A pseudo-complemented lattice is a lattice $A$ with 0 such that to each $a \in A$, the largest annihilating element of $a$ exists in $A$. That is, there exists an element $a^* \in A$ such that for any $x \in A$, $a \land x = 0$ if and only if $x \leq a^*$. Here $a^*$ is called the pseudo-complement of $a$. For each element $a$ of a pseudo-complemented lattice $A$, $a^*$ is uniquely determined by $a$ and hence $*$ can be regarded as a unary operation on $A$. Moreover, a pseudo-complemented lattice contains the greatest element, namely, $0^*$. Thus, it follows that every pseudo-complemented lattice $A$ can be regarded as an algebra $(A, \lor, \land, *, 0, 1)$ of type $(2, 2, 1, 0, 0)$. The set $A^* = \{a^* \mid a \in A\}$, where $*$ is a pseudo-complementation on $A$, becomes a Boolean algebra.

U.M. Swamy, G.C. Rao and G. Nanaji Rao introduced the concept of a pseudo-complementation in an ADL [54] and they observed that an Almost Distributive Lattice (ADL) $A$ can have more than one pseudo-complementation. In fact, they proved that there is a one-to-one correspondence between the set of all maximal elements of an ADL $A$ and the set of all pseudo-complements on $A$. Also, they proved that if $*$ is a pseudo-complementation on an ADL $A$, then the set $A^* = \{a^* \mid a \in A\}$ is a Boolean algebra.
under suitable operations and that the pseudo-complementation \( * \) on \( A \) is equationally definable.

The set \( B(A) \) of all complemented elements of a distributive lattice \( A \) is called the Birkhoff center of \( A \) and it is a sublattice of \( A \) which is also a Boolean algebra. Analogously, U.M. Swamy and S. Ramesh [53] defined the Birkhoff center of an ADL \( A \) as

\[
B(A) = \{ x \in A \mid \text{there exists } y \in A \text{ such that } x \land y = 0 \text{ and } x \lor y \text{ is a maximal element of } A \}
\]

and proved that \( B(A) \) is a subADL of \( A \) and is a relatively complemented ADL.

In [12], G. Epstein defined the concept of a pseudo-supplement of an element \( x \) in a distributive lattice \( A \) as the greatest element \( b \) in the center of \( A \) such that \( b \leq x \). In this chapter, we extend the concept of pseudo-supplementation to an ADL \( A \) with the help of the Birkhoff center \( B(A) \) of \( A \). Since the dual of an ADL is not an ADL, in general, we also introduce the concept of a dual pseudo-supplementation in an ADL and study its properties.

Chapter - I is divided into 4 sections. In section 1.1, we introduce the concept of a pseudo-supplemented Almost Distributive Lattice (or, simply a PSADL) as a generalization of a pseudo-supplemented lattice and derive the properties of a PSADL. We obtain necessary and sufficient conditions for an ADL to become a PSADL. Unlike in lattices, the dual of an ADL is not an ADL, in general. For this reason, in section 1.2, we introduce the concept of a dual pseudo-supplemented ADL (or, simply a dual PSADL) and derive some important properties of a dual PSADL. Also, we obtain many important characterizations of a dual PSADL. In section 1.3, we introduce the concept of a dual pseudo-complemented ADL (or, simply a dual PCADL) and prove that dual PCADL is
equationally definable. Also, we obtain different characterizations of a dual PCADL. If $\star$ is a dual pseudo-complementation on an ADL $A$, we prove that the set $A_\star = \{x_\star \land m \mid x \in A\}$ is a Boolean algebra under suitable operations. A Stone ADL [55] is an ADL with a pseudo-complementation $\ast$ on it such that $x^* \lor x^{**} = 0^*$ for all $x \in A$. In section 1.4, we introduce the concept of a dual Stone Almost Distributive Lattice (or, simply a dual Stone ADL) and obtain its properties. We also derive necessary and sufficient conditions for an ADL to become a dual Stone ADL. Also, we prove that a dual PCADL $A$ is a dual Stone ADL if and only if it is a dually normal ADL.

1.1 Pseudo-Supplemented ADLs

The concept of a pseudo-complementation in an ADL was introduced and studied by U.M. Swamy, G.C. Rao and G. Nanaji Rao in [54]. In this section, we introduce the concept of a pseudo-supplementation in an ADL and study its properties. The results of this section are accepted for publication in form of a paper entitled “Pseudo-Supplemented Almost Distributive Lattices” in Southeast Asian Bulletin of Mathematics (SEAMS). First, we give some important properties of the center of an ADL which are required to study the properties of a pseudo-supplemented ADL.

We begin with the following definition which is taken from [12].

**Definition 1.1.1.** [12] If $A$ is a bounded distributive lattice, then the set of all complemented elements of $A$ forms a Boolean algebra and it is called the Birkhoff center of $A$.

In [53], U.M. Swamy and S. Ramesh extended this concept to the class of ADL as
Definition 1.1.2. \([53]\) Let \(A\) be an ADL with a maximal element and
\[
B(A) = \{ a \in A \mid a \land b = 0 \text{ and } a \lor b \text{ is maximal for some } b \in A \}.
\]
Then \((B(A), \lor, \land)\) is a relatively complemented ADL and it is called the Birkhoff center of \(A\). We use the symbol \(B\) instead of \(B(A)\) when there is no ambiguity.

It can be observed that, for any \(b \in B\), \(b \land m\) is a complemented element in the distributive lattice \([0, m]\) whose complement will be denoted by \(b^m\). Throughout this section, \(A\) represents an ADL with a maximal element \(m\) and \(B\), its Birkhoff center.

Theorem 1.1.3. Let \(a \in A\). Then \(a \in B\) if and only if, \(a \land x \in B([0, x])\) for each \(x \in A\).

Proof. Let \(a \in B\) and \(x \in A\). Then there exists \(c \in A\) such that \(a \land c = 0\) and \(a \lor c\) is maximal. Now, \(c \land x \in [0, x]\) and \((a \land x) \land (c \land x) = a \land c \land x\) (by Theorem 0.2.3)
\[
\begin{align*}
&= 0 \land x \\
&= 0
\end{align*}
\]
and \((a \land x) \lor (c \land x) = (a \lor c) \land x = x\) (since \(a \lor c\) is maximal). Hence \(a \land x \in B([0, x])\).

Conversely, suppose \(a \in A\) and for each \(x \in A\), \(a \land x \in B([0, x])\). In particular, \(a \land m\) is a complemented element of \([0, m]\). Let \(b\) be the complement of \(a \land m\) in \([0, m]\). Then \(a \land m \land b = 0\) and \((a \land m) \lor b = m\).

So that \(a \land b = 0\) and \((a \lor b) \land m = (a \land m) \lor (b \land m)\)
\[
= (a \land m) \lor b \text{ (since } b \in [0, m]\text{)}
\]
\[
= m.
\]
Hence \(a \lor b\) is maximal (by Theorem 0.2.7). Thus \(a \in B\). \(\blacksquare\)
In the above theorem, if \( x \in B \), we have the following.

**Theorem 1.1.4.** If \( x \in B \), \( B([0, x]) \subseteq B \) and hence \( B([0, x]) = \{ a \wedge x \mid a \in B \} \).

**Proof.** Let \( b \in B([0, x]) \). Then there exists \( c \in [0, x] \) such that \( b \wedge c = 0 \) and \( b \lor c = x \).

Also, since \( x \in B \), there exists \( a \in A \) such that \( x \wedge a = 0 \) and \( x \lor a \) is a maximal element of \( A \). Now \( b \wedge (c \lor a) = (b \wedge c) \lor (b \wedge a) = 0 \lor (b \wedge a) \)

\[
= b \wedge x \wedge a \quad \text{(since } b \in B([0, x]))
\]

\[
= 0.
\]

Also \( (b \lor (c \lor a)) \wedge (x \lor a) = ((b \lor c) \lor a) \wedge (x \lor a) \)

\[
= (x \lor a) \wedge (x \lor a)
\]

\[
= x \lor a.
\]

Since \( x \lor a \) is a maximal element of \( A \), by Theorem 0.2.7, we get that \( b \lor (c \lor a) \) is also maximal element of \( A \). Thus \( b \in B \) and \( B([0, x]) \subseteq B \). On the other hand, \( a \land x \in B([0, x]) \) for all \( a \in B \).

If \( A \) is an ADL with a maximal element \( m \), then the set \( \text{PI}(A) \) of all principal ideals of \( A \) forms a bounded distributive lattice [51]. In the following theorem, we characterize the Birkhoff center \( \text{B(PI}(A)) \) of the lattice \( \text{PI}(A) \).

**Theorem 1.1.5.** \( B(\text{PI}(A)) = \{(b) \mid b \in B\} \).

**Proof.** Suppose \( x \in A \) such that \( (x) \in B(\text{PI}(A)) \). Then there exists \( (y) \in \text{PI}(A) \) such that \( (x) \cap (y) = (0) \) and \( (x) \lor (y) = A \). Therefore \( x \land y = 0 \). Now \( A = (x) \lor (y) = (x \lor y) \) and hence \( x \lor y \) is maximal. Thus \( x \in B \). Conversely, suppose \( b \in B \). Then there exists
an element \( c \in A \) such that \( b \land c = 0 \) and \( b \lor c \) is maximal. Now, \( (b] \cap (c] = (b \land c] = (0] \) and \( (b] \lor (c] = (b \lor c] = A \) (since \( b \lor c \) is maximal). Hence \( (b] \in B(PI(A)) \).

Now, we prove the following.

**Theorem 1.1.6.** Let \( x, y \in A \) such that \( (x] = (y] \). Then \( x \in B \) if and only if \( y \in B \).

**Proof.** Let \( x, y \in A \) such that \( (x] = (y] \). Then \( y \land x = x \) and \( x \land y = y \). Suppose \( x \in B \).

Then there exists \( d \in A \) such that \( x \land d = 0 \) and \( x \lor d \) is a maximal element of \( A \).

Since \( y \land d = x \land y \land d = y \land x \land d = y \land 0 = 0 \) and

\[
(y \lor d) \land (x \lor d) = ((y \lor d) \land x) \lor ((y \lor d) \land d)
= ((y \lor d) \land x) \lor d
= (((y \land x) \lor (d \land x)) \lor d
= (x \lor (d \land x)) \lor d
= x \lor d.
\]

Since \( x \lor d \) is maximal, by Theorem 0.2.7(iii), \( y \lor d \) is maximal. Therefore \( y \in B \) and the rest follows by symmetry.

