline{N}_0^0(r, \alpha; f, g \mid \geq k + 1) = S(r, g)$$

or if $k = 0$ and

$$\overline{N}(r, \alpha; f) - \overline{N}_0(r, \alpha; f, g) = S(r, f), \overline{N}(r, \alpha; g) - \overline{N}_0(r, \alpha; f, g) = S(r, g),$$

then we say $f, g$ weakly share $\alpha$ with weight $k$. Here we write $f, g$ share “$(\alpha, k)$” to mean that $f, g$ weakly share $\alpha$ with weight $k$.

Obviously if $f, g$ share “$(\alpha, k)$”, then $f, g$ share “$(\alpha, p)$” for any integer $p$, $0 \leq p < k$.

Also we note that $f, g$ share $\alpha$ “IM” or “CM” if and only if $f, g$ share “$(\alpha, 0)$” or “$(\alpha, \infty)$” respectively.

In 2008, Banerjee and Murkherjee [2] proved the following theorem.

**Theorem 5.1.1.** Let $f$ and $g$ be two transcendental meromorphic functions such that $f^n(a f^2 + b f + c) f'$ and $g^n(a g^2 + bg + c) g'$ where $a \neq 0$ and $|b| + |c| \neq 0$ share “$(\alpha, 2)$”. Then the following holds:

(i) If $b \neq 0, c = 0$ and $n > \max [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min \{\Theta(\infty; f), \Theta(\infty; g)\}], \frac{4}{\Theta(\infty; f) + \Theta(\infty; g)} - 2$, be an integer, where $\Theta(\infty; f) + \Theta(\infty; g) > 0$, then $f \equiv g$.

(ii) If $b \neq 0, c \neq 0$ and $n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min \{\Theta(\infty; f), \Theta(\infty; g)\}]$, the roots of the equation $az^2 + bz + c = 0$ are distinct and one of $f$ and $g$ is non entire.
meromorphic function having only multiple poles, then \( f \equiv g \).

(iii) If \( b \neq 0, c \neq 0 \) and \( n > \lfloor 12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\} \rfloor \)
and the roots of the equation \( ax^2 + bx + c = 0 \) coincides, then \( f \equiv g \).

(iv) If \( b = 0, c \neq 0 \) and \( n > \lfloor 12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\} \rfloor \),
then either \( f \equiv g \) or \( f \equiv -g \). If \( n \) is an even integer then the possibility \( f \equiv -g \) does not arise.

In this section, we study the uniqueness of meromorphic functions when two nonlinear differential polynomials of more general form namely \( f^n(af^3 + bf^2 + cf + d)f' \) and \( g^n(af^3 + bg^2 + cg + d)g' \) where \( a \neq 0 \) and \( b, c, d \neq 0 \) share a small function.

Though we use the standard notations and definitions of the value distribution theory available in [18], we explain some definitions and notations which are used in the paper.

**Definition 5.1.4 ([32]).** For \( a \in C \cup \{\infty\} \) we denote by \( N(r, a; f) \leq 1 \) the counting function of simple \( a \) points of \( f \). For a positive integer \( m \) we denote by \( N(r, a; f \mid \leq m) \) \( (N(r, a; f \mid \geq m)) \) the counting function of those \( a \) points of \( f \) whose multiplicities are not greater (less) than \( m \) where each \( a \) point is counted according to its multiplicity.

\[ \overline{N}(r, a; f \mid \leq m) \quad (\overline{N}(r, a; f \mid \geq m)) \]
are defined similarly, where in counting the \( a \) points of \( f \) we ignore the multiplicities.

Also \( N(r, a; f \mid < m), \overline{N}(r, a; f \mid > m) \) and \( \overline{N}(r, a; f \mid > m) \) are defined analogously.

**Definition 5.1.5 ([34], cf. [61]).** We denote by \( N_2(r, a; f) \) the sum
\[ \overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) \].

**Definition 5.1.6 ([34]).** Let \( f \) and \( g \) be two nonconstant meromorphic functions such that \( f \) and \( g \) share the value 1 IM. Let \( z_0 \) be a one-point of \( f \) with multiplicity \( p \), a one-point of \( g \) with multiplicity \( q \). We denote by \( \overline{N}_L(r, 1; f) \) the counting function of those one-points of \( f \) and \( g \) for which \( p > q \), each point in this counting functions is counted only once. In the same way we can define \( \overline{N}_L(r, 1; g) \).

**Definition 5.1.7 ([35]).** Let \( a, b \in C \cup \{\infty\} \). We denote by \( N(r, a; f \mid g = b) \) the counting
function of those \(a\)-points of \(f\), counted according to multiplicity, which are \(b\)-points of \(g\).

**Definition 5.1.8 ([35]).** Let \(a, b \in C \cup \{\infty\}\). We denote by \(N(r; a; f \mid g \neq b)\) the counting function of those \(a\)-points of \(f\), counted according to multiplicity, which are not \(b\)-points of \(g\).

### 5.1.2 LEMMAS

In this section, we present some lemmas which are needed in the sequel. Let \(f, g, F_1, G_1\) be four nonconstant meromorphic functions. Henceforth we shall denote by \(h\) and \(H\) the following two functions.

\[
h = \left( \frac{f''}{f} - \frac{2f'}{f-1} \right) - \left( \frac{g''}{g} - \frac{2g'}{g-1} \right) \quad \text{and} \quad H = \left( \frac{F_1''}{F_1} - \frac{2F_1'}{F_1-1} \right) - \left( \frac{G_1''}{G_1} - \frac{2G_1'}{G_1-1} \right).
\]

**Lemma 5.1.1.** (\([2]\)) If \(f, g\) be share “(1,1)" and \(h \neq 0\). Then

\[
N(r, 1; f \mid \leq 1) \leq N(r, 0; h) + S(r, f) \leq N(r, \infty; h) + S(r, f) + S(r, g).
\]

**Lemma 5.1.2.** (\([2]\)) If \(f, g\) be share “(1,1)" and \(h \neq 0\). Then

\[
N(r, \infty; h) \leq \overline{N}(r, 0; f \mid \geq 2) + \overline{N}(r, 0; g \mid \geq 2) + \overline{N}(r, \infty; f \mid \geq 2) + \overline{N}(r, \infty; g \mid \geq 2) + \overline{N}_L(r, 1; f) + \overline{N}_L(r, 1; g) + \overline{N}_0(r, 0; f') + \overline{N}_0(r, 0; g') + S(r),
\]

where \(\overline{N}_0(r, 0; f')\) is the reduced counting function of those zeros of \(f'\) which are not the zeros of \(f(f - 1)\) and \(\overline{N}_0(r, 0; g')\) is similarly defined.

**Lemma 5.1.3.** (\([2]\)) If for a positive integer \(k\), \(N_k(r, 0; f' \mid f \neq 0)\) denotes the counting function of those zeros of \(f'\) which are not the zeros of \(f\), where a zero of \(f'\) with multiplicity \(m\) is counted \(m\) times if \(m \leq k\) and \(k\) times if \(m > k\) then

\[
N_k(r, 0; f' \mid f \neq 0) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - \sum_{p=k+1}^{\infty} \overline{N} \left( r, 0; \frac{f'}{f} \mid \geq p \right) + S(r, f).
\]

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Lemma 5.1.4. ([2]) Let \( f, g \) be share \("(1,2)"\) and \( h \neq 0 \). Then
\[
T(r, f) \leq N_2(r, 0; f) + N_2(r, \infty; f) + N_2(r, 0; g) + N_2(r, \infty; g) - \sum_{p=3}^{\infty} N\left(r, 0; \frac{g'}{g} \mid \geq p\right) + S(r, f) + S(r, g).
\]

Lemma 5.1.5. ([42]) Let \( f \) be nonconstant meromorphic function and let
\[
R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}
\]
be an irreducible rational function in \( f \) with constant coefficients \( \{a_k\} \) and \( \{b_j\} \) where \( a_n \neq 0 \) and \( b_m \neq 0 \). Then \( T(r, R(f)) = dT(r, f) + S(r, f) \), where \( d = \max\{n, m\} \).

