Dissipative dynamical systems exhibiting complex or chaotic behaviour occur quite commonly in condensed matter physics, fluid physics, accelerator physics, chemical reactions, electrical circuits and so on. For such multidimensional systems, the volume of the phase space contracts in all directions and the trajectory converges to a limit cycle. One direction will have the slowest convergence and hence these systems have asymptotic motions that can be modelled by one dimensional noninvertible maps. The differential equations representing these systems can be reduced to one dimensional maps by means of Poincaré sections, stroboscopic methods or return maps corresponding to the maximum values of the variable. Such maps also arise naturally as simple models of system behaviour as in population studies of certain species. In general, one dimensional maps of the form

\[ x_{n+1} = f_\lambda(x_n) \]  

... (2.1)
have a tuning parameter $\lambda$ whose variation drives
the system through a series of bifurcations. This
period-doubling route, usually called the Feigenbaum
Scenario, forms the most common and the most well
studied route from periodic to aperiodic behaviour.
The sequence of bifurcations exhibits some striking
universal features near the infinite bifurcation
limit $\lambda_\infty$ characterised by two scaling constants
$\alpha$ and $\delta$. Thus the measurable properties of any
system in the aperiodic limit can be described in a
way that essentially bypasses the details of the
equations governing the system. The universality
theory was developed in the context of quadratic
maps by Feigenbaum [43]. Soon after this work,
several numerical and theoretical studies lent
support to the universality picture in a number of
models in various dimensions. It was first tested
experimentally in 1980 in an actual turbulence
experiment [44]. The universality theory was later
extended to circle maps [45] and hamiltonian
maps [46] and later generalised to include period
n-tuplings in complex iterative maps [47]. It also
plays an important role in the intermittent route
to chaos [48].
2.1 The Universality theory

The key to understanding repeated period-doublings is the introduction of a doubling transformation $T$ which carries a map $f$ to one obtained by

i) composing $f$ with itself,

ii) restricting to an appropriate subdomain and

iii) changing the co-ordinates to magnify the subdomain to the original domain. To account for the universality, one has to show that $T$ has a fixed point and that in the neighbourhood of the fixed point $T$ is expanding in one direction and contracting in all others.

To define the doubling operator $T$, we consider one-hump maps of the form

$$x_{n+1} = 1 - \lambda |x_n|^z \quad \ldots \ (2.2)$$

where $z$ is the order of the local maximum. This map has the following general features:

1) For $\lambda_{n-1} < \lambda < \lambda_n$, there exists a stable $2^{n-1}$
cycle with elements $x_0^*, x_1^*, \ldots x_{2^{n-1}}^*$ which is characterized by

$$\prod_{i=1}^{2^{n-1}} f'_{\lambda_i}(x_i^*) < 1 \quad \ldots \quad (2.3)$$

where the prime denotes differentiation of the function $f_{\lambda_i}$ with respect to $x_i$.

2) At $\lambda_n$, all points of the $2^{n-1}$ cycle become unstable via pitchfork bifurcations leading to a new stable $2^n$ cycle for $\lambda_n < \lambda < \lambda_{n+1}$.

3) A $2^n$ superstable cycle is defined by

$$\prod_{i=1}^{2^{n-1}} f'_{\lambda_i}(x_i^*) = 0 \quad \ldots \quad (2.4)$$

This implies it always contains $x_0^* = 0$ as a cycle element.

4) The distance $d_n$ of the point in a $2^n$ supercycle which is closest to $x = 0$ is given by

$$d_n = \frac{2^{n-1}}{\lambda_n} (o) \quad \ldots \quad (2.5)$$

We have $\frac{d_n}{d_{n+1}} = -\alpha \quad \ldots \quad (2.6)$
where $\alpha$ is the scale factor introduced in the preceding chapter.