The concept of a pseudo-supplemented lattice was introduced by G. Epstein and A. Horn [12] as follows.

**Definition 1.1.7.** [12] Let \( A \) be a distributive lattice with 0, 1 and Birkhoff center \( B \). If, for each \( x \in A \), there exists a greatest element \( b \in B \) such that \( b \leq x \), then \( A \) is called a pseudo-supplemented lattice. The element \( b \) is denoted by \( x! \) and it is called the pseudo

supplement of \( x \).
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Now we extend this definition to the class of an ADL as follows.

**Definition 1.1.8.** Let $A$ be an ADL with a maximal element $m$ and Birkhoff center $B$. $A$ is called a pseudo-supplemented Almost Distributive Lattice (or, simply a PSADL) if, for each $x \in A$, there exists $b \in B$ such that

- $P_1 : x \land b = b$
- $P_2 : \text{if } c \in B \text{ and } x \land c = c, \text{ then } b \land c = c$.

First we observe the following.

**Lemma 1.1.9.** Let $x \in A$ and suppose there exists $b_1, b_2 \in B$ satisfying $P_1$ and $P_2$ of Definition 1.1.8, then $b_1 \land m = b_2 \land m$.

Then, in the Definition 1.1.8, $b \land m$ is uniquely determined by $x$ and it is denoted by $x!$. We call $x!$, the pseudo-supplement of $x$. Also, we observe that $x! \in B([0, m])$. With this notation, we give the characterizing properties of $x!$ in the following.

**Lemma 1.1.10.** Let $A$ be a PSADL with a maximal element $m$ and Birkhoff center $B$. Then, for any $x \in A$, we have the following:

(i) $x \land x! = x!$.

(ii) If $c \in B$ and $x \land c = c$, then $x! \land c = c$.

**Proof.** Let $x \in A$. Then, by Definition 1.1.8, there exists $b \in B$ satisfying $P_1$ and $P_2$. Then $x! = b \land m$.

(i) $x \land x! = x \land b \land m = b \land m$ (by $P_1$ of the Definition 1.1.8) = $x!$. 
(ii) Let \( c \in B \) such that \( x \land c = c \). Then, by \( P_2 \) of the Definition 1.1.8, \( b \land c = c \).

Now \( x! \land c = b \land m \land c = b \land c = c \).

\[ \square \]

In the following lemma, we state some important fundamental properties of \( x! \).

**Lemma 1.1.11.** Let \( A \) be a PSADL with a maximal element \( m \) and Birkhoff center \( B \).

Then, for any \( x, y \in A \) and \( b \in B \), we have the following:

(i) \( x \lor x! = x \).

(ii) \( 0! = 0 \).

(iii) \( x \land m = m \iff x! = m \).

(iv) \( m_1! = m \) for any maximal element \( m_1 \) of \( A \).

(v) If \( b \in B \), then \( b! = b \land m \).

(vi) \( x! = x!! \).

(vii) If \( y \land x = x \), then \( y! \land x! = x! \).

**Proof.** Let \( x, y \in A \) and \( b \in B \).

(i) Since \( x \land x! = x! \), by Theorem 0.2.3(iv), we get \( x \lor x! = x \).

(ii) From (i) above, we get \( 0 = 0 \lor 0! = 0! \).

(iii) Suppose \( x \land m = m \). Since \( m \in B \), by Definition 1.1.8, we get \( x! \land m = m \) and hence \( x! = m \). Conversely, if \( x! = m \), then \( x \land m = x \land x! = x! = m \).
(iv) Follows from (iii) directly.

(v) Let \( b \in B \). Then, by Lemma 1.1.10, \( b \wedge b! = b! \). Also since \( b \wedge b = b \), we get from Lemma 1.1.10, \( b! \wedge b = b \). Now \( b \wedge m = b! \wedge b \wedge m = b! \wedge m = b! \). Also since \( b \wedge b = b \), we get from Lemma 1.1.10, \( b! \wedge b = b \).

(vi) since \( x! \in B \), by (v) above, we get \( x!! = x! \wedge m = x! \).

(vii) Suppose \( y \wedge x = x \). Then \( y \wedge x! = y \wedge x \wedge x! = x \wedge x! = x! \). Since \( x! \in B \), by Lemma 1.1.10, we get \( y! \wedge x! = x! \).

In the following theorem, we give some more important properties of a pseudo-supplement.

**Theorem 1.1.12.** Let \( A \) be a PSADL with a maximal element \( m \) and Birkhoff center \( B \).

Then, for any \( x, y \in A \), we have the following:

(i) \( (x \wedge y)! = x! \wedge y! \).

(ii) \( (x \wedge y)! = (y \wedge x)! \).

(iii) \( x! \vee y! \leq (x \vee y)! \).

(iv) \( (x \vee y)! = (y \vee x)! \).

(v) \( (x \wedge m)! = x! \).

Proof. Let \( x, y \in A \).

(i) We have \( x! \wedge y! \in B \) and \( x \wedge y \wedge (x! \wedge y!) = (x \wedge x!) \wedge (y \wedge y!) = x! \wedge y! \). Hence, by Lemma 1.1.11.(vii), we get \( (x \wedge y)! \wedge x! \wedge y! = x! \wedge y! \). Thus \( x! \wedge y! \leq (x \wedge y)! \).
On the other hand, since \( x \land (x \land y) = x \land y \), again by Lemma 1.1.11(vii), we get
\[ x! \land (x \land y)! = (x \land y)! \] and hence \( (x \land y)! \leq x! \). Similarly, we get \( (x \land y)! \leq y! \).

Therefore \( (x \land y)! \leq x! \land y! \) and hence \( (x \land y)! = x! \land y! \).

(ii) \( (x \land y)! = x! \land y! \) = \( x! \land y! \land m \)
\[ = y! \land x! \land m \text{ (by Theorem 0.2.3.(viii))} \]
\[ = y! \land x! \]
\[ = (y \land x)!. \]

(iii) We have \( x! \lor y! \in B \) and

\[ (x \lor y) \land (x! \lor y!) = ((x \lor y) \land x!) \lor ((x \lor y) \land y!) \]
\[ = ((x \land x!) \lor (y \land x!)) \lor ((x \land y!) \lor (y \land x!)) \]
\[ = (x! \lor (y \land x!)) \lor ((x \land y!) \lor y!) \]
\[ = x! \lor y!. \]

Then, by Lemma 1.1.11(vii), we get \( (x \lor y)! \land (x! \lor y!) = x! \lor y! \) and hence \( x! \lor y! \leq (x \lor y)! \).

(iv) Since \( (x \lor y) \land (y \lor x)! = (y \lor x) \land (y \lor x)! = (y \lor x)! \) and \( (y \lor x)! \in B \), by Lemma 1.1.10, we get \( (x \lor y) \land (y \lor x)! = (y \lor x)! \). Hence \( (y \lor x)! \leq (x \lor y)! \). Similarly, we get \( (x \lor y)! \leq (y \lor x)! \) and hence \( (x \lor y)! = (y \lor x)! \).

(v) \( (x \land m)! = (m \land x)! = x! \).
It can be observed that the uniqueness of \( x! \) depends on the maximal element \( m \) we have taken into consideration.

**Theorem 1.1.13.** Let \( A \) be an ADL with a maximal element \( m \) and Birkhoff center \( B \). Then \( A \) is a PSADL if and only if \([0, m]\) is a pseudo-supplemented lattice.

**Proof.** Suppose \( A \) is a PSADL. Let \( x \in [0, m] \). Then there exists an element \( b \in B \) satisfying \( P_1 \) and \( P_2 \) of the Definition 1.1.8. That is \( x \land b = b \) and if \( c \in B \) such that \( x \land c = c \), then \( b \land c = c \). Define \( x! = b \land m \). Then \( x! \in B([0, m]) \) (by Theorem 1.1.3). Now \( x \land x! = x \land b \land m = b \land m = x! \). Let \( c \in B([0, m]) \) such that \( x \land c = c \). Then, by Theorem 1.1.4, \( c \in B \) and hence \( b \land c = c \). Now \( x! \land c = b \land m \land c = b \land c = c \). Therefore \([0, m]\) is a pseudo-supplemented lattice. Conversely, suppose \([0, m]\) is a pseudo-supplemented lattice in which the pseudo-supplement of \( x \) is denoted by \( x! \). Let \( x \in A \). Then \( x \land m \in [0, m] \). Write \( b = (x \land m)! \). Then \( b \in B([0, m]) \) and hence by Theorem 1.1.4, we get \( b \in B \). Now \( x \land b = x \land m \land b = (x \land m) \land (x \land m)! = (x \land m)! = b \). Let \( c \in B \) such that \( x \land c = c \). Then, by Theorem 1.1.3, \( c \land m \in B([0, m]) \) and \( x \land c \land m = c \land m \). Then \( c \land m \leq x \land m \) and hence \( c \land m \leq (x \land m)! = b \). Thus

\[
\begin{align*}
b \land c &= b \land c \land c = b \land c \land m \land c \\
&= c \land m \land c \\
&= c.
\end{align*}
\]

Hence \( A \) is a PSADL. \( \blacksquare \)

**Note 1.1.14.** If \( b \in B([0, m]) \), then \( b = c \land m \) for some \( c \in B \). By Theorem 1.1.4, \( b \in B \) and hence \( b! = b \land m = c \land m = b \). Thus \( B([0, m]) = \{ x! \mid x \in A \} \).
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A PSADL becomes a pseudo-supplemented lattice once it becomes a lattice. Therefore we get a number of equivalent conditions for a PSADL to become a pseudo-supplemented lattice as a consequence of Theorem 0.2.9. These conditions are stated in the following theorem.

**Theorem 1.1.15.** Let \((A, \lor, \land, !, 0, m)\) be a PSADL with Birkhoff center \(B\). Then the following are equivalent:

(i) \(A\) is a pseudo-supplemented lattice.

(ii) \(A\) is a distributive lattice.

(iii) \((A, \leq)\) is directed above.

(iv) \(\lor\) is commutative.

(v) \(\land\) is commutative.

(vi) \(\lor\) is right distributive over \(\land\).

(vii) The relation \(\theta = \{(x, y) \in A \times A \mid y \land x = x\}\) on \(A\) is antisymmetric.