Lemma 5.1.6. Let \( F_1 = \frac{f^n(a f^2 + b f^2 + c f + d) f'}{a} \) and \( G_1 = \frac{f^n(a f^3 + b f^2 + c g + d) g'}{a} \), where \( a \neq 0 \) and \( |a| + |b| + |c| + |d| \neq 0 \). Then \( S(r, F_1) = S(r, f) \) and \( S(r, G_1) = S(r, g) \).

Proof. Using Lemma 5.1.5 we see that
\[
T(r, F_1) \leq (n + 3)T(r, f) + T(r, f') + S(r, f) = (n + 5)T(r, f) + S(r, f).
\]

and
\[(n + 3)T(r, f) = T(r, f^n(a f^2 + b f^2 + c f + d)) + O(1) \leq T(r, F_1) + T(r, f') + S(r, f),
\]
that is, \( T(r, F_1) \geq (n + 1)T(r, f) + S(r, f) \). Hence \( S(r, F_1) = S(r, f) \).

In the same way we can prove \( S(r, G_1) = S(r, g) \).

Lemma 5.1.7. ([62]) If \( h \equiv 0 \) and
\[
\limsup_{r \to \infty} \frac{N(r, 0; f) + N(r, \infty; f) + N(r, 0; g) + N(r, \infty; g)}{T(r)} < 1,
\]
\( r \in I \) then \( f = g \) or \( f, g = 1 \).
Lemma 5.1.8. Let \( f, g \) be two nonconstant meromorphic functions. Then

\[
f^n(af^3 + bf^2 + cf + d)f'g^n(af^3 + bg^2 + cg + d)g' \neq \alpha^2,
\]

where \( \alpha \neq 0 \) and \( |b| + |c| + |d| \neq 0 \) and \( n(> 9) \) is an integer.

Proof. If possible, let

\[
f^n(af^3 + bf^2 + cf + d)f'g^n(af^3 + bg^2 + cg + d)g' = \alpha^2. \quad (5.1.1)
\]

We consider the following cases.

Case 1. Let the roots of the equation \( az^3 + bz^2 + cz + d = 0 \) are distinct and suppose they are \( \beta_1, \beta_2 \) and \( \beta_3 \).

Subcase 1.1. Out of \( \beta_1, \beta_2 \) and \( \beta_3 \) say \( \beta_2 = \beta_3 = 0 \). Then (5.1.1) reduces to

\[
a^2 f^{n+2}(f - \beta_1)f'g^{n+2}(g - \beta_1)g' = \alpha^2.
\]

Let \( z_0 \) be a zero of \( f \) with multiplicity \( p(\geq 1) \) which is not a zero or pole of \( \alpha \). Clearly \( z_0 \) is a pole of \( g \) with multiplicity \( q(\geq 1) \) such that

\[
(n + 2)p + p - 1 = (n + 3)q + q + 1, \quad (5.1.2)
\]

i.e., \( q = (n + 3)(p - q) - 2 \geq n + 1 \).

Again from (5.1.2) we get

\[
(n + 3)p = (n + 3)q + q + 2 \geq (n + 3)(n + 2), \quad i.e., \quad p \geq n + 2.
\]

Noting that \( \alpha \) is a small function we obtain

\[
N(r, 0; f) \geq (n + 2)N(r, 0; f) + S(r, f).
\]

Next we suppose \( z_1 \) be a zero of \( f - \beta_1 \) with multiplicity \( p(\geq 1) \) which is not a zero or
pole of \( \alpha \). Then \( z_1 \) be a pole of \( g \) with multiplicity \( q(\geq 1) \) such that

\[
3p - 1 = (n + 4)q + 1 \quad \text{i.e.,} \quad p \geq \frac{n + 6}{3}.
\]

Let \( \mathcal{N}(r, 0; f') (\mathcal{N}(r, 0; g')) \) denotes the reduced counting function of those zeros of \( f'(g') \) which are not the zeros of \( f(f - \beta_1)(g(g - \beta_1)) \). Since a pole of \( f \) is either a zeros of \( g(g - \beta_1) \) or a zero or pole of \( g' \) or a zero or pole of \( \alpha \) we note that

\[
\mathcal{N}(r, \infty; f) \leq \mathcal{N}(r, 0; g) + \mathcal{N}(r, \beta_1; g) + \mathcal{N}(r, 0; g') + S(r)
\]

\[
\leq \frac{1}{n + 2} \mathcal{N}(r, 0; g) + \frac{3}{n + 6} \mathcal{N}(r, \beta_1; g) + \mathcal{N}(r, 0; g') + S(r)
\]

\[
\leq \left( \frac{1}{n + 2} + \frac{3}{n + 6} \right) T(r, g) + \mathcal{N}(r, 0; g') + S(r).
\]

By the second fundamental theorem we get

\[
T(r, f) \leq \mathcal{N}(r, 0; f) + \mathcal{N}(r, \beta_1; f) + \mathcal{N}(r, \infty; f) - \mathcal{N}(r, 0; f') + S(r, f)
\]

\[
\leq \frac{1}{n + 2} \mathcal{N}(r, 0; f) + \frac{3}{n + 6} \mathcal{N}(r, \beta_1; f) + \left( \frac{1}{n + 2} + \frac{3}{n + 6} \right) T(r, g)
\]

\[
+ \mathcal{N}(r, 0; g') - \mathcal{N}(r, 0; f') + S(r).
\]

i.e.,

\[
\left( 1 - \frac{1}{n + 2} - \frac{3}{n + 6} \right) T(r, f) \leq \left( \frac{1}{n + 2} + \frac{3}{n + 6} \right) T(r, g)
\]

\[+ \mathcal{N}(r, 0; g') - \mathcal{N}(r, 0; f') + S(r). \tag{5.1.3}
\]

In a similar manner we get

\[
\left( 1 - \frac{1}{n + 2} - \frac{3}{n + 6} \right) T(r, g) \leq \left( \frac{1}{n + 2} + \frac{3}{n + 6} \right) T(r, f)
\]

\[+ \mathcal{N}(r, 0; f') - \mathcal{N}(r, 0; g') + S(r). \tag{5.1.4}
\]

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Adding (5.1.3) and (5.1.4) we get
\[
\left(1 - \frac{2}{n+2} - \frac{6}{n+6}\right)\left\{T(r, f) + T(r, g)\right\} \leq S(r),
\]
which is a contradiction for \(n > 9\). Hence this subcase does not hold.