Then

$$\lim_{n \to \infty} (-\alpha)^n d_{n+1} = d_1 \quad \ldots \quad (2.7)$$

ie.

$$\lim_{n \to \infty} (-\alpha)^n f_2^n \Lambda_{n+1} (\omega) = d_1 \quad \ldots \quad (2.8)$$
This can be generalised to the whole interval \((-1,1)\) and the rescaled functions then converge to a limiting function \([28]\).

\[
\lim_{n \to \infty} (-\alpha)^n \frac{f^{2^n}}{\Lambda_{n+1}} \frac{x}{\alpha^n} = g_1(x) \quad \ldots \quad (2.9)
\]

\(g_1(x)\) is determined by the behaviour of \(f^{2^n}(x)\) \(\Lambda_{n+1}\) around \(x = 0\) and should be universal for all \(f\) with the same \(z\) \([49]\).
We introduce a whole family of functions

\[ g_i(x) = \lim_{n \to \infty} (\frac{-a}{a})^n \frac{f^{2n}}{a^n} \left( \frac{x}{a^n} \right) \]  \hspace{1cm} (2.10)

where the \( g_i \) for \( i > 1 \) are iterates of \( g_1 \). All such functions are related by

\[ g_{i-1}(x) = -\alpha g_i(g_i(\frac{x}{a})) = T g_i(x) \] \hspace{1cm} (2.11)

In the limit \( i \to \infty \), we define

\[ g(x) = \lim_{i \to \infty} g_i(x) \] \hspace{1cm} (2.12)

Clearly \( g(x) \) satisfies the equation

\[ g(x) = -\alpha g(g(x/a)) \] \hspace{1cm} (2.13)

Thus \( g \) is the fixed point function of the doubling transformation \( T \). Since the equation does not fix absolute scales, we introduce a normalisation condition,

\[ g(0) = 1 \] \hspace{1cm} (2.14)

Equation (2.13) determines \( \alpha \) universally as

\[ \alpha = -\frac{1}{g(1)} \] \hspace{1cm} (2.15)
\( g(x) \) is obtained as the limit of \( f^{2^n} \)'s at the value of \( \Lambda_{-\infty} \). This is the unique value of \( \Lambda \) at which repeated applications of \( T \) will lead to a convergent function.

We define \([26, 43]\)

\[ \Delta g_i = g_{i+1}(x) - g_i(x) \quad \ldots (2.16) \]

Then equation (2.11) becomes,

\[ g_i(x) = -\alpha(g_i + \Delta g_i) \left[ g_i(x/\alpha) + \Delta g_i(x/\alpha) \right] \]

\[ = g_{i-1}(x) - \alpha \left[ \Delta g_i(g_i(x/\alpha)) \right. \]

\[ + \left. g_i'(g_i(x/\alpha))\Delta g_i(x/\alpha) \right] + O((\Delta g_i)^2) \]

\[ \ldots (2.17) \]

ie. \( \Delta g_{i-1}(x) = -\alpha \left[ \Delta g_i(g_i(x/\alpha)) + g_i'(g_i(x/\alpha)) \right. \]

\[ x \Delta g_i(x/\alpha) \right] + O((\Delta g_i)^2) \quad \ldots (2.18) \]

In the limit \( i \to \infty \), \( g_i \to g \) so that \( \Delta g_i \to 0 \).

So we write,

\[ \Delta g_i(x) = \eta_i h(x) \quad \ldots (2.19) \]

with the condition \( \eta_i \to 0 \) as \( i \to \infty \).
Then (2.18) gives a closed equation for \( h(x) \) and \( n_i \):

\[
\begin{align*}
\eta_{i-1} &= \delta \eta_i, \quad \ldots \ (2.20) \\
h(x) &= -(\alpha/\delta) \left[ h(g(x/\alpha)) + g'(g(x/\alpha)) \right] x h(x/\alpha) \quad \ldots \ (2.21)
\end{align*}
\]

Equation (2.20) can be trivially solved to give

\[
\eta_i = \delta^{-i} \quad \ldots \ (2.22)
\]

Then (2.12) is satisfied if \( \delta > 1 \). It can be shown that [26]

\[
\Lambda_{n+1} - \Lambda_n \approx \delta^{-n} \quad \ldots \ (2.23)
\]

logarithmicly so that the original definition of \( \delta \) given in Chapter 1 automatically follows.

We observe that universality arises from the fact that bifurcation is a local phenomenon. Here we consider the region near \( x = 0 \) where distances between points scale as \( \alpha \), while points near \( x = 1 \) are found to scale as \( \alpha^2 \). It is also possible to consider scaling and universal behaviour centered around points other than these two [50].
2.2 The Renormalisation Group equations and calculation of $\alpha$ and $\delta$

The essence of the renormalisation group (RG) analysis given above is that different maps with the same $z$ value have the same values for $\alpha$ and $\delta$. The RG equations (2.13) and (2.21) can be used to evaluate $\alpha$ and $\delta$ for a given $z$ value. However the structure and solutions of these equations are not yet fully studied. Numerical evaluation of $\alpha$ and $\delta$ shows that there exists universality classes characterised by $\alpha$ and $\delta$ for different $z$ values [51].

