Chapter 6

Simultaneous Iterative Methods for Split Equality Fixed Point Problems in Banach Spaces

6.1 Introduction

During the recent past, several split type problems, namely, Moudafi's split feasibility problem [131], new implicit feasibility null-point problem [133], generalized split feasibility problem [173], split common fixed point problem [49], split common null point problem [33], etc., have been investigated and analyzed because of their applications in different areas of science, engineering, management and medical sciences. Since Hilbert spaces possess nice geometrical properties, several iterative methods for these problems have been investigated in the setting of Hilbert spaces. However, there are few papers on the iteration of these problems in the setting of Banach spaces.

In this chapter, we consider the following split equality fixed point problem (in short, SEFPP) in the setting of Banach spaces.

Let X_1, X_2 and X_3 be uniformly convex and uniformly smooth real Banach spaces, $A : X_1 \to X_3$ and $B : X_2 \to X_3$ be bounded linear operators. The SEFPP is defined as follows:

$$\text{Find } x \in \text{Fix}(T) \text{ and } y \in \text{Fix}(S) \quad \text{such that} \quad Ax = By, \tag{6.1.1}$$

where $T : X_1 \to X_1$ and $S : X_2 \to X_2$ be (nonlinear) mappings such that $\text{Fix}(T) \neq \emptyset$ and $\text{Fix}(S) \neq \emptyset$. We denote by Θ the set of solutions of SEFPP (6.1.1), and assume that $\Theta \neq \emptyset$.

When X_1, X_2 and X_3 are Hilbert spaces, SEFPP (6.1.1) is considered and studied by Moudafi and Shemas [134] for firmly nonexpansive mappings. Further, this problem is studied for quasi-nonexpansive mappings in [58, 199], for demicontractive mappings in [62] and for Lipschitz hemi-contractive mappings in [142] in the setting of Hilbert spaces.

Furthermore, if $B \equiv I$ is the identity mapping, then SEFPP (6.1.1) reduces to the following split common fixed point problem (in short, SCFPP):

$$\text{Find } x \in \text{Fix}(T) \text{ such that } Ax \in \text{Fix}(S). \tag{6.1.2}$$

This problem extensively studied in the literature, see, for example, [37, 49, 56, 95, 128, 129, 194] and the references therein. Very recently, SCFPP (6.1.2) is studied in [177] in the setting of Banach spaces.

In this chapter, we develop simultaneous iterative algorithms for solving SEFPP (6.1.1) and study the weak convergence of the sequences generated by the proposed algorithms either for firmly nonexpansive type mappings or for relative nonexpansive mappings. Further, we apply our results for some well known nonlinear problems. The algorithms and results of this chapter improve and generalize several known results in the setting of Hilbert spaces.

6.2 Algorithms and Convergence Results

Rest of the chapter, unless otherwise specified, we assume that X_1, X_2 and X_3 are uniformly convex and 2-uniformly smooth real Banach spaces having smoothness constant κ satisfying $0 < \kappa \leq \frac{1}{\sqrt{2}}$, $A : X_1 \to X_3$ and $B : X_2 \to X_3$ are bounded linear operators and J_1, J_2 and J_3 are duality mappings on X_1, X_2 and X_3, respectively, such that J_1 and J_2 are weakly sequentially continuous.

We propose the following simultaneous iterative algorithm for computing the approximate solutions of SEFPP (6.1.1).

Algorithm 6.2.1. Initialization: *Take arbitrary $x_1 \in X_1$ and $y_1 \in X_2$.*
Iterative Steps: *For a given current $x_n \in X_1$, $y_n \in X_2$ compute*

$$x_{n+1} = T\left(J_1^{-1}\left(J_1 x_n - \gamma A^* J_3(Ax_n - By_n)\right)\right), \tag{6.2.1a}$$

$$y_{n+1} = S\left(J_2^{-1}\left(J_2 y_n + \gamma B^* J_3(Ax_n - By_n)\right)\right), \; n \in \mathbb{N}, \tag{6.2.1b}$$

where $\gamma \in \left(0, \min\left(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\right)\right)$.
Last Step: *Update $n := n + 1$.*

Next we prove the weak convergence of the sequences generated by Algorithm 6.2.1.