**Subcase 1.2.** The roots \(\beta_1, \beta_2\) and \(\beta_3\) are non zero.

Let \(z_0\) be a zero of \(f\) with multiplicity \(p(\geq 1)\) which is not a zero or pole of \(\alpha\). Then from (5.1.1) we get \(z_0\) is a pole of \(g\) with multiplicity \(q(\geq 1)\) such that
\[
np + p - 1 = (n + 4)q + 1
\]
(5.1.5)
i.e., \(q \geq \frac{n-1}{3}\). So from (5.1.5) we get
\[
(n + 1)p \geq \frac{(n + 2)(n + 1)}{3}, \text{ i.e., } p \geq \frac{(n + 2)}{3} 
\]
So from above we have
\[
N(r, 0; f) \geq \frac{n+2}{3}N(r, 0; f) + S(r, f), \text{ and so } \Theta(0; f) \geq 1 - \frac{3}{n+2}.
\]
Next suppose \(z_1\) be a zero of \(f - \beta_1\) with multiplicity \(p(\geq 1)\) and it is not a zero or pole of \(\alpha\). Then \(z_1\) be pole of \(g\) with multiplicity \(q(\geq 1)\) such that
\[
3p - 1 = (n + 4)q + 1, \text{ i.e., } p = \frac{(n + 4)q + 2}{3} \geq \frac{n + 6}{3}.
\]
\[
N(r, \beta_1; f) \geq \frac{n+6}{3}N(r, 0; f) + S(r, f), \text{ and so } \Theta(\beta_1; f) \geq 1 - \frac{3}{n+6},
\]
Similarly we can deduce that \(\Theta(\beta_2; f) \geq 1 - \frac{3}{n+6}, \ Theta(\beta_2; f) \geq 1 - \frac{3}{n+6},\)
Since \(\Theta(0; f) + \Theta(\beta_1; f) + \Theta(\beta_2; f) + \Theta(\beta_3; f) \leq 3\), it follows that
\[
4 - \frac{3}{n+2} - \frac{9}{n+6} \leq 3, \text{ or } \frac{3}{n+2} + \frac{9}{n+6} \geq 1
\]
which is a contradiction for \(n > 9\). Hence this subcase also does not hold.
Case 2. Let the roots of the equation \( az^3 + bz^2 + cz + d = 0 \) are equal say \( \beta_1 = \beta_2 = \beta_3 = \beta \). Let \( z_0 \) be a zero of \( f \) with multiplicity \( q \geq 1 \) which is not a zero or pole of \( \alpha \). Then \( z_0 \) is a pole of \( g \) with multiplicity \( q \geq 1 \) such that \( np + p - 1 = (n + 4)q + 1 \), i.e., \( q \geq \frac{n-1}{3} \) and so \( p \geq \frac{(n+1)}{3} \).

Hence \( N(r, 0; f) \geq \frac{n+2}{3} N(r, 0; f) + S(r, f) \).

Next suppose \( z_1 \) be a zero of \( f - \beta \) with multiplicity \( p \geq 1 \) and it is not a zero or pole of \( \alpha \). Then \( z_1 \) be pole of \( g \) with multiplicity \( q \geq 1 \) such that

\[
4p - 1 = (n + 4)q + 1 \geq n + 5, \quad \text{i.e.,} \quad p \geq \frac{n + 6}{4}.
\]

Let \( \overline{N}_\otimes(r, 0; f') (\overline{N}_\otimes(r, 0; g')) \) denotes the reduced counting function of those zeros of \( f'(g') \) which are not the zeros of \( f(f - \beta) (g(g - \beta)) \). Now proceeding in the same way as done in subcase 1.1 we note that

\[
\overline{N}(r, \infty; f) \leq \left( \frac{3}{n + 2} + \frac{4}{n + 6} \right) T(r, g) + \overline{N}_\otimes(r, 0; g') + S(r).
\]

By the second fundamental theorem we get

\[
T(r, f) \leq \overline{N}(r, 0; f) + \overline{N}(r, \beta; f) + \overline{N}(r, \infty; f) - \overline{N}_\otimes(r, 0; f') + S(r, f)
\]

\[
\leq \frac{3}{n + 2} N(r, 0; f) + \frac{4}{n + 6} N(r, \beta; f) + \left( \frac{3}{n + 2} + \frac{4}{n + 6} \right) T(r, g)
\]

\[
+ \overline{N}_\otimes(r, 0; g') - \overline{N}_\otimes(r, 0; f') + S(r).
\]

i.e.,

\[
\left( 1 - \frac{3}{n + 2} - \frac{4}{n + 6} \right) T(r, f) \leq \left( \frac{3}{n + 2} + \frac{4}{n + 6} \right) T(r, g)
\]

\[
+ \overline{N}_\otimes(r, 0; g') - \overline{N}_\otimes(r, 0; f') + S(r).
\]

(5.1.6)
In a similar manner we get

\[
\left(1 - \frac{3}{n+2} - \frac{4}{n+6}\right) T(r, g) \leq \left(\frac{3}{n+2} + \frac{4}{n+6}\right) T(r, f) + \overline{N}(r, 0; f') - \overline{N}(r, 0; g') + S(r).
\]

(5.1.7)

Adding (5.1.6) and (5.1.7) we get

\[
\left(1 - \frac{6}{n+2} - \frac{8}{n+6}\right) \{T(r, f) + T(r, g)\} \leq S(r),
\]

which is a contradiction for \(n > 9\). This proves the Lemma.

**Lemma 5.1.9.** Let \(F = f^{n+1}\left[\frac{a^3}{n+4} + \frac{b^2}{n+3} + \frac{c^1}{n+2} + \frac{d}{n+1}\right]\) and \(G = g^{n+1}\left[\frac{a^3}{n+4} + \frac{b^2}{n+3} + \frac{c^1}{n+2} + \frac{d}{n+1}\right]\)

where \(n(\geq 6)\) is an integer \(a \neq 0, |b| + |c| + |d| \neq 0\). Then \(F' \equiv G'\) implies \(F \equiv G\).

**Proof.** Let \(F' \equiv G'\), then \(F = G + e\) where \(e\) is constant. If possible let \(e \neq 0\). Then by second fundamental theorem and Lemma 5.1.5 we get

\[
(n + 4)T(r, f) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, e; F) + S(r, F)
\]

\[
\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \overline{N}(r, \beta_1; f) + \overline{N}(r, \beta_2; f) + \overline{N}(r, \beta_3; f)
\]

\[
+ \overline{N}(r, 0; g) + \overline{N}(r, \beta_1; g) + \overline{N}(r, \beta_2; g) + \overline{N}(r, \beta_3; g) + S(r, f)
\]

\[
(n + 4)T(r, f) \leq 5T(r, f) + 4T(r, g) + S(r, f)
\]

(5.1.8)

where \(\beta_1, \beta_2\) and \(\beta_3\) are the roots of the equation \(az^3 + bz^2 + cz + d = 0\). In a similar manner we get

\[
(n + 4)T(r, g) \leq 5T(r, g) + 4T(r, f) + S(r, g).
\]

(5.1.9)

Adding (5.1.8) and (5.1.9) we get \((n - 5)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),

which is a contradiction for \(n > 6\). So \(e = 0\) and the Lemma follows.
Lemma 5.1.10. \((12)\) Let

\[
Q(\omega) = (n - 1)^2(\omega^n - 1)(\omega^{n-2} - 1) - n(n - 2)(\omega^{n-1} - 1)^2,
\]

then

\[
Q(\omega) = (\omega - 1)^4(\omega - \beta_1)(\omega - \beta_2)...(\omega - \beta_{2n-6}),
\]

where \(\beta_j \in C \setminus \{0, 1\} (j = 1, 2, ..., 2n - 6)\), which are distinct respectively.

Lemma 5.1.11. Let \(F\) and \(G\) be given as in Lemma 5.1.9 and \(n(\geq 4)\) be an integer. Suppose \(F \equiv G\). Then the following holds.

(i) If \(b \neq 0, c = d = 0\) and \(\Theta(\infty; f) + \Theta(\infty; g) > \frac{6}{n+3}\), then \(f \equiv g\).

(ii) If \(b \neq 0, c \neq 0, d \neq 0\) and the roots of the equation \(az^3 + bz^2 + cz + d = 0\) are distinct and one of \(f\) and \(g\) is non entire meromorphic function having only multiple poles, then \(f \equiv g\).

(iii) If \(b \neq 0, c \neq 0, d \neq 0\) and the roots of the equation \(az^3 + bz^2 + cz + d = 0\) coincides, then \(f \equiv g\).

(iv) If \(b = 0, c \neq 0, d \neq 0\) then either \(f \equiv g\) or \(f \equiv -g\). If \(n\) is an even integer then the possibility \(f \equiv -g\) does not arise.

Proof. We consider the following cases.