Among the available analytic methods, we mention the eigenvalue matching RG method [52] and Helleman's scheme [53]. The former has the drawback that it is difficult to extend it beyond the second or fourth order of renormalisation. Moreover it provides us with the $\delta$ values only. There is no direct way of getting $\alpha$ and $g(x)$. The Helleman scheme has been applied to quadratic maps. But for maps with $z>2$, the algebra involved is very cumbersome.
A method of evaluating $\xi$ and $\delta$ using equations (2.13) and (2.21) was reported by Delbourgo et al which involves the truncation of $g(x)$ to first order in $|x|^2$ [54]. For second or higher order, numerical methods must be resorted to. Moreover the asymptotic expressions in their approach refer to the limit $N \rightarrow \infty$ for $N$-replication. Even though multifurcations other than bifurcations are usual, we feel that the large $N$ limit is rather unphysical. A computational iterative procedure based on the nested structure of $g(x)$ is used by van der Weele et al and is found to have a rapid convergence for small $z$ values [55].

Perturbative scheme for evaluation of universal parameters

An analytic method based on a perturbative scheme was developed by Singh [56] to solve equation (2.13) for $\alpha$ and $g(x)$. Here $g(x)$ is expanded into an infinite series and the coefficients of expansion are replaced by a perturbative series in inverse powers of $\alpha$. The method has been applied to quadratic
maps giving good results. We have extended this scheme so that (2.13) and (2.21) can be solved simultaneously for $\alpha, \delta, g(x)$ and $h(x)$ for any general $z$. Here the functional equations are replaced by infinite dimensional nonlinear vector equations.

To this end we write $g(x)$ as

$$g(x) = 1 + \sum_{n=1}^{\infty} P_n \frac{x^{nz}}{|x|} \quad \ldots (2.24)$$

with the normalisation given in (2.14). In the neighbourhood of the extremum at $x = 0$, $g(x)$ is positive for any $z$ and $g(g(x))$ can be expanded into a similar power series. Thus

$$g(g(x)) = 1 + \sum_{r=1}^{\infty} P_r + (P_1 \sum_{r=1}^{\infty} rzP_r) \frac{x^z}{|x|} + (P_2 \sum_{r=1}^{\infty} rP_r + P_1^2 \sum_{r=1}^{\infty} \frac{rz(rz-1)}{2} x P_r) \frac{|x|^{2z}}{|x|} + \ldots \quad \ldots (2.25)$$

We redefine the coefficients of expansion in (2.24) as

$$P_n a^n = S_n |\alpha|^z \quad \ldots (2.26)$$
Using (2.24) - (2.26) in (2.13) and equating coefficients of $|x|^n z$ we get

$$\frac{1}{\alpha} + 1 + |\alpha|^z \sum_{r=1}^{\infty} \frac{S_r}{\alpha^r} = 0, \quad n = 0 \quad \ldots (2.27)$$

$$\frac{1}{z} \sum_{r=1}^{\infty} \frac{rS_r}{\alpha^r-1} = 0, \quad n = 1 \quad \ldots (2.28)$$

$$S_n \left[ 1 - \frac{1}{|\alpha|^z(n-1)} \right] + \frac{n}{1} \sum_{l \geq 2} \frac{\sum_{r \geq 1} (rz) S_r}{l \alpha^{r-1}}$$

$$x \sum_{m_1 \geq 1} \cdots m_l \geq 1 \frac{S_{m_1} S_{m_2} \cdots S_{m_l}}{|\alpha|^z(n-l)} \delta_{m_1+\cdots+m_l,n} = 0; \quad n = 2, 3, 4, \ldots \quad \ldots (2.29)$$

These form an infinite set of coupled nonlinear equations. To solve these, we expand $S_n$ in inverse powers of $\alpha$,

$$S_n(\alpha) = \sum_{m=0}^{\infty} \frac{S_{nm}}{\alpha^m} \quad \ldots (2.30)$$

Using this expansion in (2.28) and (2.29), equating coefficients of equal powers of $1/\alpha$ to zero, we obtain
a hierarchy of equations which can be solved successively for the coefficients $S_{nm}$. These are used in (2.27) to yield the equation for $\alpha$:

$$\frac{1}{\alpha} + 1 + |\alpha|^z \sum_{r=1}^{\infty} \sum_{m=0}^{\infty} \frac{S_{rm}}{\alpha^{r+m}} = 0 \quad \ldots \quad (2.31)$$

