SYNOPSIS

The thesis work is mainly centered on the asymptotic behaviour of nonlinear and nonintegrable dissipative dynamical systems. It is found that completely deterministic nonlinear differential equations describing such systems can exhibit random or chaotic behaviour. Theoretical studies on this chaotic behaviour can enhance our understanding of various phenomena such as turbulence, nonlinear electronic circuits, erratic behaviour of heart and brain, fundamental molecular reactions involving DNA, meteorological phenomena, fluctuations in the cost of materials and so on.

Chaos is studied mainly under two different approaches - the nature of the onset of chaos and the statistical description of the chaotic state. Successive period doubling bifurcations constitute one of the most common mechanisms for the onset of chaos in many systems representing a wide variety of physical phenomena. Irrespective of the differences
in the underlying physics, such systems are found to have some common behaviour as far as onset of chaos is considered. This universal behaviour was reported by M.J. Feigenbaum in the case of one dimensional maps that represent the Poincare' sections of higher dimensional flows [26]. He found that the period doubling route to chaos is characterised by two universal constants, the bifurcation rate $\delta$ and the scaling factor $\alpha$. The computation of $\alpha$ and $\delta$ for any map is usually done numerically [51].

We have developed an analytic algorithm involving a perturbative scheme, which provides fairly accurate values for $\alpha$ and $\delta$. This is given in the first section of the thesis. We consider maps of the form $x_{n+1} = 1 - \lambda |x_n|^z$, where $z$ is the order of the local maximum of the map. The universality theory is based on two renormalisation group (RG) equations for the fixed point function $g(x)$ at the accumulation point $\lambda_\infty$ of the period-doubling bifurcations [49]. Expanding $g(x)$ into a general power series in $|x|^z$ and substituting in one of the RG equations, an infinite set of coupled nonlinear
equations for the coefficients of expansion $S_n$ are obtained. To solve them, $S_n$ are expanded into a perturbative series in inverse powers of $\alpha$. The equations can then be successively solved for the coefficients $S_{nm}$. Thus $\alpha$ as well as $g(x)$ can be expressed explicitly in terms of $S_{nm}$ and therefore computed analytically. Using the other RG equation, $\delta$ can be computed in a similar way. We find that the equations for $\delta$ can be cast in the form of a matrix eigenvalue equation and the highest positive and real eigenvalue furnishes $\delta$.

In general for any $z$ value, the first three coefficients $S_{nm}$ can be written down explicitly. Using them and using the technique of Padé approximants to sum the series, we get $\alpha(z)$ and $\delta(z)$ for any $z$ value. These equations define the different universality classes and a universal relation connecting $\alpha$, $\delta$ and $z$ is also derived.

For a given $z$, it is possible to carry out the calculations to any order of accuracy. Thus for a quartic map and a cubic map, we computed $\alpha$ and $\delta$ to different orders and found that the agreement with the numerical values is excellent. However, the perturbation series is not highly convergent but
asymptotic in nature and therefore the truncation of
the series is crucial in giving good results.

When \( z \) is not an integer, the above method is
rather cumbersome to apply. In such cases we find
that the general method can be side stepped and the
expressions for \( \alpha \) and \( \delta \) can be expanded in \( \epsilon \), where
\( 0 < |\epsilon| < 1 \). Any noninteger \( z \) value can be written as
\( z = z' + \epsilon \), where \( z' \) is the integer nearest to \( z \).
Then the \( \epsilon \)-expansions can be used to compute \( \alpha \) and \( \delta \).
We observe that this expansion procedure does not
affect the accuracy of the values much. Using the
iterates of the computed universal function and some
scaling arguments, we have derived expressions for the
important dimensions characterising the attractor at
\( \chi_\infty \) in one dimensional maps [31]. The agreement with
available numerical values is good. The behaviour of
\( D_0 \) and \( D_1 \) with \( z \) is different for large \( z \) values. \( D_1(z) \)
shows a dip as \( z \) increases while \( D_0(z) \) shows
saturation.

The second section of the thesis deals with
investigations on the onset of stochastic behaviour in
a driven pendulum with van der Pol like dissipation.
Such a system with some modifications can model nonlinear
electronic circuits, Josephson junction with interference of tunnelling currents, and laser systems. Predicting the exact transition point where chaos sets in, in such systems is still a challenging problem. Till now, the only analytic method available for this has been the Melnikov criterion [4]. The Melnikov analysis gives the distance between the stable and unstable manifolds of the perturbed system based on calculations involving trajectories of the unperturbed system. We have derived this function for the above system, followed by a detailed numerical analysis including phase portraits, Poincaré sections and power spectra.

The system we considered has essentially three control parameters, the damping constant $\beta$, the driving amplitude $A$ and the driving frequency $\omega$. In our numerical computations, we mostly kept $\beta$ at 0.2. Then the behaviour of the system as $A$ is varied was studied at three different frequencies. We found the following interesting results.

i) At very low frequencies i.e $\omega = 0.04$, the transition to chaos is quite obvious. Just as the Melnikov criterion is crossed, the invariant curve in phase space loses its smoothness. As $A$ is increased, it develops distortions and twists and finally reaches the strange attractor.
ii) At medium frequencies, $\omega = 0.4$, we found that as $A$ increases, the limit cycle develops a stochastic band of increasing thickness.

iii) At frequencies $\omega \approx 1$, the behaviour is qualitatively similar. However as $A$ increases, the stochastic band splits again into periodic trajectories.

iv) Because of the nature of the dissipation, the centre $(0,0)$ of the perturbed system is no longer stable. Numerical studies reveal that the centre becomes unstable through a bifurcation and for sufficient perturbations, orbits near $(0,0)$ spiral away to the limit cycle.

The effect of an additive white noise in the above system is interesting from a practical point of view [86]. We made a numerical analysis of the influence of noise especially near the transition point. We find that for low noise amplitudes, the presence of noise smooths out the stochasticity to some extent although the approach to chaos is accelerated by noise.