Case 1. Suppose \(d = 0, c = 0\) and \(b \neq 0\). Then \(F \equiv G\) implies

\[
f^{n+3} \left[ \frac{a}{n+4} f + \frac{b}{n+3} \right] \equiv g^{n+3} \left[ \frac{a}{n+4} g + \frac{b}{n+3} \right] \tag{5.1.10}
\]

Let us assume \(f \neq g\). We consider two cases.

Subcase 1.1. Let \(y = \frac{a}{b}\) be a constant. Since \(y \neq 1\), from (5.1.10) it follows that \(y^{n+4} \neq 1, y^{n+3} \neq 1, y^{n+2} \neq 1\) and \(f \equiv -\frac{b(n+4)(1-y^{n+3})}{a(n+3)(1-y^{n+4})}\), a constant, which is impossible.

Subcase 1.2. Let \(y = \frac{a}{b}\) be a nonconstant. Noting that \(f \neq g\) clearly the poles of \(f\)
comes from the zeros of \( y - u_k \) where \( u_k = \exp(\frac{2\pi i}{n+4}) \), \( k = 1, 2, \ldots, n+3 \). So we have

\[
\sum_{k=1}^{n+3} N(r, u_k; y) \leq N(r, \infty; f).
\]

By second fundamental theorem and Lemma 5.1.5 we get

\[
nT(r, y) \leq \sum_{k=1}^{n+3} N(r, u_k; y) + S(r, y) \leq N(r, \infty; f) + S(r, y) \\
\leq (1 - \Theta(\infty; f) + \varepsilon)T(r, f) + S(r, y) \\
= (n + 3)(1 - \Theta(\infty; f) + \varepsilon)T(r, y) + S(r, y),
\]

i.e.,

\[
\left[ \frac{n}{n+3} - 1 + \Theta(\infty; f) - \varepsilon \right] T(r, y) \leq S(r, y), \quad (5.1.11)
\]

where \( \varepsilon > 0 \) be arbitrary. In a similar manner we can obtain

\[
\left[ \frac{n}{n+3} - 1 + \Theta(\infty; g) - \varepsilon \right] T(r, y) \leq S(r, y), \quad (5.1.12)
\]

Adding (5.1.11) and (5.1.12) we get

\[
\left[ \Theta(\infty; f) + \Theta(\infty; g) - \frac{6}{n+3} - 2\varepsilon \right] T(r, y) \leq S(r, y). \quad (5.1.13)
\]

Since \( \Theta(\infty; f) + \Theta(\infty; g) > \frac{6}{n+3} \) we can choose \( \delta > 0 \) such that

\[
\Theta(\infty; f) + \Theta(\infty; g) = \frac{6}{n+3} + \delta.
\]

So for \( 0 < \varepsilon < \frac{\delta}{2} \) from (5.1.13) we can deduce a contradiction. Hence \( f \equiv g \).

**Case 2.** Suppose \( b \neq 0 \), \( c \neq 0 \) and \( d \neq 0 \). Then \( F \equiv G \) implies

\[
Af^{n+4} + Bf^{n+3} + Cf^{n+2} + Df^{n+1} = Ag^{n+4} + Bg^{n+3} + Cg^{n+2} + Dg^{n+1}, \quad (5.1.14)
\]
where \( A = \frac{a}{n+4}, \ B = \frac{b}{n+3}, \ C = \frac{c}{n+2} \) and \( D = \frac{d}{n+1} \).

Let us assume that \( f \neq g \).

**Subcase 2.1.** Suppose the roots of equation \( az^3 + bz^2 + cz + d = 0 \) are distinct. Since (5.1.14) implies \( f, g \) share \((\infty, \infty)\) without loss of generality we may assume that \( g \) has some multiple poles. Putting \( \eta = \frac{f}{g} \) in (5.1.14) we get

\[
Ag^3(\eta^{n+4} - 1) + Bg^2(\eta^{n+3} - 1) + Cg(\eta^{n+2} - 1) + Dg(\eta^{n+1} - 1) = 0,
\]

i.e.,

\[
Ag^3 \equiv \frac{-g}{(\eta^{n+4} - 1)} \left[ Bg(\eta^{n+3} - 1) + C(\eta^{n+2} - 1) \right] - \frac{D(\eta^{n+1})}{(\eta^{n+4} - 1)}.
\]  \(5.1.15\)

Let \( z_0 \) be a pole of \( g \) which is not a root of \( \eta - u_k = 0 \), where \( u_k = \exp\left(\frac{2k\Pi i}{n+4}\right) \),

\( k = 1, 2, ..., n + 3 \) with multiplicity \( p \). Then from (5.1.15) we have

\[
3p = 2p \quad \text{i.e.,} \quad p = 0.
\]

which is impossible. The other poles of the right hand side of (5.1.15) are the roots of \( \eta - u_k = 0 \) where \( u_k = \exp\left(\frac{2k\Pi i}{n+4}\right) \), \( k = 1, 2, ..., n + 3 \). Suppose \( z_1 \) is a zero of \( \eta - u_k \) of multiplicity \( r \). From (5.1.15) we see that \( z_1 \) is a pole of \( g \) with multiplicity \( s \) (say) such that

\[
3s = 2s + r \quad \text{i.e.,} \quad s = r.
\]

Since \( g \) has no simple pole it follows that \( \eta - u_k \) has no simple zero for \( k = 1, 2, ..., n+3 \).

Hence \( \Theta(u_k; \eta) \geq \frac{1}{3} \) for \( k = 1, 2, ..., n + 3 \).

Since \( \sum_{k=1}^{n+3} \Theta(u_k; \eta) \leq 3 \) and \( n \geq 4 \) we arrive at a contradiction.

**Subcase 2.2.** Suppose the roots of the equation \( az^3 + bz^2 + cz + d = 0 \) coincides and
so we obtain by putting \( \eta = \frac{f}{g} \) in (5.1.14) we get

\[
a(n + 3)(n + 2)(n + 1)g^3(\eta^{n+4} - 1) + b(n + 4)(n + 2)(n + 1)g^2(\eta^{n+3} - 1) \\
+c(n + 4)(n + 3)(n + 1)g(\eta^{n+2} - 1) + d(n + 4)(n + 3)(n + 2)(\eta^{n+1} - 1) \equiv 0.
\]

(5.1.16)

Since \( \eta \) is not constant using Lemma 5.1.10 we get from (5.1.16) that

\[
\left[ (n + 3)(n + 2)(n + 1)g^3(\eta^{n+4} - 1) + \frac{c}{2b}(n + 4)(n + 2)(n + 1)(\eta^{n+3} - 1) \right] \\
= -\frac{d}{b}(n + 4)(n + 2)(n + 1)Q(\eta),
\]

(5.1.17)

where \( Q(\eta) = (\eta - 1)^4(\eta - \beta_1)(\eta - \beta_2)\ldots(\eta - \beta_{2n}) \) and \( \beta_j \in C \setminus \{0, 1\} \ (j = 1, 2, \ldots, 2n) \), which are distinct. This implies that every zero of \( \eta - \beta_j \) \((j = 1, 2, \ldots, 2n)\) has a multiplicity of at least 3, i.e., \( \Theta(\beta_j; \eta) \geq \frac{1}{3} \) for \((j = 1, 2, \ldots, 2n)\). But \( \sum_{j=1}^{2n} \Theta(\beta_j; \eta) \leq 3 \) which implies \( n \leq 3 \). This is a contradiction. So \( \eta \) is constant and from (5.1.15) we have \((\eta^{n+1} - 1) = 0, (\eta^{n+2} - 1) = 0 \) and \((\eta^{n+3} - 1) = 0 \) which implies \( \eta = 1 \) and \( \eta = 1 \) and so \( f \equiv g \).