Finally, we conclude this section with the following.

**Theorem 1.1.16.** Let \(A\) be an ADL with a maximal element \(m\) and Birkhoff center \(B\). Then \(A\) is a PSADL if and only if \(PI(A)\) is a pseudo-supplemented lattice.

**Proof.** Suppose \(A\) is a PSADL. For any \(x \in A\), define \((x)! = (x)!\). Since \(x! \in B\), by Theorem 1.1.5, we get \((x)! \in B(PI(A))\) and \((x] \cap (x)! = (x] \cap (x)! = (x)! = (x)\) and hence \((x)! \subseteq (x)\). Let \((b) \in B(PI(A))\) such that \((b) \subseteq (x)\). Then, by Theorem 1.1.5, \(b \in B\) and
$x \land b = b$. So that $x \land b = b$ and $(x!] \cap (b] = (b]$. Hence $PI(A)$ is a pseudo-supplemented lattice. Conversely, suppose $PI(A)$ is a pseudo-supplemented lattice. For any $x \in A, (x!] \in B(PI(A))$ and hence $(x]! = (b]$ for some $b \in B$. Then $(b] \subseteq (x]$ so that $b \in (x]$ and hence $x \land b = b$. Let $c \in B$ such that $x \land c = c$. Now $(c] \in B(PI(A))$, $(x] \cap (c] = (x \land c] = (c]$. That is $(c] \subseteq (x]$ and hence $(c] \subseteq (x!] = (b]$. Thus $b \land c = c$ and hence $A$ is a PSADL. ■

1.2 Dual Pseudo-Supplemented ADLs

The concept of a dual pseudo-supplementation in a distributive lattice is given by G. Epstein and A. Horn in [12]. Since the dual of a distributive lattice ia again a distributive lattice, dual pseudo-complemented lattice or dual pseudo-supplemented distributive lattice need not be investigate. But the dual of an ADL is not an ADL, in general. For this reason, in this section, we define a dual pseudo-supplemented ADL and its properties.

**Definition 1.2.1.** Let $A$ be a distributive lattice with 0, 1 and Birkhoff center $B$. If, for any $x \in A$, there exists a smallest element $b \in B$ such that $b \geq x$, then $A$ is called a dual pseudo-supplemented lattice. The element $b$ is denoted by $x_!$ and $x_!$ is called the dual pseudo-supplement of $x$.

Now we extend this concept to the class of an ADLs as follows.

**Definition 1.2.2.** Let $A$ be an ADL with a maximal element $m$ and Birkhoff center $B$. $A$ is said to be a dual pseudo-supplemented Almost Distributive Lattice (or, simply a dual PSADL) if, for each $x \in A$, there exists $b \in B$ satisfying:

(i): $b \land x = x$

(ii): if $c \in B$ such that $c \land x = x$, then $c \land b = b$. 


In this case, $b \land m$ is uniquely determined by $x$ and it is denoted by $x_i$. We can observe that $x_i \in B([0, m])$. We call $x_i$, the dual pseudo-supplement of $x$.

**Note 1.2.3.** From the Definition 1.2.2, for any $x \in A$, we get the following:

$D_1 : x_i \land x = x$.

$D_2 :$ If $c \in B$ such that $c \land x = x$, then $c \land x_i = x_i$.

$D_3 : x_i \land m = x_i$.

Throughout this section, $A$ stands for a dual PSADL $(A, \lor, \land, i, 0, m)$ with a maximal element $m$ and Birkhoff center $B$, unless otherwise stated.

In the following theorem, we give some fundamental properties of a dual PSADL.

**Theorem 1.2.4.** For any $x, y \in A$ and $b \in B$, we have the following:

(i) $x_i \lor x = x_i$.

(ii) $m_i = m$.

(iii) $0_i = 0$.

(iv) If $b \in B$, then $b_i = b \land m$.

(v) $x_i = x_{jj}$.

(vi) If $y \land x = x$, then $x_i \leq y_i$.

**Proof.** Let $x, y \in A$ and $b \in B$.

(i) Since $x_i \land x = x$, by Theorem 0.2.3, we get $x_i \lor x = x_i$. 
(ii) Since \( x \land x = x \), by replacing \( x \) by \( m \), we get \( m \land m = m \) and hence \( m = m \).

(iii) Since \( 0 \in B \) and \( 0 \land 0 = 0 \), by \( D_2 \) of Note 1.2.3, we get \( 0 \land 0 = 0 \) and hence \( 0 = 0 \).

(iv) By \( D_1 \) of Note 1.2.3, \( b \land b = b \) and hence \( b \land m \leq b \). On the other hand, since \( b \in B \) and \( (b \land m) \land b = b \), by \( D_2 \) of Note 1.2.3, we get \( (b \land m) \land b i = b \) and hence \( b \leq b \land m \).

Therefore \( b \land m \).

(v) Since \( x \in B \), by (iv) above, we get \( x \land m = x \land m = x \).

(vi) Suppose \( y \land x = x \). Then \( x \land m \leq y \land m \leq y \) and hence \( x \land m = x \land m \land y = x \land y \).

Now \( x = x \land m \land x = x \land y \land x = y \land x \). Then, by \( D_2 \) of Note 1.2.3, we get \( y \land x = x \land m \land x \land y \) and hence \( x \leq y \).

In the following theorem, we give the compatibility of a dual pseudo-supplement with the binary operations \( \lor \) and \( \land \).

**Theorem 1.2.5.** For any \( x, y \in A \), we have the following:

(i) \( (x \lor y) \lor = x \lor y \).

(ii) \( (x \lor y) \lor = (y \lor x) \lor \).

(iii) \( x \land y \leq (x \lor y) \lor \).

(iv) \( (x \land y) \land = (y \land x) \land \).

(v) \( (x \land m) \land = x \land m \).
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Proof. Let \( x, y \in A \).

(i) Since \( x_i \lor y_i \in B \) and

\[
(x_i \lor y_i) \land (x \lor y) = ((x_i \lor y_i) \land x) \lor ((x_i \lor y_i) \land y)
\]

\[
= ((x \land x) \lor (y \land x)) \lor ((x \land y) \lor (y \land y))
\]

\[
= (x \lor (y \land x)) \lor ((x \land y) \lor y) \quad (\text{by Theorem 1.2.4(i)})
\]

\[
= x \lor y,
\]

we get \((x_i \lor y_i) \land (x \lor y)_i = (x \lor y)_i\). On the other hand, since \((x \lor y) \land x = x\) and \((x \lor y) \land y = y\), by Theorem 1.2.4(vi), we get \(x_i \leq (x \lor y)_i\) and \(y_i \leq (x \lor y)_i\) and hence \(x_i \lor y_i \leq (x \lor y)_i\). Therefore \((x_i \lor y_i) = (x_i \lor y_i) \land (x \lor y)_i = (x \lor y)_i\).

(ii) \((x \lor y)_i = x_i \lor y_i = y_i \lor x_i \quad \text{since} \ x_i, y_i \leq m\)

\[
= (y \lor x)_i.
\]

(iii) Since \(x \land (x \land y) = x \land y\), by Theorem 1.2.4(vi), we get \((x \land y)_i \leq x_i\). Similarly \(y \land (x \land y) = x \land y\), implies \((x \land y)_i \leq y_i\). Hence \((x \land y)_i \leq x_i \land y_i\).

(iv) Since \((x \land y) \land y \land x = y \land x\), by Theorem 1.2.4(vi), we get \((y \land x)_i \leq (x \land y)_i\). On the other hand, since \((y \land x) \land x \land y = x \land y\), again by Theorem 1.2.4(vi), we get \((x \land y)_i \leq (y \land x)_i\) and hence \((x \land y)_i = (y \land x)_i\).

(v) \((x \land m)_i = (m \land x)_i \quad \text{by (iv) above} \) = \(x_i\).

\(\blacksquare\)
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Theorem 1.2.6. Let $A$ be an ADL with a maximal element $m$ and Birkhoff center $B$. Then $A$ is a dual PSADL if and only if $[0, m]$ is a dual pseudo-supplemented lattice.

Proof. Suppose $A$ is a dual PSADL. Let $x \in [0, m]$. Then $x \in B([0, m])$ and $x \geq x$ (since $x \wedge x = x$). Let $c \in B([0, m])$ such that $c \geq x$. Write $c = b \wedge m$ for some $b \in B$. Since $b \wedge x = b \wedge m \wedge x = c \wedge x = x$, we get $b \wedge x = x$. Now $c \wedge x = b \wedge m \wedge x = b \wedge x = x$ and hence $c \geq x$. Therefore $[0, m]$ is a dual pseudo-supplemented lattice. Conversely, Suppose $[0, m]$ is a dual pseudo-supplemented lattice in which the dual pseudo-supplement of an element $y$ is defined by $y_!$. For $x \in A$, define $x^! = (x \wedge m)_!$. We have $x^! \geq x \wedge m$, so that $x \wedge m = x^! \wedge x \wedge m$ and hence $x^! \wedge x = x$. Let $b \in B$ such that $b \wedge x = x$. Then $b \wedge m \in B([0, m])$ and $b \wedge m \wedge x \wedge m = x \wedge m$. So that $b \wedge m \geq x \wedge m$. Hence $b \wedge m \geq x^!$. Now, $b \wedge x^! = b \wedge m \wedge x^! = x^!$. Hence $A$ is a dual PSADL.

Finally, we conclude this section with the following theorem which proof is analogues to the Theorem 1.1.16.

Theorem 1.2.7. Let $A$ be an ADL with a maximal element $m$ and Birkhoff center $B$. Then $A$ is a dual PSADL if and only if $PI(A)$ is a dual pseudo-supplemented lattice.

1.3 Dual Pseudo-Complemented ADLs

The concept of a pseudo-complement was introduced and extensively studied in semi-lattice and particularly in distributive lattices by O. Frink [10] and G. Birkhoff [4]. Later, the concept of a Stone lattices has been studied by by several authors like R. Bables, O. Frink, G. Gratzer, M.E. Adams, R. Beazer, C. Varlet, V. Koubek, J. Sichler, T. Katrik,
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T. Hecht, H.P. Sankappanavar and etc. The concept of a Pseudo-complementation in an ADL was introduced and their important properties were given by U.M. Swamy, G.C. Rao and G. Nanaji Rao [54]. Also, the concept of a Stone ADLs was introduced by the same authors in [55].