The universal function $g(x)$ is given by

$$g(x) = 1 + \sum_{n=1}^{\infty} \left[ |\alpha|^{z-n} \sum_{m=0}^{\infty} \frac{S_{nm}}{\alpha^m} \right] |x|^nz \quad \ldots \quad (2.32)$$

In our work, we have expanded $h(x)$ also into a power series as in (2.24). We substitute this in (2.21) and equate coefficients of $|x|^nz$ on both sides to get,

$$- \alpha \sum_{r=1}^{\infty} \left[ l + h_r + |\alpha|^z \frac{rz S_r}{\alpha^r} \right] = \delta ; \quad n = 0 \quad \ldots \quad (2.33)$$

$$- \alpha \sum_{r=1}^{\infty} \left[ \binom{rz}{1} \frac{h_r S_1}{\alpha} + 2 \binom{rz}{2} \frac{|\alpha|^{z-l} S_r S_l}{\alpha^r} ight. \left. + h_1 \binom{rz}{1} \frac{S_r}{\alpha^r} \right] = \delta h_1 ; \quad n = 1 \quad \ldots \quad (2.34)$$
The expansion given in (2.30) can be used here also and the $S_{nm}$ coefficients determined earlier can be substituted. We have reduced the above set of equations into a matrix eigenvalue equation of the form.

$$D \ h = \delta \ h \quad \ldots \quad (2.36)$$

where $h$ is the column $[1, h_1, h_2, \ldots, h_n]$. The largest real eigenvalue of $D$ furnishes the relevant $\delta$ value.
In practice, we can work only with truncated series of $S_\tau$ and $h_\tau$. We determine the $S_{nm}$ coefficient which are required to retain terms upto a definite power of $1/\alpha$ in the equation for $\alpha$, (2.31). Then the maximum number of coefficients $h_\tau$ that can be included are used in the calculation of $\delta$.

However we find that the perturbation series is not highly convergent but shows some of the basic characteristics of asymptotic series. The details of the calculations given in the next chapter indicate that the absolute values of the successive $S_{nm}$ coefficients, for a given $n$, do not decrease steadily, but decrease for a few $m$ values, then increase, beyond some $m$ value. The corresponding $\alpha$ values calculated using the perturbative series also show some fluctuations about the numerical value, instead of converging to it uniformly. Therefore the selection of the truncation point is rather crucial. It is found that there is definite advantage if we use Padé approximants [57] to sum the series in the expressions for $\alpha$ and $\delta$. 
2.4 Universal relations for general $z$

It is now well established that one-hump maps fall into different universality classes labelled by the order of their extremum $z$.

For any $z$ value, the first few coefficients $S_{nm}$ using (2.28) and (2.29) work out to be

$$S_{10} = -\frac{1}{z} ; \quad S_{11} = \frac{-\left(z-1\right)}{z^2} ; \quad S_{20} = \frac{(z-1)}{2z^2} \quad \ldots \quad (2.37)$$

The equation for $\alpha$ (2.31) using these coefficients is,

$$\frac{1}{\alpha} + 1 = |\alpha|^z \left[ \frac{1}{z^2\alpha} + \frac{(z-1)}{2z^2\alpha^2} + \ldots \right] \quad (2.38)$$

The series in the brackets can be replaced by its Padé approximant. Thus considering the lowest approximation, we use the $[1/1]$ approximant to yield

$$\frac{\alpha^z}{z} = (1+\alpha) \left[ 1 - \frac{(z-1)}{2z\alpha} \right] \quad \ldots \quad (2.39)$$

In the limit $z \rightarrow 1$, we find $\alpha \rightarrow \infty$. For a given $z>1$, (2.39) can be solved for $\alpha$. Fig 2.1 shows
Fig. 2.1 - The values of $\alpha$ for different values of $z$. The computed values using (2.39) are shown by triangles while numerical values are indicated by circles.
the values of \( \alpha \) thus obtained. The numerical values are also plotted. It is clear that the agreement is quite good. The computed values are provided in Table 2.1.