**Case 3.** Suppose \( b = 0 \) and \( c \neq 0, d \neq 0 \). Then (5.1.14) reduces to

\[
f^{n+1} \left[ \frac{af^3}{n+4} + \frac{cf}{n+2} + \frac{d}{n+1} \right] \equiv g^{n+1} \left[ \frac{ag^3}{n+4} + \frac{cg}{n+2} + \frac{d}{n+1} \right]
\]

Now proceeding in the line of Lemma 5.1.4 in [37] we can prove \( f \equiv g \) and \( f \equiv -g \) and if \( n \) is an even integer then the possibility of \( f \equiv -g \) does arise.

**Lemma 5.1.12.** Let \( F \) and \( G \) be given as in Lemma 5.1.9 and \( F_1, G_1 \) be given by Lemma 5.1.6. If \( \gamma_1, \gamma_2, \gamma_3 \) are the roots of \( \frac{a}{n+4}z^3 + \frac{b}{n+3}z^2 + \frac{c}{n+2}z + \frac{d}{n+1} = 0 \) and \( \beta_1, \beta_2, \beta_3 \) are the roots of \( az^3 + bz^2 + cz + d = 0 \). Then

\[
T(r, F) \leq T(r, F_1) + N(r, 0; f) + N(r, \gamma_1; f) + N(r, \gamma_2; f) + N(r, \gamma_3; f) \\
- N(r, \beta_1; f) - N(r, \beta_2; f) - N(r, \beta_3; f) - N(r, 0; f') + S(r).
\]

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Proof. Clearly $F' = \alpha F_1$ and $G' = \alpha G_1$. By the first fundamental theorem and Lemmas 5.1.5, 5.1.6 we obtain

$$T(r, F) = T \left( r, \frac{1}{F} \right) + O(1) = N(r, 0; F) + m \left( r, \frac{1}{F} \right) + O(1)$$

$$\leq N(r, 0; F) + m \left( r, \frac{F'}{F} \right) + m(r, 0; F') + O(1)$$

$$= T(r, F') + N(r, 0; F') - N(r, 0; F') + S(r, F)$$

$$\leq T(r, F_1) + (n + 1)N(r, 0; f) + N(r, \gamma_1; f) + N(r, \gamma_2; f) + N(r, \gamma_3; f) - nN(r, 0; f) - N(r, \beta_1; f) - N(r, \beta_2; f) - N(r, \beta_3; f) - N(r, 0; f') + S(r)$$

$$\leq T(r, F_1) + N(r, 0; f) + N(r, \gamma_1; f) + N(r, \gamma_2; f) + N(r, \gamma_3; f) - N(r, \beta_1; f) - N(r, \beta_2; f) - N(r, \beta_3; f) - N(r, 0; f') + S(r).$$

5.1.3 STATEMENT AND PROOF OF MAIN RESULTS

Theorem 5.1.1. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that $f^n(a f^3 + b f^2 + c f + d) f'$ and $g^n(a g^3 + b g^2 + c g + d) g'$ where $a \neq 0$ and $|b| + |c| + |d| \neq 0$ share "($\alpha, 2^n$)". Then the following holds.

(i) If $b \neq 0$, $c = d = 0$ and $n > \max [13 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min \{\Theta(\infty; f), \Theta(\infty; g)\}]$, be an integer, where $\Theta(\infty; f) + \Theta(\infty; g) > 0$, then $f \equiv g$.

(ii) If $b \neq 0$, $c \neq 0$, $d \neq 0$ and $n > [13 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min \{\Theta(\infty; f), \Theta(\infty; g)\}]$, the roots of the equation $az^3 + bz^2 + cz + d = 0$ are distinct and one of $f$ and $g$ is non-entire meromorphic function having only multiple poles, then $f \equiv g$.

(iii) If $b \neq 0$, $c \neq 0$, $d \neq 0$ and $n > [13 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min \{\Theta(\infty; f), \Theta(\infty; g)\}]$ and the roots of the equation $az^3 + bz^2 + cz + d = 0$ coincide, then $f \equiv g$.

(iv) If $b = 0$, $c \neq 0$, $d \neq 0$ and $n > [13 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min \{\Theta(\infty; f), \Theta(\infty; g)\}]$, then either $f \equiv g$ or $f \equiv -g$. If $n$ is an even integer then the possibility $f \equiv -g$ does not arise.

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Proof. Let $F$, $G$ be defined as in Lemma 5.1.9 and $F_1$ and $G_1$ be defined as in Lemma 5.1.6. Then it follows that $F'$ and $G'$ share $(\alpha, 2)$" and hence $F_1$ and $G_1$ share "$(1, 2)$". Suppose $H \neq 0$. Then by Lemma 5.1.4, 5.1.6 we get

$$T(r, F_1) \leq N_2(r, 0; F_1) + N_2(r, \infty; F_1) + N_2(r, 0; G_1)$$
$$+ N_2(r, \infty; G_1) + S(r, f) + S(r, g)$$
$$\leq 2\overline{N}(r, 0; f) + N(r, \beta_1; f) + N(r, \beta_2; f) + N(r, \beta_3; f)$$
$$+ 2\overline{N}(r, 0; g) + N(r, \beta_1; g) + N(r, \beta_2; g) + N(r, \beta_3; g)$$
$$+ 2\overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) + \overline{N}(r, 0; f') + \overline{N}(r, 0; g') + S(r). \quad (5.1.18)$$

Now from Lemma 5.1.5, 4.1.1 and 5.1.12 we can obtain from (5.1.18) for $\varepsilon(> 0)$

$$(n + 4)T(r, f) \leq 2\overline{N}(r, 0; f) + 2\overline{N}(r, \infty; f) + 4T(r, f) + 2\overline{N}(r, 0; g)$$
$$+ 2\overline{N}(r, \infty; g) + 3T(r, g) + N(r, 0; g') + S(r).$$
$$\leq 6T(r, f) + 6T(r, g) + 2\overline{N}(r, \infty; f)$$
$$+ 3\overline{N}(r, \infty; g) + S(r). \quad (5.1.19)$$

In a similar manner we can obtain

$$(n + 4)T(r, g) \leq (17 - 2\Theta(\infty; f) - 3\Theta(\infty; g) + 2\varepsilon) T(r) + S(r). \quad (5.1.20)$$

From (5.1.19) and (5.1.20) we get

$$[n - 13 + 2\Theta(\infty; f) + 2\Theta(\infty; g) + \min\{\Theta(\infty; f); \Theta(\infty; g)\} - 2\varepsilon] T(r) \leq S(r). \quad (5.1.21)$$

Since $\varepsilon(\geq 0)$ is arbitrary, (5.1.21) implies a contradiction. Hence $H \equiv 0$. Since for $\varepsilon > 0$
we have

\[
\overline{N}(r, 0; f') \leq T(r, f') - m\left(r, \frac{1}{f'}\right) \\
\leq m(r, f) + N(r, \infty; f) + \overline{N}(r, \infty; f) - m\left(r, \frac{1}{f'}\right) + S(r, f) \\
\leq (2 - \Theta(\infty; f) + \varepsilon)T(r, f) - m\left(r, \frac{1}{f'}\right) + S(r, f).
\]

We note that

\[
\overline{N}(r, 0; F_1) + \overline{N}(r, \infty; F_1) + \overline{N}(r, 0; G_1) + \overline{N}(r, \infty; G_1) \\
\leq \overline{N}(r, 0; f) + \overline{N}(r, \beta_1; f) + \overline{N}(r, \beta_2; f) + \overline{N}(r, \beta_3; f) \\
+ \overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + \overline{N}(r, 0; g) + \overline{N}(r, \beta_1; g) \\
+ \overline{N}(r, \beta_2; g) + \overline{N}(r, \beta_3; g) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; g') \\
\leq (13 - 2\Theta(\infty; f) - 2\Theta(\infty; g) + 2\varepsilon)T(r) - m(r, 0; f') \\
- m(r, 0; g') + S(r).
\]  