The following definition is taken from [54].

**Definition 1.3.1.** [54] Let \((A, \lor, \land, 0)\) be an ADL. Then a unary operation \(x \mapsto x^*\) on \(A\) is called a pseudo-complementation on \(A\) if, for any \(x, y \in A\), it satisfies the following conditions:

(i) \(x \land y = 0 \Rightarrow x^* \land y = y.\)

(ii) \(x \land x^* = 0.\)

(iii) \((x \lor y)^* = x^* \land y^*.\)

Unlike in lattices, the dual of an ADL is not an ADL, in general. For this reason, in this section, we introduce the concept of a dual pseudo-complementation in the class of ADLs. We obtain some important properties of a dual pseudo-complemented ADL(or, simply a dual PCADL). Also, we derive different characterizations of a dual PCADL and we prove that dual pseudo-complementation is equationally definable.

**Definition 1.3.2.** A distributive lattice \(A\) with 0, 1 is said to be dually pseudo-complemented if, there is a unary operation \(*\) on \(A\) such that \(x \lor y = 1\) if and only if \(y \geq x_*\) for all \(x, y \in A\).

The following theorem derive directly from the above definition.
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**Theorem 1.3.3.** If \((A, \lor, \land, \ast, 0, 1)\) is a dual pseudo-complemented lattice, then for any \(x, y \in A\), \((x \land y) \ast = x \lor y \ast\).

Now, we extend this concept to the class of an ADLs as follows.

**Definition 1.3.4.** Let \((A, \lor, \land)\) be an ADL with a maximal element \(m\). Then a unary operation \(\ast\) on an ADL \(A\) is called a dual pseudo-complementation on \(A\) if, for \(x, y \in A\), it satisfies the following conditions:

\[
d_1 : x \lor x^\ast \text{ is a maximal element of } A.
\]

\[
d_2 : \text{If } y \in A \text{ and } x \lor y = m, \text{ then } (x^\ast \lor y) \land m = y \land m.
\]

\[
d_3 : (x \land y)^\ast = x^\ast \lor y^\ast.
\]

An ADL \(A\) with a dual pseudo-complementation is called a dually pseudo-complemented Almost Distributive Lattice (or, simply a dual PCADL).

Throughout this section, \(A\) stands for a dual PCADL \((A, \lor, \land, \ast, 0, m)\). In the following theorem, we prove some important fundamental properties of \(\ast\) which will be frequently used.

**Theorem 1.3.5.** For any \(x, y \in A\), we have the following:

(i) If \(x\) is a maximal, then \(x^\ast = 0\).

(ii) \((x \land m)^\ast = x^\ast\).

(iii) \((x \land y)^\ast = (y \land x)^\ast \text{ and } (x \lor y)^\ast = (y \lor x)^\ast\).

(iv) \(0^\ast \land m = m\).
(v) \( x^{**} \land m \leq x \land m \) and \( x \land x^{**} = x^{**} \).

**Proof.** Let \( x, y \in A \).

(i) Since \( x \) is a maximal, we have \( x \lor 0 \) is also a maximal. Hence, by Definition 1.3.4,

\[
(x_* \lor 0) \land m = 0 \land m.
\]

Thus \( x_* \land m = 0 \) and hence \( x_* = 0 \).

(ii) \( (x \land m)_* = x_* \lor m_* = x_* \lor 0 \) (by (i) above)

\[
= x_*.
\]

(iii) \( (x \land y)_* = (x \land y \land m)_* \) (by (ii) above)

\[
= (y \land x \land m)_*
\]

\[
= (y \land x)_*.
\]

Also \( (x \lor y)_* = ((x \lor y) \land m)_* \) (by (ii) above)

\[
= ((y \lor x) \land m)_*
\]

\[
= (y \lor x)_*.
\]

(iv) Since \( 0 \lor 0_* \) is maximal, we get \( (0 \lor 0_*) \land m = m \) and hence \( 0_* \land m = m \). Therefore \( 0_* \) is maximal.

(v) By \( d_1 \) of the Definition 1.3.4, we have \( x_* \lor x \) is a maximal and hence by \( d_2 \) of the Definition 1.3.4, \( (x^{**} \lor x) \land m = x \land m \). Thus \( x^{**} \land m \leq x \land m \) and hence \( x \land x^{**} = x^{**} \).

\[\blacksquare\]

**Theorem 1.3.6.** For any \( x, y \in A \), we have the following:

(i) If \( y \land x = x \), then \( x_* \geq y_* \) and \( x^{**} \leq y^{**} \).
(ii) \( x_\ast = x_{\ast\ast\ast} \).

(iii) \( x_\ast = 0 \iff x_{\ast\ast} \land m = m \).

(iv) \( x \land m = m \iff x_{\ast\ast} \land m = m \).

\textbf{Proof.} Let \( x, y \in A \).

(i) Suppose \( y \land x = x \). Then \( x_\ast = (y \land x)_\ast = y_\ast \lor x_\ast \) and hence \( y_\ast \leq x_\ast \).

Now \( y_{\ast\ast} = (y_\ast \land x_\ast)_\ast = (x_\ast \land y_\ast)_\ast \) (by Theorem 1.3.5.(iii))

\[ = x_{\ast\ast} \lor y_{\ast\ast} \] and hence \( x_{\ast\ast} \leq y_{\ast\ast} \).

(ii) Since \( x_{\ast\ast} = x \land x_{\ast\ast} \), we get \( x_{\ast\ast\ast} = (x \land x_{\ast\ast})_\ast = x_\ast \lor x_{\ast\ast\ast} \)

\[ = x_\ast \text{ (since } x_\ast \land x_{\ast\ast\ast} = x_{\ast\ast\ast} \text{).} \]

(iii) Suppose \( x_\ast = 0 \). Then \( x_{\ast\ast} = 0_\ast \) and hence \( x_{\ast\ast} \land m = 0_\ast \land m = m \). Conversely, suppose \( x_{\ast\ast} \land m = m \). Then \( 0 = m_\ast = (x_{\ast\ast} \land m)_\ast \)

\[ = x_{\ast\ast\ast} \lor m_\ast \]

\[ = x_{\ast\ast\ast} \lor 0 \]

\[ = x_{\ast\ast\ast} \]

\[ = x_\ast. \]

(iv) Suppose \( x \land m = m \). Then \( (x \land m)_\ast = m_\ast \implies (x_\ast \lor m_\ast) = m_\ast \)

\[ \implies x_\ast = 0 \]

\[ \implies x_{\ast\ast} \land m = 0_\ast \land m \]

\[ \implies x_{\ast\ast} \land m = m \text{ (by Theorem 1.3.5 (iv)).} \]
Conversely, suppose $x_{**} \land m = m$. Then $m = x_{**} \land m \leq x \land m$ (by Theorem 1.3.5.(v)) and hence $x \land m = m$.

\[ \square \]

**Corollary 1.3.7.** For any $x, y \in A$, we have the following:

(i) $x_{**} = (x \land y)_{**} \lor x_{**}$.

(ii) $x_{**} = x_{**} \land (x \lor y)_{**}$.

In a distributive lattice, the dual pseudo-complementation is unique (if it exists). But, in an ADL, there can be several dual pseudo-complementations. In the following result, we prove the relation between the dual pseudo complementations in $A$.

**Lemma 1.3.8.** Let $A$ be an ADL with a maximal element $m$ and $\ast, \perp$ be dual pseudo-complementations on $A$. Then, for any $x, y \in A$, we have the following:

(i) $x_\perp \land x_\ast = x_\ast$ and $x_\perp \lor x_\ast = x_\perp$.

(ii) $x_* = x_{\perp \perp}$.

(iii) $x_\ast = y_\ast$ if and only if $x_\perp = y_\perp$.

(iv) $x_\ast$ is a maximal $\iff x_\perp$ is a maximal.

(v) $x_\perp = x_\ast \land 0_\perp$.

(vi) $x_\ast \land x_{**} = 0 \iff x_\perp \land x_{\perp \perp} = 0$.

**Proof.** Let $x, y \in A$. 

\[ \square \]
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(i) Since \((x \lor x) \land m = m\), by the Definition 1.3.4, we get \((x \lor x) \land m = x \land m\).

Now \(x \land x = x \land m \land x = (x \lor x) \land m \land x = (x \lor x) \land x = x\) and hence
\(x \lor x = x\).

(ii) Now \(x \land x = (x \land x) \land = (x \land x) \land = x\).

(iii) Suppose \(x = y\). Now, \(x = x \land x\) (by Theorem 1.3.6(ii))
\begin{align*}
&= x \land x (by \ (ii) \ above) \\
&= y \land y \\
&= y \land y \\
&= y.
\end{align*}

By symmetry, we get the converse.

(iv) Suppose \(x\) is a maximal. By (i) above, we have \(x = x \lor x\) and hence \(x\) is a maximal. By symmetry, we get the converse.

(v) \(x \land 0 = x \land x \land 0\) (by (i) above)
\begin{align*}
&= x \land x \land 0 \\
&= x \land 0 (by \ (i) \ above) \\
&= x (since \ 0 \leq x, \ we \ get \ x \leq 0).
\end{align*}

(vi) Suppose \(x \land x = 0\). We have \(x = x \land x\) (by (i) above). Also, by (ii) above,
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\[ x_{\perp\perp} = x_{\perp} = x_{\ast\ast} \wedge x_{\perp}. \] Now

\[ x_{\perp} \wedge x_{\perp\perp} = x_{\ast} \wedge x_{\perp} \wedge x_{\ast\ast} \wedge x_{\perp}\]

\[ = x_{\ast} \wedge x_{\ast\ast} \wedge x_{\perp} \wedge x_{\ast\perp}\]

\[ = 0. \]

By symmetry, we get the converse.

In the following result, we establish a one-to-one correspondence between the set of all dual pseudo complementations on \( A \) and the set of all maximal elements of \( A \).

**Theorem 1.3.9.** Let \( N \) be the set of all maximal elements in an ADL \( A \) and \( DPC(A) \) be the set of all dual pseudo-complementations on an ADL \( A \). For any \( n \in N \), define \( \ast_n : A \rightarrow A \) by \( x_{\ast_n} = x_{\ast} \wedge n \) for all \( x \in A \). Then \( (A, \vee, \wedge, \ast_n, 0, n) \) is a dual PCADL and the map \( \phi : N \rightarrow DPC(A) \) defined by \( \phi(x) = x_{\ast_n} \) is a bijection.