Theorem 6.2.1. *Let $T : X_1 \to X_1$ and $S : X_2 \to X_2$ be firmly nonexpansive type mappings such that* $\mathrm{Fix}(T) \neq \emptyset$ *and* $\mathrm{Fix}(S) \neq \emptyset$. *Then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm 6.2.1 converges weakly to an element* $(x, y) \in \Theta$.

Proof. Let $(p, q) \in \Theta$. Then $p \in \mathrm{Fix}(T)$, $q \in \mathrm{Fix}(S)$ and $Ap = Bq$. Let $u_n = J_1^{-1}(J_1 x_n - \gamma A^* J_3(Ax_n - By_n))$, then by Lemma 2.4.6, we have

$$\begin{aligned}
&\phi(p, u_n) \\
&= \phi(p, J_1^{-1}(J_1 x_n - \gamma A^* J_3(Ax_n - By_n)) \\
&= \|p\|^2 - 2\langle p, J_1 x_n - \gamma A^* J_3(Ax_n - By_n)\rangle + \|J_1 x_n - \gamma A^* J_3(Ax_n - By_n)\|^2 \\
&= \|p\|^2 - 2\langle p, J_1 x_n - \gamma A^* J_3(Ax_n - By_n)\rangle + \|x_n - \gamma J_1^{-1} A^* J_3(Ax_n - By_n)\|^2 \\
&\leq \|p\|^2 - 2\langle p, J_1 x_n\rangle + 2\gamma\langle Ap, J_3(Ax_n - By_n)\rangle \\
&\quad + 2\|kx_n\|^2 + \|\gamma J_1^{-1} A^* J_3(Ax_n - By_n)\|^2 - 2\langle x_n, \gamma A^* J_3(Ax_n - By_n)\rangle \\
&\leq \|p\|^2 - 2\langle p, J_1 x_n\rangle + 2\gamma\langle Ap, J_3(Ax_n - By_n)\rangle \\
&\quad + \|x_n\|^2 + \gamma^2\|A\|^2\|Ax_n - By_n\|^2 - 2\gamma\langle Ax_n, J_3(Ax_n - By_n)\rangle \\
&= \phi(p, x_n) - 2\gamma\langle Ax_n - Ap, J_3(Ax_n - By_n)\rangle + \gamma^2\|A\|^2\|Ax_n - By_n\|^2. \qquad (6.2.2)
\end{aligned}$$

Similarly, let $v_n = J_2^{-1}(J_2 y_n + \gamma B^* J_3(Ax_n - By_n))$, we have

$$\phi(q, v_n) \leq \phi(q, y_n) + 2\gamma\langle By_n - Bq, J_3(Ax_n - By_n)\rangle + \gamma^2\|B\|^2\|Ax_n - By_n\|^2. \quad (6.2.3)$$

From (6.2.1a), in view of inequality (6.2.2), we have

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, Tu_n) \\
&\leq \phi(p, u_n) \\
&\leq \phi(p, x_n) - 2\gamma\langle Ax_n - Ap, J_3(Ax_n - By_n)\rangle \\
&\quad + \gamma^2\|A\|^2\|Ax_n - By_n\|^2. \qquad (6.2.4)
\end{aligned}$$

Similarly, in view of (6.2.3), we have

$$\begin{aligned}
\phi(q, y_{n+1}) &= \phi(q, Sv_n) \\
&\leq \phi(q, v_n) \\
&\leq \phi(q, y_n) + 2\gamma\langle By_n - Bq, J_3(Ax_n - By_n)\rangle \\
&\quad + \gamma^2\|B\|^2\|Ax_n - By_n\|^2. \qquad (6.2.5)
\end{aligned}$$