Also using Lemma 5.1.5 we get

\[
T(r, F') + m\left(r, \frac{1}{f'}\right) = m(r, f'^n(af^3 + bf^2 + cf + d)f') + m\left(r, \frac{1}{f'}\right) \\
+ N(r, \infty; f'^n(af^3 + bf^2 + cf + d)f') \\
\geq m(r, f'^n(af^3 + bf^2 + cf + d)) + N(r, \infty; f'^n(af^3 + bf^2 + cf + d)) \\
= T(r, f'^n(af^3 + bf^2 + cf + d)) = (n + 3)T(r, f) + O(1).
\]

Similarly

\[
T(r, G') + m\left(r, \frac{1}{g'}\right) \geq (n + 3)T(r, g) + O(1).
\]
From (5.1.23) and (5.1.24) we get

$$\max\{T(r, F_1), T(r, G_1)\} \geq (n + 3)T(r) - m\left(r, \frac{1}{f'}\right) - m\left(r, \frac{1}{g'}\right) + O(1). \quad (5.1.25)$$

By (5.1.22) and (5.1.25) applying Lemma 5.1.7 we get either $F_1 \equiv G_1$ or $F_1 G_1 \equiv 1$.

Now from Lemma 5.1.8 it follows that $F_1 G_1 \neq 1$. Again $F_1 \equiv G_1$ implies $F' \equiv G'$. So from lemmas 5.1.9 and 5.1.11 the theorem follows.

From Theorem 5.1.1 we can immediately deduce the following corollaries.

**Corollary 5.1.1.** Let $f$ and $g$ be two transcendental meromorphic functions such that $\Theta(\infty; f) + \Theta(\infty; g) > \frac{6}{n+3}$, and $n(\geq 14)$ be an integer. If $f^n(a f^3 + b f^2) f'$ and $g^n(a g^3 + b g^2) g'$ share \((\alpha, 2)\) then $f \equiv g$.

**Corollary 5.1.2.** Let $f$ and $g$ be two transcendental meromorphic functions and one of $f$ and $g$ is non entire meromorphic function having only multiple poles, such that $n > \left[13 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\right]$ be an integer. If $a f^n(f - \beta_1)(f - \beta_2)(f - \beta_3)f'$ and $a g^n(g - \beta_1)(g - \beta_2)(g - \beta_3)g'$ share \((\alpha, 2)\), where $\beta_1, \beta_2$ and $\beta_3$ are the distinct roots of the equation $az^3 + bz^2 + cz + d = 0$ with $|\beta_1| \neq |\beta_2| \neq |\beta_3|$, then $f \equiv g$.

**Corollary 5.1.3.** Let $f$ and $g$ be two transcendental meromorphic functions such that $n > \left[13 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\right]$ be an integer. If $a f^n(f + k)^3 f'$ and $a g^n(g + k)^3 g'$ share \((\alpha, 2)\) where $k$ is a nonzero constant then $f \equiv g$.

**Corollary 5.1.4.** Let $f$ and $g$ be two transcendental meromorphic functions such that $n > \left[13 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\right]$ be an integer. If $f^n(a f^3 + cf + d)f'$ and $g^n(a g^3 + cg + d)g'$ share \((\alpha, 2)\) then either $f \equiv g$ or $f \equiv -g$. If $n$ is an even integer then the possibility $f \equiv -g$ does not arise.
5.2 UNIQUENESS OF MEROMORPHIC FUNCTIONS
WITH THEIR NONLINEAR DIFFERENTIAL POLYNOMIALS SHARE A SMALL FUNCTION

5.2.1 INTRODUCTION AND RESULTS

In this section, we deal with the uniqueness of meromorphic functions when two nonlinear differential polynomials generated by two meromorphic functions share a small function. We consider the case for some general differential polynomials \([f^n P(f), f']\) where\(P(f)\) is a polynomial which generalize some result due to Abhijit Banerjee and Sonali Mukherjee [2].

In 2002 Fang and Fang [16] and in 2004 Lin-Yi [40] independently proved the following result.

**Theorem 5.2.A.** Let \(f\) and \(g\) be two nonconstant meromorphic functions and \(n(\geq 13)\) be an integer. If \(f^n(f-1)^2 f'\) and \(g^n(g-1)^2 g'\) share \(1\ CM\), then \(f \equiv g\).

In 2004 Lin-Yi [41] improved Theorem 5.2.A by generalizing it in view of fixed point. Lin-Yi [41] proved the following result.

**Theorem 5.2.B.** Let \(f\) and \(g\) be two transcendental meromorphic functions and \(n(\geq 13)\) be an integer. If \(f^n(f-1)^2 f'\) and \(g^n(g-1)^2 g'\) share \(\omega\ CM\), then \(f \equiv g\).

With the notion of weighted sharing of value recently the first author [3] improved Theorem 5.2.A as follows.

**Theorem 5.2.C.** Let \(f\) and \(g\) be two nonconstant meromorphic functions and \(n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\Theta(\infty; f), \Theta(\infty; g)]\), is an integer. If \(f^n(f-1)^2 f'\) and \(g^n(g-1)^2 g'\) share \((1,2)\) then \(f \equiv g\).

In the mean time Lahiri and Sarkar [37] also studied the uniqueness of meromorphic functions corresponding to nonlinear differential polynomials which are different from that of previously mentioned and proved the following.
Theorem 5.2.D. Let \( f \) and \( g \) be two nonconstant meromorphic functions such that \( f^n(f^2 - 1)f' \) and \( g^n(g^2 - 1)g' \) share \((1,2)\), where \( n(\geq 13) \) is an integer then either \( f \equiv g \) or \( f \equiv -g \). If \( n \) is an even integer then the possibility of \( f \equiv -g \) does not arise.

In 2008, Banerjee and Muralikrishna [2] proved the following theorem.

Theorem 5.2.E. Let \( f \) and \( g \) be two transcendental meromorphic functions such that \( f^n(a f^2 + b f + c)f' \) and \( g^n(a g^2 + bg + c)g' \) where \( a \neq 0 \) and \( |b| + |c| \neq 0 \) share \((\alpha,2)\). Then the following holds:

(i) If \( b \neq 0 \), \( c = 0 \) and \( n > \max[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}], \frac{4}{\Theta(\infty; f) + \Theta(\infty; g)} - 2, \) be an integer, where \( \Theta(\infty; f) + \Theta(\infty; g) > 0 \), then \( f \equiv g \).

(ii) If \( b \neq 0 \), \( c \neq 0 \) and \( n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}], \) the roots of the equation \( a z^2 + bz + c = 0 \) are distinct and one of \( f \) and \( g \) is non entire meromorphic function having only multiple poles, then \( f \equiv g \).

(iii) If \( b \neq 0 \), \( c \neq 0 \) and \( n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}], \) and the roots of the equation \( az^2 + bz + c = 0 \) coincides, then \( f \equiv g \).

(iv) If \( b = 0 \), \( c \neq 0 \) and \( n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}], \) then either \( f \equiv g \) or \( f \equiv -g \). If \( n \) is an even integer then the possibility \( f \equiv -g \) does not arise.

In this section, we obtain unicity theorem when \( [f^n P(f)f'] \) and \( [g^n P(g)g'] \) share a small function.