**Proof.** Let \( A \) be an ADL with a maximal element \( m \) and for any \( n \in N \), define \( \ast_n : A \rightarrow A \) by \( x_{\ast_n} = x_{\ast} \wedge n \) for all \( x \in A \). First we prove that \( \phi \) is well-defined. Let \( x, y \in A \) such that \( x \lor y \) is a maximal. Then \( (x_{\ast_n} \lor y) \wedge n = ((x_{\ast} \wedge n) \lor y) \wedge m \wedge n = y \wedge m \wedge n = y \wedge n \) and \( (x \lor x_{\ast_n}) \wedge n = (x \lor y) \lor (x_{\ast} \wedge n) = (x \lor x_{\ast}) \wedge n = m \wedge n = n. \) Therefore \( x \lor x_{\ast_n} \) is a maximal. Now \( (x \wedge y)_{\ast_n} = (x \wedge y)_{\ast} \wedge n = (x_{\ast} \lor y_{\ast}) \wedge n = (x_{\ast} \wedge n) \lor (y_{\ast} \wedge n) = x_{\ast_n} \wedge y_{\ast_n}. \) Therefore \( \ast_n \) is a dual pseudo-complementation on \( A \). Let \( n_1 \) and \( n_2 \) be two maximal elements such that \( \ast_{n_1} = \ast_{n_2}. \) Then \( n_1 = 0_{\ast} \wedge n_1 = 0_{n_1} = 0_{n_2} = 0 \wedge n_2 = n_2. \) Finally, we prove \( \phi \) is
onto. Let \( \bot \in \text{DP}(A) \). Then \( n_0 = 0 \bot \) and for any \( x \in A \), \( x \star n_0 = x \star 0 \bot = x \bot \) (by lemma 1.3.8(v)). Thus \( \phi \) is a bijection.

Let \((A, \lor, \land)\) be a finite distributive lattice. If we define, for any \( x \in A \),
\[ x \star = \bigwedge\{y \in A \mid x \lor y = 1\} \]
then \( \star \) is a dual pseudo-complementation on \( A \). Using this, we prove the following.

**Theorem 1.3.10.** If \( A \) is a finite ADL with a maximal element \( m \), then \( A \) is a dual PCADL.

**Proof.** Let \( A \) be a finite ADL with a maximal element \( m \). Then \(([0, m], \lor, \land)\) is a finite distributive lattice and hence a dual pseudo-complemented lattice. Let \( \star \) denote the dual pseudo-complementation on \([0, m]\). For any \( x \in A \), define \( x \bot = (x \land m) \star \). Now, we verify that \( \bot \) is a dual pseudo-complementation on \( A \). Let \( x \in A \). Now
\[
(x \lor x \bot) \land m = (x \land m) \lor (x \bot \land m)
\]
\[
= (x \land m) \lor ((x \land m) \star \land m)
\]
\[
= (x \land m) \lor (x \land m) \star
\]
\[
= m.
\]
Suppose \( y \in A \) such that \( x \lor y = m \). Then
\[
(x \bot \lor y) \land m = ((x \land m) \star \land m) \lor (y \land m)
\]
\[
= (x \land m) \star \lor (y \land m)
\]
\[
= y \land m
\]
(since \( (x \land m) \lor (y \land m) = (x \lor y) \land m = m \)).

Finally, for any \( x, y \in A \),
\[
(x \land y) \bot = ((x \land y) \land m) \star
\]
\[(x \wedge m)_* \lor (y \wedge m)_*\]
\[= x_\perp \lor y_\perp.
\]

Hence \(A\) is a dual PCADL. 

**Theorem 1.3.11.** Let \(A\) be a dual PCADL with a maximal element \(m\). Then, for any \(x, y \in A\), the following are equivalent:

(i) \((x \lor y) \wedge m = m\).

(ii) \((x_* \lor y) \wedge m = m\).

(iii) \((x_* \lor y_{**}) \wedge m = m\).

(iv) \((x \lor y_{**}) \wedge m = m\).

**Proof.** Let \(x, y \in A\).

(i)\(\implies\)(ii):

Suppose \((x \lor y) \wedge m = m\). Then \((x \wedge m) \lor (y \wedge m) = m\) and hence, by definition 1.3.4, \((x \wedge m)_* \lor (y \wedge m)) \wedge m = y \wedge m\). Thus \((x_* \lor y) \wedge m = y \wedge m\). Now

\[(x_* \lor y) \wedge m = (x_* \lor m) \lor (y \wedge m)\]
\[= (x_* \lor m) \lor ((x_* \lor y) \wedge m)\]
\[= (x_* \lor x_* \lor y) \wedge m\]
\[= m\] (since \(x_* \lor x_*\) is a maximal, we get \(x_* \lor x_* \lor y\) is also a maximal).

(ii)\(\implies\)(iii):

Suppose \((x_* \lor y) \wedge m = m\). Then \((y \lor x_{**}) \wedge m = m\) and hence \((y_{**} \lor x_{**}) \wedge m = m\) as in
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the above ((i)⇒(ii)). Therefore \((x_{**} \lor y_{**}) \land m = m\).

(iii)⇒(iv):

Suppose \((x_{**} \lor y_{**}) \land m = m\). Then \((x \lor x_{**} \lor y_{**}) \land m = (x \land m) \lor m\) and hence \((x \lor y_{**}) \land m = m\). Similarly, we get (iv)⇒(i). \(\blacksquare\)

Using the above theorem, we prove the following.

**Theorem 1.3.12.** Let \(A\) be a dual PCADL with a maximal element \(m\). Then, for any \(x, y \in A\), \((x \lor y)_{**} = x_{**} \lor y_{**}\).

**Proof.** Let \(x, y \in A\). Since \(x \leq x \lor y\) and \(y \leq y \lor x\), we get \(x_{**} \leq (x \lor y)_{**}\) and \(y_{**} \leq (y \lor x)_{**} = (x \lor y)_{**}\). Hence \((x_{**} \lor y_{**}) \leq (x \lor y)_{**}\). On the other hand, since, for any \(x, y \in A\), \(((x \lor y) \lor (x \lor y)) \land m = m\), by Theorem 1.3.11, we get \((x_{**} \lor y_{**}) \lor (x \lor y)_{**} \land m = m\) and hence \(((x \lor y)_{**} \lor x_{**} \lor y_{**}) \land m = (x_{**} \lor y_{**}) \land m\). Thus \((x \lor y)_{**} \leq x_{**} \lor y_{**}\) and hence \((x \lor y)_{**} = x_{**} \lor y_{**}\). \(\blacksquare\)

In the following theorem, we prove that a dual PCADL is equationally definable.

**Theorem 1.3.13.** Let \(A\) be an ADL with a maximal element \(m\). A unary operation \(*\) on \(A\) is a dual pseudo-complementation on \(A\) if and only if, for any \(x, y \in A\), the following conditions hold:

(i) \((x \lor x_{*}) \land m = m\).

(ii) \((x_{**} \lor x) \land m = x \land m\).

(iii) \((x \lor y)_{**} = x_{**} \lor y_{**}\).
(iv) \( (m_\ast \lor x) \land m = x \land m \).

(v) \( (x \land y)_\ast = x_\ast \lor y_\ast \).

Proof. Suppose \( A \) satisfies conditions (i) to (v). Then, by Definition 1.3.4, it is enough to show that \( A \) satisfies \( d_1 \) of Definition 1.3.4. For this, first we prove that the following.

(a). \( (x \land m)_\ast \land m = (x_\ast \lor m_\ast) \land m = (m_\ast \lor x_\ast) \land m = x_\ast \land m \) (by condition (iv)).

(b). \( x^{**} \land m = (x^{***} \lor m_\ast) \land m \) (by condition (iv))

\[
= (x^{**} \land m)_\ast \land m \quad \text{(by condition (v))}
\]

\[
= (x^{**} \land x \land m)_\ast \land m \quad \text{(by condition (i))}
\]

\[
= (x^{***} \lor x_\ast \lor m_\ast) \land m
\]

\[
= x_\ast \land m \quad \text{(by (a) and condition (iv)).}
\]

Now, let \( y \in A \) such that \( x \lor y = m \).

Then \( y \land m = (m_\ast \lor y) \land m \) (by condition (iv) )

\[
= (((x_\ast \lor x^{**}) \land m)_\ast \lor y) \land m \quad \text{(by condition (i))}
\]

\[
= (((x_\ast \lor x^{**}) \land m)_\ast \land m) \lor (y \land m)
\]

\[
= (x_\ast \lor x^{**}) \land m \lor (y^{**} \land y) \land m \quad \text{(by (a), conditions (ii) and (iv))}
\]

\[
= ((x \land x_\ast) \lor y^{**} \lor y) \land m \quad \text{(by condition (v))}
\]

\[
= (((x \land x_\ast) \lor y)^{**} \lor y) \land m \quad \text{(by condition (iii))}
\]

\[
= (((x \lor y) \land (x_\ast \lor y)^{**} \lor y) \land m
\]

\[
= ((m \land (x_\ast \lor y)^{**} \lor y) \land m \quad \text{(since } x \lor y = m)
\]

\[
= ((x_\ast \lor y)^{**} \lor y) \land m
\]

\[
= (x^{***} \lor y^{**} \lor y) \land m \quad \text{(by condition (iii))}
\]
= (x_\ast \lor y) \land m \text{ (by (b) and condition (ii)).}

Hence $A$ is a dual PCADL. Conversely, if $A$ is a dual PCADL, then conditions (i) to (v) are already proved.

Now, we give a different set of equations that equivalent to a dual pseudo-complementation on an ADL.

**Theorem 1.3.14.** Let $A$ be an ADL with a maximal element $m$. A unary operation $\ast$ on $A$ is a dual pseudo-complementation if and only if, for any $x, y \in A$, it satisfies the following conditions:

(i) $(x_\ast \lor y) \land m = ((x \lor y)_\ast \lor y) \land m$.

(ii) $(m_\ast \lor x) \land m = x \land m$.

(iii) $m_{\ast\ast} \land m = m$.

(iv) $(x \land y)_\ast = x_\ast \lor y_\ast$.