Using the coefficients in (2.37) in the equations for \( \delta \) (2.33) – (2.35) we get the approximate expression,

\[
2\delta = a^z - \alpha + 2 + \sum_{n=1}^{\infty} \left( \frac{z-1}{z\alpha} \right)^n + 4|\alpha|^{z+1} \left\{ \left( \frac{z-1}{z\alpha} \right) - \frac{(z-1)}{z^2\alpha^2} - \frac{z(z-1)}{z^3\alpha^3} \right\}^{1/2}
\]

\[\ldots (2.40)\]

Replacing the series in the last term by its \([1/2]\) Pade approximant, we get

\[
2\delta = a^z - \alpha + 2 + \sum_{n=1}^{\infty} \left( \frac{z-1}{z\alpha} \right)^n + 4|\alpha|^{z+2} \frac{z(z-1)}{z\alpha(z\alpha+1)+1+z(z-1)}^{1/2}
\]

\[\ldots (2.41)\]

As \( z \) increases the first term within the square brackets increases faster than the second, so we write
Table 2.1 - Computed values of $\alpha$ using (2.39).
The numerical values are also given for comparison [51,55,58].

<table>
<thead>
<tr>
<th>$z$</th>
<th>Computed values of $\alpha$</th>
<th>Numerical $\alpha$ values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>7.978673</td>
<td>7.97</td>
</tr>
<tr>
<td>1.2</td>
<td>5.390312</td>
<td>5.37</td>
</tr>
<tr>
<td>1.5</td>
<td>3.405767</td>
<td>3.39</td>
</tr>
<tr>
<td>2.0</td>
<td>2.517021</td>
<td>2.5029</td>
</tr>
<tr>
<td>2.1</td>
<td>2.422142</td>
<td>2.4084</td>
</tr>
<tr>
<td>2.2</td>
<td>2.340300</td>
<td>2.3269</td>
</tr>
<tr>
<td>2.3</td>
<td>2.268835</td>
<td>2.2557</td>
</tr>
<tr>
<td>2.4</td>
<td>2.205783</td>
<td>2.1928</td>
</tr>
<tr>
<td>2.5</td>
<td>2.149665</td>
<td>2.1368</td>
</tr>
<tr>
<td>3.0</td>
<td>1.940393</td>
<td>1.9277</td>
</tr>
<tr>
<td>4.0</td>
<td>1.704310</td>
<td>1.69</td>
</tr>
<tr>
<td>5.0</td>
<td>1.571511</td>
<td>1.56</td>
</tr>
<tr>
<td>6.0</td>
<td>1.485068</td>
<td>1.467</td>
</tr>
<tr>
<td>7.0</td>
<td>1.423759</td>
<td>1.41</td>
</tr>
<tr>
<td>8.0</td>
<td>1.377734</td>
<td>1.358</td>
</tr>
<tr>
<td>10.0</td>
<td>1.312754</td>
<td>1.2914</td>
</tr>
<tr>
<td>100.0</td>
<td>1.047940</td>
<td>1.03373</td>
</tr>
</tbody>
</table>
When \( z \) is large the last term in (2.42) is \( \approx 1 \).
So we can recover the universality relation
derived by Delbourgo [58] from (2.42) i.e.

\[
\delta \approx \alpha^z - \alpha + 1 \quad \ldots (2.43)
\]

For very large values of \( z \), \( \delta = \alpha^z - \alpha + c \) where \( c<1 \). So we can infer the inequality

\[
\alpha^z > \delta > \alpha^z - \alpha \quad \ldots (2.44)
\]

These limiting expressions were derived earlier
using entirely different arguments [55].

The values of \( \delta \) for different maps computed
using equation (2.41) are shown along with numerical
values in Fig. 2.2. We observe that the computed
values agree well with numerical values as is evident.
Fig. 2.2 - The $\delta$ values of maps with different $z$ values. The triangles correspond to values obtained using (2.41) while circles indicate numerical values.
from Table 2-II.

In the next chapter, we apply the above perturbative scheme to specific cases viz. quadratic, cubic and quartic maps. For non-polynomial maps, we present an expansion in terms of a small parameter $\epsilon$. 
Table 2.II - Computed values of $\delta$ using (2.41). The numerical values from [51,55,58] are included.

<table>
<thead>
<tr>
<th>$z$</th>
<th>Computed values of $\delta$</th>
<th>Numerical $\delta$-values</th>
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</thead>
<tbody>
<tr>
<td>1.1</td>
<td>2.818065</td>
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<td>1.2</td>
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<td>2.4</td>
<td>5.322243</td>
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<td>12.350160</td>
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<td>27.75</td>
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