5.2.2 LEMMAS

In this section we present some lemmas which will be needed in the sequel. Let \( f, g, F_1, G_1 \) be four nonconstant meromorphic functions. Henceforth we shall denote by \( h \) and \( H \) the following two functions.

\[
h = \left( \frac{f''}{f'} - \frac{2f'}{f-1} \right) - \left( \frac{g''}{g'} - \frac{2g'}{g-1} \right) \quad \text{and} \quad H = \left( \frac{F_1''}{F_1'} - \frac{2F_1'}{F_1-1} \right) - \left( \frac{G_1''}{G_1'} - \frac{2G_1'}{G_1-1} \right).
\]

Lemma 5.2.1. Let \( F_1 = \frac{f^n P(f)f'}{\alpha} \) and \( G_1 = \frac{g^n P(g)g'}{\alpha} \), where \( \alpha(\neq 0, \infty) \) is a small function of \( f \) and \( g \). Then \( S(r, F_1) = S(r, f) \) and \( S(r, G_1) = S(r, g) \).
Proof. Using Lemma 1.1.2 we see that

\[ T(r, F_1) \leq (n + m)T(r, f) + T(r, f') + S(r, f) = (n + m + 2)T(r, f) + S(r, f). \]

and

\[ (n + m)T(r, f) = T(r, f^n P(f)) + O(1) \leq T(r, F_1) + T(r, f') + S(r, f), \]

that is, \( T(r, F_1) \geq (n + m - 2)T(r, f) + S(r, f) \). Hence \( S(r, F_1) = S(r, f) \).

In the same way we can prove \( S(r, G_1) = S(r, g) \). This proves the Lemma.

Lemma 5.2.2. Let \( f, g \) be two nonconstant meromorphic functions. Then

\[ f^n P(f) f' g^n P(g) g' \neq \alpha^2, \]

where \( n + m (\geq 6) \) is an integer.

Proof. Let

\[ f^n P(f) f' g^n P(g) g' = \alpha^2. \tag{5.2.1} \]

Let \( z_0 \) be a \( 1 \)-point of \( f \) with multiplicity \( p (\geq 1) \). Then \( z_0 \) is a pole of \( g \) with multiplicity \( q (\geq 1) \) such that \( np + p - 1 = nq + q + mq + 1 \),

i.e.,

\[ mq + 2 = (n + 1)(p - q) \tag{5.2.2} \]

From (5.2.2) we get \( q \geq \frac{n - 1}{m} \) and again from (5.2.2) we obtain

\[ p \geq \frac{1}{n + 1} \left[ \frac{(n + m + 1)(n - 1)}{m} + 2 \right] = \frac{n + m - 1}{m}. \]

Let \( z_1 \) be a zero of \( P(f) \) with multiplicity \( p_1 (\geq 1) \). Then \( z_1 \) is a pole of \( g \) with multiplicity
\( q_1 (\geq 1) \), say, so from (5.2.1) we get

\[
2p_1 - 1 = (n + m + 1)q + 1 \\
\geq (n + m + 2)
\]

i.e.,

\[
p_1 \geq \frac{(n + m + 3)}{2}
\]

Since a pole of \( f \) is either a zero of \( g^n P(g) \) or a zero of \( g' \), we have

\[
\overline{N}(r, \infty; f) \leq \overline{N}(r, 0; g) + \overline{N}(r, 0; g^m) + \overline{N}_0(r, 0; g') + S(r, f) + S(r, g) \\
\leq \frac{m}{n + m - 1} \overline{N}(r, 0; g) + \frac{2}{n + m + 3} \overline{N}(r, 0; g^m) + \overline{N}_0(r, 0; g') + S(r, f) + S(r, g)
\]

where \( \overline{N}_0(r, 0; g') \) denotes the reduced counting function of those zeros of \( g' \) which are not the zeros of \( g P(g) \).

As \( P(f) - \alpha_m f^m + \alpha_{m-1} f^{m-1} + \ldots + \alpha_1 f + \alpha_0 \) where \( \alpha_m, \alpha_{m-1}, \ldots, \alpha_0 \) are \( m \) distinct complex numbers. Then by second fundamental theorem of Nevanlinna we get

\[
mT(r, f) \leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \sum_{j=1}^{m} \overline{N}(r, a_j; f) - \overline{N}_0(r, 0; f') + S(r, f) \\
\leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + \overline{N}(r, a; f^m) - \overline{N}_0(r, 0; f') + S(r, f) \\
\leq \left( \frac{m}{n + m - 1} + \frac{2m}{n + m + 3} \right) \{ T(r, g) + T(r, f) \} + \overline{N}_0(r, 0; g') \\
- \overline{N}_0(r, 0; f') + S(r, f) + S(r, g). \quad (5.2.3)
\]

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Similarly, we have

\[ mT(r, g) \leq \left( \frac{m}{n + m - 1} + \frac{2m}{n + m + 3} \right) \{ T(r, g) + T(r, f) \} + \overline{N}_0(r, 0; f') \]
\[- \overline{N}_0(r, 0; g') + S(r, f) + S(r, g). \]

(5.2.4)

Adding (5.2.3) and (5.2.4) we obtain

\[ \left( 1 - \frac{2}{n + m - 1} - \frac{4}{n + m + 3} \right) \{ T(r, f) + T(r, g) \} \leq S(r, f) + S(r, g). \]

which is a contradiction. This proves the Lemma.

**Lemma 5.2.3.** Let \( f \) and \( g \) be two transcendental meromorphic function and

\[ F = f^{n+1} \left[ \frac{a_m}{m+n+1} f^m + \frac{a_{m-1}}{m+n} f^{m-1} + \ldots + \frac{a_0}{n+1} \right] \]
\[ G = g^{n+1} \left[ \frac{a_m}{m+n+1} g^m + \frac{a_{m-1}}{m+n} g^{m-1} + \ldots + \frac{a_0}{n+1} \right] \]

where \( n(> m + 2) \) is an integer. Then \( F' \equiv G' \) implies that \( F \equiv G \).

**Proof.** Let \( F' \equiv G' \), then \( F = G + c \) where \( c \) is constant. Let \( c \neq 0 \). Then by second fundamental theorem and Lemma 1.1.2 we get

\[ T(r, F) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, c; F) + S(r, F) \]
\[ \leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \overline{N} \left( r, \frac{a_m}{m+n+1}; f^m \right) + \overline{N}(r, 0; g) + \overline{N} \left( r, \frac{a_m}{m+n+1}; g^m \right) + S(r, f) \]
\[ \leq 2T(r, f) + mT(r, f) + T(r, g) + mT(r, g) + S(r, f). \]

Hence we get

\[ (m + n + 1)T(r, f) \leq (2 + m)T(r, f) + (m + 1)T(r, g) + S(r, f). \]

(5.2.5)
Similarly, we have

\[(m + n + 1)T(r, g) \leq (2 + m)T(r, g) + (m + 1)T(r, f) + S(r, g).\] (5.2.6)

Adding (5.2.5) and (5.2.6) we obtain

\[(m + n + 1)\{T(r, f) + T(r, g)\} \leq (3 + 2m)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).\]

i.e., \((n - m - 2)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g)\) which is a contradiction.

So \(c = 0\) and the Lemma is proved.