**Proof.** Suppose $\ast$ is a dual pseudo-complementation on $A$. Then, we have proved that the conditions (ii), (iii) and (iv) hold. Since $(x \lor y \lor (x \lor y)_\ast) \land m = m$, we get $(x_\ast \lor y \lor (x \lor y)_\ast) \land m = (y \lor (x \lor y)_\ast) \land m$. Now, since $x = x \land (x \lor y)$, we get $(x_\ast \lor (x \lor y)_\ast) \land m = x_\ast \land m$ and hence $(y \lor (x \lor y)_\ast) \land m = (x_\ast \lor y) \land m$. Conversely, suppose that $\ast$ satisfies conditions...
(i) to (iv). Then $A$ satisfies $d_3$ of the Definition 1.3.4. Let $y \in A$ such that $x \lor y = m$.

Then $(x \star y) \land m = ((x \lor y) \star y) \land m$ (by condition (i))

$$= (m \lor y) \land m$$

$$= y \land m$$ (by condition (ii)).

Thus we get $d_1$ of definition 1.3.4.

Finally, $(x \star x) \land m = ((x \land m) \lor x) \land m$

$$= (((m \lor x) \land m) \lor x) \land m$$

(by condition (ii) and from (a) of Theorem 1.3.13)

$$= ((m \lor x) \lor x) \land m$$ (since $(x \land m) \land m = x \land m$)

$$= (m \lor x) \land m$$ (by condition(i))

$$= (m \lor x) \land (x \land m)$$

$$= m \lor (x \land m)$$ (by condition (iii))

$$= m.$$

Hence $A$ is a dual PCADL. \[\square\]

The concept of an annihilator was introduced by U.M. Swamy, G.C. Rao and G. Nanaji Rao [54]. The following definition is taken from [54].

**Definition 1.3.15.** [54] Let $A$ be an ADL and $S$ be any non-empty subset of $A$. Define $(S)^* = \{a \in A \mid s \land a = 0 \text{ for all } s \in S\}$. Then $(S)^*$ is called the annihilator of $S$ in $A$. 
Note 1.3.16. 1. It can be verified that \((S)^*\) is an ideal of \(A\).

2. We write \((s)^*\) for \((S)^*\) if \(S = \{s\}\).

G.C. Rao and S. Ravi Kumar was introduced the concept of a dual annihilator of a subset of \(A\) in [39] as follows.

**Definition 1.3.17.** [39] Let \(A\) be an ADL with a maximal element \(m\) and \(S\) is any non-empty subset of \(A\). Define \((S)^+ = \{a \in A \mid s \lor a\) is a maximal element in \(A\) for all \(s \in S\}\). Then \((S)^+\) is called the dual annihilator of \(S\) in \(A\).

**Note 1.3.18.** It can be verified that \((S)^+\) is a filter of \(A\) and we write \((s)^+\) for \((S)^+\) if \(S = \{s\}\).

Now we prove the following.

**Theorem 1.3.19.** Let \(A\) be a dual PCADL. Then, for any \(x \in A\), \((x)^+ = [x_*]\).

*Proof.* Since, for any \(x \in A\), \(x \lor x_*\) is a maximal, we get \(x_* \in (x)^+\). Also, since \((x)^+\) is a filter in \(A\), we get \([x_*] \subseteq (x)^+\). On the other hand, if \(y \in (x)^+\), then \(x \lor y\) is a maximal and hence \((x_* \lor y) \land m = y \land m\). Also, since \(x_* \lor y \in [x_*]\), we get \(y \in [x_*]\) and hence \((x)^+ \subseteq [x_*]\). \(\blacksquare\)

**Corollary 1.3.20.** Let \(A\) be an ADL with a maximal element \(m\) and \(\ast, \bot\) be dual pseudo-complementations on \(A\). Then, for any \(x \in A\), \([x_*] = [x_\bot]\).

Let \(A\) be an ADL. It may be recall that \((A, \leq)\) is a partial ordered set if we define \(x \leq y\) if and only if \(x \land y = x\) or equivalently \(x \lor y = y\), for any \(x, y \in A\). Now we prove the following.
Theorem 1.3.21. Let $A$ be a dual PCADL and $A_* = \{a_* \land m \mid a \in A\}$. Then $(A_*, \leq)$ is a Boolean algebra.

Proof. For any $x \in A$, we have $0 \leq x$ and hence $x_* \leq 0_*$. So that, for any $x, y \in A$, $(x_* \land m) \lor (y_* \land m) = (x_* \lor y_*) \land m = (x \land y)_* \land m \in A_*$ and hence $(x \land y)_* \land m$ is the l.u.b of $x_* \land m$, $y_* \land m$ in $(A_*, \leq)$. Also, since $x \land m \leq (x \lor y)_* \land m$ and $y \land m \leq (x \lor y)_* \land m$, we get $x_* \land m \geq (x \lor y)_* \land m$ and $y_* \land m \geq (x \lor y)_* \land m$. Thus $(x \lor y)_* \land m$ is the lower bound of $x_* \land m$, $y_* \land m$ in $(A_*, \leq)$. Suppose $t_* \land m \in A_*$ such that $t_* \land m \leq x_* \land m$ and $t_* \land m \leq y_* \land m$. Then $x_* \land m \leq t_* \land m$, $y_* \land m \leq t_* \land m$ and hence $(x_* \lor y_*)_* \land m \leq t_* \land m$. Thus $t_* \land m \leq (x_* \lor y_*)_* \land m = (x \lor y)_* \land m$. Then $(x \lor y)_* \land m$ is the g.l.b of $x_* \land m$, $y_* \land m$ in $(A_*, \leq)$. Define $(x_* \land m) \overline{\land} (y_* \land m) = (x \lor y)_* \land m$. Hence $(A_*, \lor, \overline{\land}, 0, m)$ is a bounded distributive lattice. Let $x_* \land m \in A_*$. Then $x_* \land m \in A_*$ and $(x_* \land m) \lor (x_* \land m) = (x_* \lor x_*) \land m = m$ and $(x_* \land m) \overline{\land} (x_* \land m) = (x_* \lor x_*)_* \land m = m_* \land m = 0$. Thus $(A_*, \lor, \overline{\land}, 0, m)$ is a complemented lattice. Finally, we prove distributivity of the lattice $(A_*, \lor, \overline{\land})$. Let $x_* \land m, y_* \land m$ and $z_* \land m \in A_*$. Then

\[
((x_* \land m) \overline{\land} (y_* \land m)) \lor ((x_* \land m) \overline{\land} (z_* \land m)) = ((x \lor y)_* \land m) \lor ((x \lor z)_* \land m) = ((x \lor y)_* \lor (x \lor z)_*) \land m = ((x \lor (y \land z))_*) \land m = (x_* \land m) \overline{\land} ((y_* \lor z_*) \land m) = (x_* \land m) \overline{\land} ((y_* \land m) \lor (z_* \land m)).
\]
Hence $(A_*, \lor, \neg, 0, m)$ is a Boolean algebra, where $(x_\star \land m) \lor (y_\star \land m) = (x \land y)_\star \land m$ and $(x_\star \land m) \land (y_\star \land m) = (x \lor y)_\star \land m$.

Using the above theorem, we prove the following two results.

**Theorem 1.3.22.** Let $A$ be an ADL with a maximal element $m$ and a dual pseudo-complementation $\star$. Then the map $f : A \to A_\star$ defined by $f(x) = x_\star \land m$ is an onto homomorphism.

**Proof.** Let $x, y \in A$. Then

$$f(x \land y) = (x \land y)_\star \land m = (x_\star \lor y_\star)_\star \land m$$

$$= (x_\star \land m) \land (y_\star \land m)$$

$$= f(x) \land f(y)$$

and $f(x \lor y) = (x \lor y)_\star \land m = (x_\star \lor y_\star) \land m = (x_\star \land m) \lor (y_\star \land m) = f(x) \lor f(y)$. Hence $f$ is a homomorphism. Since every element of $A_\star$ is of the form $x_\star \land m = x_{\star\star} \land m$, we get that $f$ is a surjection.

**Theorem 1.3.23.** Let $A$ be an ADL with two dual pseudo-complementations $\star$ and $\perp$. Then the map $f : A_\star \to A_{\perp}$ defined by $f(x_\star \land m) = x_{\perp} \land m$, is an isomorphism of the Boolean algebras $(A_\star, \lor, \neg)$ and $(A_{\perp}, \lor, \neg)$.

**Proof.** Let $x, y \in A$. By Lemma 1.3.8.(iii), for any $x, y \in A$, $x_\star = y_\star$ and hence $x_\star \land m = y_\star \land m$ if and only if $x_{\perp} \land m = y_{\perp} \land m$. Therefore $f$ is well defined and one-one. Clearly,
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$f$ is onto. Let $x,y \in A$. Then

\[
\begin{align*}
    f((x \land m) \land (y \land m)) &= f((x \lor y) \land m) = (x \lor y) \land \land m \\
    &= (x \land m) \land (y \land m) \\
    &= f(x \land m) \land f(y \land m)
\end{align*}
\]

and

\[
\begin{align*}
    f((x \land m) \lor (y \land m)) &= f((x \lor y) \land m) = (x \land y) \land \land m \\
    &= (x \land m) \lor (y \land m) \\
    &= f(x \land m) \lor f(y \land m).
\end{align*}
\]

Hence $f$ is an isomorphism of $A^\ast$ onto $A^\perp$.

We conclude this section with the following theorems which give different characterizations of a dual PCADL.

**Theorem 1.3.24.** Let $A$ be an ADL with a maximal element $m$. Then the following are equivalent:

(i) $A$ is a dual PCADL.

(ii) $[a, a \lor m]$ is a dual pseudo-complemented lattice for all $a \in A$.

(iii) $[0, m]$ is a dual pseudo-complemented lattice.

**Proof.** Let $A$ be an ADL with a maximal element $m$.

(i)\(\implies\) (ii):

Suppose $A$ is a dual PCADL and $a \in A$. Let $m_1 = a \lor m$ and $x \in [a, m_1]$, define $x_\perp = a \lor (x \land m)$. Then $x_\perp \in [a, m_1]$. Let $y \in [a, m_1]$ such that $x \lor y = m_1$. Then
(x ∨ y) ∧ m = m_1 ∧ m = m and hence (x_⋆ y) ∧ m = y ∧ m. Thus (x_⋆ y) ∧ m_1 = y ∧ m_1 = y.