**Lemma 5.2.4.** Let \(F\) and \(G\) be given as in Lemma 5.2.3 and \(F_1, G_1\) be given by Lemma 5.2.1. Then

\[(i)\ T(r, F) \leq T(r, F_1) + N(r, 0; f) + N(r, b_1; f) + N(r, b_2; f) + \ldots + N(r, b_m; f)\]

\[\quad - N(r, c_1; f) - N(r, c_2; f) - \ldots - N(r, c_m; f) - N(r, 0; f') + S(r, f).\]

\[(ii)\ T(r, G) \leq T(r, G_1) + N(r, 0; g) + N(r, b_1; g) + N(r, b_2; g) + \ldots + N(r, b_m; g)\]

\[\quad - N(r, c_1; g) - N(r, c_2; g) - \ldots - N(r, c_m; g) - N(r, 0; g') + S(r, g).\]

where \(b_1, b_2, \ldots, b_m\) are roots of the algebraic equation

\[\frac{a_m}{m + n + 1}z^m + \frac{a_{m-1}}{m + n}z^{m-1} + \ldots + \frac{a_0}{n + 1} = 0\]

and \(c_1, c_2, \ldots, c_m\) are roots of the algebraic equation

\[a_mz^m + a_{m-1}z^{m-1} + \ldots + a_0 = 0.\]
Proof. By the Nevanlinna’s first fundamental theorem and Lemmas 1.1.2 we obtain

\[ T(r, F) = T\left( r, \frac{1}{F} \right) + O(1) = N(r, 0; F) + m\left( r, \frac{1}{F} \right) + O(1) \]
\[ \leq N(r, 0; F) + m\left( r, \frac{F'}{F} \right) + m(r, 0; F') + O(1) \]
\[ = T(r, F') + N(r, 0; F) - N(r, 0; F') + S(r, F) \]
\[ \leq T(r, F_1) + N(r, 0; f) + N(r, b_1; f) + \ldots + N(r, b_m; f) \]
\[ - N(r, c_1; f) - \ldots - N(r, c_m; f) - N(r, 0; f') + S(r, f). \]

Similarly, we have

\[ T(r, G) \leq T(r, G_1) + N(r, 0; g) + N(r, b_1; g) + N(r, b_2; g) + \ldots + N(r, b_m; g) \]
\[ - N(r, c_1; g) - N(r, c_2; g) - \ldots - N(r, c_m; g) - N(r, 0; g') + S(r, g). \]

where \( b_1, b_2, \ldots, b_m \) are roots of the algebraic equation

\[ \frac{a_m}{m+n+1}z^m + \frac{a_{m-1}}{m+n}z^{m-1} + \ldots + \frac{a_0}{n+1} = 0 \]

and \( c_1, c_2, \ldots, c_m \) are roots of the algebraic equation

\[ a_mz^m + a_{m-1}z^{m-1} + \ldots + a_0 = 0. \]

This proves the Lemma.

5.2.3 STATEMENT AND PROOF OF MAIN RESULTS

Theorem 5.2.1. Let \( f \) and \( g \) be two transcendental meromorphic functions. Let \( P(f) = a_m f^m + a_{m-1} f^{m-1} + \ldots + a_1 f + a_0 \), \( (a_m \neq 0) \), and \( a_i(i = 0, 1, \ldots, m) \) is the first nonzero
coefficient from the right, and \( n, m \) be a positive integer with

\[
n > [m + 10 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f); \Theta(\infty; g)\}] .
\]

If \( [f^n P(f) f'] \) and \( [g^n P(g) g'] \) share \("(\alpha, 2)\) then \( f \equiv g \).

**Proof.** Let \( F, G \) be defined as in Lemma 5.2.3 and \( F_1 \) and \( G_1 \) be defined as in Lemma 5.2.1. Then it follows that \( F' \) and \( G' \) share \"(\alpha, 2)\" and hence \( F_1 \) and \( G_1 \) share \"(1, 2)\". Suppose \( H \neq 0 \). Then by Lemmas 5.1.4, 5.2.1 and (5.2.3) we get

\[
T(r, F_1) \leq \text{const} \bigg( N_2(r, 0; F_1) + N_2(r, \infty; F_1) + N_2(r, 0; G_1) + N_2(r, \infty; G_1) + S(r, f) + S(r, g) \\
< \text{const} \bigg( 2N(r, 0; f) + 2N(r, \infty; f) + N(r, c_1; f) + N(r, c_2; f) + \ldots + N(r, c_m; f) \\
+ N(r, 0; f') + 2N(r, \infty; g) + 2N(r, 0; g) + N(r, c_1; g) + N(r, c_2; g) + \ldots + N(r, c_m; g) \\
+ N(r, 0; g') + \text{const} + S(r, f) + S(r, g) \bigg). 
\]

(5.2.7)

Now from Lemmas 1.1.2, 4.1.1 and 5.2.4 we can obtain from (5.2.7) for \( \varepsilon(>0) \)

\[
(n + m + 1)T(r, f) \leq 2N(r, 0; f) + 2N(r, \infty; f) + mT(r, f) + N(r, 0; f) \\
+ 2N(r, 0; g) + 2N(r, \infty; g) + mT(r, g) + N(r, 0; g') \\
\leq (m + 3) [T(r, f) + T(r, g)] + 2N(r, \infty; f) \\
+ 3N(r, \infty; g) + S(r, f) + S(r, g).
\]

(5.2.8)

\[(n + m + 1)T(r, f) \leq (2m + 11 - 3\Theta(\infty; g) - 2\Theta(\infty; f) + 2\varepsilon) T(r) + S(r). \quad (5.2.8)\]

**In a similar manner we can obtain**

\[
(n + m + 1)T(r, g) \leq (2m + 11 - 3\Theta(\infty; g) - 3\Theta(\infty; f) + 2\varepsilon) T(r) + S(r). \quad (5.2.9)
\]
From (5.2.8) and (5.2.9) we get

\[ [n - m - 10 + 2\Theta(\infty; f) + 2\Theta(\infty; g) + \min\{\Theta(\infty; f); \Theta(\infty; g)\} - 2\varepsilon] T(r) \leq S(r). \]  

(5.2.10)

Since \( \varepsilon(> 0) \) is arbitrary, (5.2.10) implies a contradiction. Hence \( H \equiv 0 \). Since for \( \varepsilon > 0 \) we have

\[
\overline{N}(r, 0; f') \leq T(r, f') - m \left( r, \frac{1}{f'} \right) \\
\leq m(r, f) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; f) - m \left( r, \frac{1}{f} \right) + S(r, f) \\
\leq (2 - \Theta(\infty; f) + \varepsilon) T(r, f) - m \left( r, \frac{1}{f} \right) + S(r, f).
\]

We note that

\[
\overline{N}(r, 0; F_1) + \overline{N}(r, \infty; F_1) + \overline{N}(r, 0; G_1) + \overline{N}(r, \infty; G_1) \\
\leq \overline{N}(r, 0; f) + \overline{N}(r, 0; f') + \overline{N}(r, 0; f') + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; f) \\
+ \overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + \overline{N}(r, 0; g) + \overline{N}(r, c_1; g) \\
+ \overline{N}(r, c_2; g) + ... + \overline{N}(r, c_m; g) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; g') \\
\leq (2m + 8 - 2\Theta(\infty; f) - 2\Theta(\infty; g) + 2\varepsilon) T(r) - m(r, 0; f') \\
- m(r, 0; g') + S(r). 
\]  

(5.2.11)

Also using Lemma 1.1.2 we get

\[
T(r, F') + m \left( r, \frac{1}{f'} \right) = m(r, f^n P(f) f') + m \left( r, \frac{1}{f'} \right) \\
+ N(r, \infty; f^n P(f) f') \\
\geq m(r, f^n P(f)) + N(r, \infty; f^n P(f)) \\
= T(r, f^n P(f)) = (n + m) T(r, f) + O(1).
\]  

(5.2.12)
Similarly
\[ T(r, G') + m \left( r, \frac{1}{g'} \right) \geq (n + m)T(r, g) + O(1). \] (5.2.13)

From (5.2.12) and (5.2.13) we get
\[ \max\{T(r, F_1), T(r, G_1)\} \geq (n + m)T(r) - m \left( r, \frac{1}{f'} \right) - m \left( r, \frac{1}{g'} \right) + O(1). \] (5.2.14)

By (5.2.11) and (5.2.14) applying Lemma 5.1.7 we get either \( F_1 \equiv G_1 \) or \( F_1 G_1 \equiv 1 \).

Now from Lemma 5.2.2 it follows that \( F_1 G_1 \neq 1 \). Again \( F_1 \equiv G_1 \) implies \( F' \equiv G'' \). So from Lemma 5.2.3 the theorem follows.