Therefore x_⋆ ∧ m_1 ≤ y and hence x_⊥ = a ∨ (x_⋆ ∧ m) ≤ y. Conversely, suppose x_⊥ ≤ y. Then

\[
x ∨ x_⊥ = (x ∨ x_⊥) ∧ m_1 = (x ∨ a ∨ (x_⋆ ∧ m)) ∧ m_1 = m_1 = a ∨ m.
\]

Hence \([a, a ∨ m]\) is a dual pseudo-complemented lattice.

(ii)⇒(iii) is trivial. Now we show that (iii)⇒(i).

Suppose \([0, m]\) is a dual pseudo-complemented lattice under the unary operation \(⊥\). For \(x \in A\), define \(x_⋆ = (x ∧ m)_⊥\). Let \(y \in A\) such that \(x ∨ y = m\). Then \(x ≤ m\). Now \(y ∧ m ∈ [0, m]\) and \(x ∨ (y ∧ m) = (x ∧ m) ∨ (y ∧ m) = (x ∨ y) ∧ m = m\). Hence \(x_⊥ ≤ y ∧ m\).

Now \((x_⋆ y) ∧ m = (x_⋆ ∧ m) ∨ (y ∧ m) = (x_⊥ ∧ m) ∨ (y ∧ m) = x_⊥ ∨ (y ∧ m) = y ∧ m\).

Now \((x ∨ x_⋆) ∧ m = (x ∨ (x ∧ m)_⊥) ∧ m = (x ∧ m) ∨ (x ∧ m)_⊥ = m\) and hence \(x ∨ x_⊥\) is a maximal. Finally,

\[
(x ∧ y)_⋆ = (x ∧ y ∧ m)_⊥ = [(x ∧ m) ∧ (y ∧ m)]_⊥ = (x ∧ m)_⊥ ∨ (y ∧ m)_⊥ \quad (\text{by Theorem 1.3.3})
\]

\(= x_⋆ ∨ y_⋆\).

Hence \((A, ∨, ∧, ⋆, 0, m)\) is a dual PCADL.
If $A$ is an ADL, then the set $\text{PI}(A)$ of all principal ideals of $A$ forms a distributive lattice [40]. Now, we prove the following.

**Theorem 1.3.25.** Let $A$ be an ADL with a maximal element $m$. Then $A$ is a dual PCADL if and only if $\text{PI}(A)$ is a dual pseudo-complemented lattice.

**Proof.** Suppose $(A, \lor, \land, \ast, 0, m)$ is a dual PCADL. For any $x \in A$, define $(x)_+ = (x_*)$. Let $x \in A$. Then $(x) \lor (x)_+ = (x) \lor (x_*) = (x \lor x_*) = A$. Now, let $y \in A$ such that $(x) \lor (y) = A$. So that $(x \lor y) \land m = m$. Hence $y \land m = (x_\ast \land m) \lor (y \land m)$. Therefore $(x_\ast) \subseteq (y)$. That is $(x)_+ \subseteq (y)$. Hence $\text{PI}(A)$ is a dual pseudo-complemented lattice. Conversely, suppose $(\text{PI}(A), \lor, \land, +)$ is a dual pseudo-complemented lattice. For $x \in A$, define $x_\ast = a \land m$ where $(x)_+ = (a)$. Since $(a) = (b)$ if and only if $a \land m = b \land m$, we get that $\ast$ is well defined. We also get that $(x)_+ = (x_\ast)$. Now $(x \lor x_\ast) = (x) \lor (x_\ast) = A$ and hence $x \lor x_\ast$ is a maximal. Let $y \in A$ such that $x \lor y = m$. Then $(x) \lor (y) = (x \lor y) = (m) = A$ and hence $(x_\ast) \subseteq (y)$. Therefore $x_\ast \land m \leq y \land m$. Finally, suppose $x, y \in A$ and $(x)_+ = (a)$, $(y)_+ = (b)$. Then $(x_\ast \lor y_\ast) = (x_\ast) \lor (y_\ast) = (x_\ast) \lor (y_\ast) = (x \land y)_+ = (x \land y)_\ast = (x \land y)_\ast$. Hence $(x_\ast \lor y_\ast) \land m = (x \land y)_\ast \land m$. Thus $(x \land y)_\ast = x_\ast \lor y_\ast$. Hence $A$ is a dual PCADL. ■

### 1.4 Dual Stone ADLs

M.H. Stone raised the problem of characterizing the class of distributive pseudo-complemented lattices $(A, \lor, \land, 0, 1)$ in which $x^\ast \lor x^{**} = 1$ hold. Later it is named as a Stone lattice by T.P. Speed and characterized the Stone lattices topologically and algebraically in 1967. Stone lattices were extended by G. Gratzer, E.T. Schmidt, J. Varlet, O. Frink and G. Bruns. Later, U.M. Swamy, G.C. Rao and G. Nanaji Rao were introduced the
concept of a Stone ADL in the class of an ADLs in [55]. The following definition of a Stone ADL is taken from [55].

**Definition 1.4.1.** [55] Let $A$ be an ADL and $*$ a pseudo-complementation on $A$. Then $A$ is called a Stone ADL if, for any $x \in A$, $x^* \lor x^{**} = 0^*$.

As observed in [54], an ADL $A$ may have a number of pseudo-complementations. But for any two pseudo-complementations $*$ and $\perp$ on $A$, $x^* \lor x^{**} = 0^*$ for all $x \in A \iff x^\perp \lor x^{\perp\perp} = 0^\perp$ for all $x \in A$. Hence, if $A$ is a Stone ADL with one pseudo-complementation, then it is a Stone lattice with any other pseudo-complementation on it.

We have observed that the dual of an ADL is an not an ADL, in general. For this reason, in this section, we introduce the concept of a dual Stonity of a dual pseudo-complemented ADL and the concept of a dual Stone ADL. We begin with the following.

**Definition 1.4.2.** Let $(A, \lor, \land, *, 0, m)$ be a dual PCADL with a maximal element $m$. Then the set \{ $x \in A \mid x_* \land x^{**} = 0$ \} is called the dual Stonity of $A$ and it is denoted by $DS(A)$.

In-view of Lemma 1.3.8. (vi), we get that the set $DS(A)$ is independent of the dual pseudo-complementation on $A$. Now we prove the following.

**Theorem 1.4.3.** Let $A$ be a dual PCADL. Then $DS(A)$ is a sub ADL of $A$, closed under dual pseudo-complementation.

**Proof.** Let $x, y \in DS(A)$. Then

$$(x \land y)_* \land (x \land y)^{**} = (x_* \lor y_*) \land (x \land y)^{**}$$
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\[ (x_\ast \land (x \land y)_\ast \ast) \lor (y_\ast \land (x \land y)_\ast \ast) \quad \ldots \ldots (1). \]

Since \( x \land y \land x = y \land x \) and \( y \land x \land y = x \land y \), by Theorem 1.3.6.(i), we get \( (y \land x)_\ast \ast \leq x_\ast \ast \) and \( (x \land y)_\ast \ast \leq y_\ast \ast \). Hence, from (1), \( (x \land y)_\ast \land (x \land y)_\ast \ast \leq (x_\ast \land x_\ast \ast) \lor (y_\ast \land y_\ast \ast) = 0 \).

Now \( (x \lor y)_\ast \land (x \lor y)_\ast \ast = (x \lor y)_\ast \land (x_\ast \lor y_\ast \ast) \)

\[
= ((x \lor y)_\ast \land x_\ast \ast) \lor ((x \lor y)_\ast \land y_\ast \ast) \quad \ldots \ldots (2).
\]

Since \( (x \lor y) \land x = x \) and \( (x \lor y) \land y = y \), by Theorem 1.3.6.(i), we get \( (x \lor y)_\ast \leq x_\ast \) and \( (x \lor y)_\ast \leq y_\ast \) and hence, from (2), we get \( (x \lor y)_\ast \land (x \lor y)_\ast \ast \leq (x_\ast \land x_\ast \ast) \lor (y_\ast \land y_\ast \ast) = 0 \) (since \( x, y \in DS(A) \)). Therefore \( x \land y, x \lor y \in DS(A) \). Also, since \( 0_\ast \land 0_\ast \ast = 0, \land 0 = 0 \), we get \( 0 \in DS(A) \). Hence DS(A) is a sub ADL containing 0. Finally, If \( x \in DS(A) \), then

\[ x_\ast \land x_\ast \ast = 0. \]

So that \( x_\ast \ast \land x_\ast \ast = 0 \) and hence \( x_\ast \in DS(A) \).

\textbf{Definition 1.4.4.} Let \( A \) be a distributive lattice with 0,1 and \( \ast \) be a dual pseudo-complementation on \( A \). Then \( A \) is said to be a dual Stone lattice, if for any \( x \in A \), \( x_\ast \land x_\ast \ast = 0 \).

Now, we introduce the concept of a dual Stone ADL.

\textbf{Definition 1.4.5.} An ADL \( A \) with a dual pseudo-complementation \( \ast \) is called a dual Stone ADL if \( DS(A) = A \), or equivalently, \( x_\ast \land x_\ast \ast = 0 \) for all \( x \in A \).

\textbf{Theorem 1.4.6.} Let \( A \) be a dual PCADL with a maximal element \( m \). Then \( A \) is a dual Stone ADL if and only if \( A_\ast = DS(A_\ast) \).

\textit{Proof.} Suppose \( A_\ast = DS(A_\ast) \). Let \( x \in A \). Then \( x_\ast \in A_\ast = DS(A_\ast) \) and hence \( x_\ast \ast \land x_\ast \ast \ast = x_\ast \land x_\ast \ast = 0 \). Thus \( x \in DS(A) \). Therefore \( A \) is a dual Stone ADL. Conversely, if \( A \) is a
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dual Stone ADL, then for any \( x \in A \), \( x_\ast \land x_{\ast\ast} = 0 \). In particular, \( x_{\ast\ast} \land x_{\ast\ast\ast} = 0 \). Then \( x_\ast \in DS(A_\ast) \) and hence \( A_\ast = DS(A_\ast) \).

Let us recall that for any \( x \in A \), the dual annihilator of \( x \) is
\[
(x)^+ = \{ y \in A \mid x \lor y \text{ is a maximal element of } A \}.
\]

Now, we prove the following.

**Theorem 1.4.7.** Let \( A \) be a dual PCADL with a maximal element \( m \). Then the following are equivalent:

(i) \( A \) is a dual Stone ADL.

(ii) \( (x)^+ \lor (y)^+ = A \) whenever \( x, y \in A \) such that \( x \lor y \) is a maximal.

(iii) \( x_\ast \land y_\ast = 0 \) whenever \( x, y \in A \) such that \( x \lor y \) is a maximal.

**Proof.** Let \( A \) be a dual PCADL with a maximal element \( m \).

(i)\(\implies\)(ii): suppose \( A \) is a dual Stone ADL and let \( x, y \in A \) such that \( x \lor y \) is a maximal. Then
\[
(x \lor y) \land m = m
\]
and hence \( (x_\ast \lor y) \land m = y \land m \). Thus \( y \land x_\ast = x_\ast \). Now \( (x_{\ast\ast} \lor y) \land m = ((y \land x_\ast) \lor y) \land m = (x_{\ast\ast} \lor y_\ast \lor y) \land m = m \) (since \( y_\ast \lor y \) is maximal, we get \( x_{\ast\ast} \lor (y_\ast \lor y) \) is also maximal). So that \( x_{\ast\ast} \lor y \) is a maximal and hence \( x_{\ast\ast} \in (y)^+ \). Since \( x \lor x_\ast \) is maximal, we get \( x_\ast \in (x)^+ \). Thus \( 0 = x_\ast \land x_{\ast\ast} \in (x)^+ \lor (y)^+ \). Hence \( (x)^+ \lor (y)^+ = A \).

(ii)\(\implies\)(iii):

Assume (ii). Let \( x, y \in A \) such that \( x \lor y \) is a maximal. Then \( (x)^+ \lor (y)^+ = A \). So that \( 0 \in (x)^+ \lor (y)^+ \). Hence \( 0 = s \land t \) for some \( s \in (x)^+ \) and \( t \in (y)^+ \). Since \( s \in (x)^+ \), \( s \lor x \) is maximal. So that \( (x_\ast \lor s) \land m = s \land m \) and hence \( (x_\ast \lor s) \land m \land x_\ast = s \land m \land x_\ast \). That is
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\[ s \land x_\ast = x_\ast. \] Similarly, \( t \land y_\ast = y_\ast. \) Now \( x_\ast \land y_\ast = s \land x_\ast \land t \land y_\ast = 0. \)

(iii)\( \implies \) (i):

Suppose \( x_\ast \land y_\ast = 0 \) whenever \( x, y \in A \) such that \( x \lor y \) is a maximal. If \( x \in A \), then \( x \lor x_\ast \)

is a maximal, and hence by (iii), \( x_\ast \land x_{\ast\ast} = 0. \) Thus \( A \) is a dual Stone ADL.

The following result can be obtained directly from Theorem 1.4.7 and Theorem 0.3.27.

**Corollary 1.4.8.** Let \( A \) be a dual PCADL with a maximal element \( m. \) Then \( A \) is a dual Stone ADL if and only if \( A \) is a dually normal ADL.

**Theorem 1.4.9.** Let \( A \) be a dual Stone ADL with a maximal element \( m. \) Then, for any \( x, y \in A, \) \( (x_\ast \lor y_\ast)_\ast = x_{\ast\ast} \land y_{\ast\ast}. \)

**Proof.** Let \( A \) be a dual Stone ADL with a maximal element \( m \) and \( x, y \in A. \) Since \( x_\ast \land m \leq (x_\ast \lor y_\ast)_\ast \land m, \) we get \( (x_\ast \lor y_\ast)_\ast \land m \leq x_{\ast\ast} \land m. \) Similarly, we get \( (x_\ast \lor y_\ast)_\ast \land m \leq y_{\ast\ast} \land m \) and hence \( (x_\ast \lor y_\ast)_\ast \land m \leq x_{\ast\ast} \land y_{\ast\ast} \land m. \) On the other hand, we have

\[ x_{\ast\ast} \land y_{\ast\ast} \land (x_\ast \lor y_\ast) = (x_{\ast\ast} \land y_{\ast\ast} \land x_\ast) \lor (x_{\ast\ast} \land y_{\ast\ast} \land y_\ast) = 0. \]

\[ \implies [x_{\ast\ast} \land y_{\ast\ast} \land (x_\ast \lor y_\ast)] \lor (x_\ast \lor y_\ast)_\ast = 0 \lor (x_\ast \lor y_\ast)_\ast \]

\[ \implies [(x_{\ast\ast} \land y_{\ast\ast}) \lor (x_\ast \lor y_\ast)_\ast] \land [(x_\ast \lor y_\ast) \lor (x_\ast \lor y_\ast)_\ast] \land m = (x_\ast \lor y_\ast)_\ast \land m \]

\[ \implies [(x_{\ast\ast} \land y_{\ast\ast}) \lor (x_\ast \lor y_\ast)_\ast] \land m = (x_\ast \lor y_\ast)_\ast \land m. \]

Hence \( x_{\ast\ast} \land y_{\ast\ast} \land m \leq (x_\ast \lor y_\ast)_\ast \land m. \) Thus \( x_{\ast\ast} \land y_{\ast\ast} \land m = (x_\ast \lor y_\ast)_\ast \land m. \) Now \( x_{\ast\ast} \land y_{\ast\ast} \land m \land 0_\ast = (x_\ast \lor y_\ast)_\ast \land m \land 0_\ast \) and hence \( x_{\ast\ast} \land y_{\ast\ast} = (x_\ast \lor y_\ast)_\ast \) (since, for any \( x \in A, \) \( 0 \leq x \), we get \( x_\ast \leq 0_\ast. \))

Now we give an important characterization of a dual Stone ADL.
Theorem 1.4.10. Let $A$ be a dual PCADL with a maximal element $m$. Then $A$ is a dual Stone ADL if and only if $(x \lor y)_* = x_* \land y_*$ for all $x, y \in A$.

Proof. Suppose $A$ is a dual Stone ADL and $x, y \in A$. Then

$$(x \lor y)_* = (x \lor y)_***$$

$$= (x_* \lor y_*)_* \text{ (by Theorem 1.3.12)}$$

$$= x*** \land y*** \quad \text{(by Theorem 1.4.9)}$$

$$= x_* \land y_*.$$ 

Conversely, suppose, for any $x, y \in A$, $(x \lor y)_* = x_* \land y_*$. Since $x \lor x_*$ is maximal, we get $0 = m_* = ((x \lor x_*) \land m)_* = x_* \land x***$ and hence $A$ is a dual Stone ADL. □

Theorem 1.4.11. Let $A$ be a dual Stone ADL with a maximal element $m$. Write $A_* = \{x_* \land m \mid x \in A\}$. Then $(A_*, \lor, \land, *, 0, m)$ is a Boolean algebra.

Proof. Since $a_* \land m \leq m$ for all $a_* \land m \in A_*$, we get $A_*$ is a bounded distributive lattice. Let $x_* \land m \in A_*$ where $x \in A$. Then $x*** \land m \in A_*$ and $x_* \land m \land x*** \land m = x_* \land x*** \land m = 0$ and $(x_* \land m) \lor (x*** \land m) = (x_* \lor x***) \land m = m$. Hence $(A_*, \lor, \land, *, 0, m)$ is a Boolean algebra. □

Now, in the following two theorems, we characterize the dual Stone ADL.

Theorem 1.4.12. Let $A$ be a dual PCADL with a maximal element $m$. Then the following are equivalent:

(i) $A$ is a dual Stone ADL.
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(ii) \([a, a \lor m]\) is a dual Stone lattice for all \(a \in A\).

(iii) \([0, m]\) is a dual Stone lattice.

**Proof.** Let \(A\) be a dual PCADL with a maximal element \(m\).

(i)\(\implies\)(ii):

Suppose \(A\) is a dual Stone ADL and \(a \in A\). For any \(x \in A\), define \(x_\perp = a \lor (x_\ast \land m)\).

Then, by Theorem 1.3.24, we get that \(\perp\) is a dual pseudo-complementation on \([a, a \lor m]\).

Now, for any \(x \in [a, a \lor m]\),

\[
x_\perp \land x_\perp = [a \lor (x_\ast \land m)] \land [a \lor (x_\ast\ast \land m)]
\]

\[
= a \lor (x_\ast \land x_\ast\ast \land m)
\]

\[
= a \lor 0
\]

\[
= a.
\]

Hence \([a, a \lor m]\) is a dual Stone lattice. (ii)\(\implies\)(iii) is trivial.

(iii)\(\implies\)(i):

Suppose \([0, m]\) is a dual Stone lattice under the unary operation \(\perp\). For \(x \in A\), define \(x_\ast = (x \land m)_\perp\). Then, by Theorem 1.3.24, we get that \(A\) is a dual PCADL. Now, for any \(x \in A\),

\[
x_\ast \land x_\ast\ast = (x \land m)_\perp \land ((x \land m)_\perp \land m)_\perp
\]

\[
= x_\perp \land (x_\perp \land m)_\perp
\]

\[
= x_\perp \land x_\perp\perp
\]

\[
= 0.
\]
Hence $A$ is a dual Stone ADL. 

Finally, we conclude this section with the following.

**Theorem 1.4.13.** Let $A$ be an ADL with a maximal element $m$. Then $A$ is a dual Stone ADL if and only if $PI(A)$ is a dual Stone lattice.

**Proof.** Suppose $(A, \lor, \land, *, 0, m)$ is a dual Stone ADL. For any $x \in A$, define $(x)_+ = (x_*)$. Then, by Theorem 1.3.25, we get that $PI(A)$ is a dual pseudo-complemented lattice. Now $(x)_+ \land (x_*)_+ = (x_* \land (x_*)_*) = (x_\land x_*) = (0)$. Hence $PI(A)$ is a dual Stone lattice. Conversely, suppose $(PI(A), \lor, \land, +, 0, A)$ is a dual Stone lattice. For $x \in A$, define $x_* = a_\land m$ where $(x)_+ = (a)$. Then, by Theorem 1.3.25, we get that $A$ is a dual PCADL. Let $x \in A$. Suppose $(a)_+ = (a)$ and $(a)_+ = (b)$. Since $(a)_+ \land (a)_+ = (0)$, we get $a_\land b = 0$ and hence $x_* \land x_* = a_\land m_\land b_\land m = 0$. Hence $A$ is a dual Stone ADL.