By adding inequalities (6.2.4) and (6.2.5) and with the help of (6.2.2) and (6.2.3), we have

$$\begin{aligned}
&\phi(p, x_{n+1}) + \phi(q, y_{n+1}) \\
&\leq \phi(p, x_n) - 2\gamma\langle Ax_n - Ap, J_3(Ax_n - By_n)\rangle \\
&\quad + \gamma^2\|A\|^2\|Ax_n - By_n\|^2 + \phi(q, y_n) \\
&\quad + 2\gamma\langle By_n - Bq, J_3(Ax_n - By_n)\rangle + \gamma^2\|B\|^2\|Ax_n - By_n\|^2 \\
&\leq \phi(p, x_n) - 2\gamma\|Ax_n - By_n\|^2 + \gamma^2\|A\|^2\|Ax_n - By_n\|^2 \\
&\quad + \phi(q, y_n) + \gamma^2\|B\|^2\|Ax_n - By_n\|^2 \\
&\leq \phi(p, x_n) + \phi(q, y_n) - \gamma(2 - \gamma(\|A\|^2 + \|B\|^2))\|Ax_n - By_n\|^2.
\end{aligned} \tag{6.2.6}$$

Let k be the minimum value of $\left(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\right)$. Since $\gamma \in \left(0, \min\left(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\right)\right)$, we have $k \leq \frac{1}{\|A\|^2}$ and $k \leq \frac{1}{\|B\|^2}$. Thus, $0 < \gamma < k$, and therefore, $0 < \gamma < \frac{1}{\|A\|^2}$ and $0 < \gamma < \frac{1}{\|B\|^2}$. Thus, $0 < \gamma\|A\|^2 < 1$ and $0 < \gamma\|B\|^2 < 1$, $0 < \gamma(\|A\|^2 + \|B\|^2) < 2$. So, we have $\gamma(2 - \gamma(\|A\|^2 + \|B\|^2)) > 0$.

Set $\Upsilon_n(p, q) = \phi(p, x_n) + \phi(q, y_n)$. Thus,

$$\Upsilon_{n+1}(p, q) \leq \Upsilon_n(p, q) - \gamma\{2 - \gamma(\|A\|^2 + \|B\|^2)\}\|Ax_n - By_n\|^2 \tag{6.2.7}$$

$$\leq \Upsilon_n(p, q). \tag{6.2.8}$$

Since $\phi(p, x_n) \geq 0$ and $\phi(q, y_n) \geq 0$, we have $\Upsilon_n(p, q) \geq 0$, and from (6.2.8), it is monotonic decreasing sequence. Therefore, $\lim_{n\to\infty} \Upsilon_n(p, q)$ exist. This further implies that $\lim_{n\to\infty} \phi(p, x_n)$ and $\lim_{n\to\infty} \phi(q, y_n)$ exist. Passing to the limit in (6.2.7), knowing that fact that $\gamma(2 - \gamma(\|A\|^2 + \|B\|^2)) > 0$, we have

$$\lim_{n\to\infty} \|Ax_n - By_n\| = 0. \tag{6.2.9}$$

Also, existence of the limit of the sequences $\{\phi(p, x_n)\}$ and $\{\phi(q, y_n)\}$ implies their boundedness. In view of inequality (2.4.3), we have the boundedness of $\{x_n\}$ and $\{y_n\}$. Since X_1 and X_2 are uniformly convex, they are reflexive [61, Milman-Pettis'theorem, Theorem 1.17]. Therefore, X_1 and X_2 are reflexive and by the boundedness of $\{x_n\}$ and $\{y_n\}$, there exists subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{y_{n_i}\}$ of $\{y_n\}$ such that $x_{n_i} \rightharpoonup x \in X_1$ and $y_{n_i} \rightharpoonup y \in X_2$ (see [7]).

Now we show that $x_n \rightharpoonup x$. In order to show this, we have to show that every subsequence of x_n converges weakly to x.

Assume to the contrary that there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in X_1$, where $x \neq z$. Since J_1 is weakly sequentially continuous, so

$J_1 x_{n_i} \overset{*}{\rightharpoonup} J_1 x$ and $J_1 x_{n_j} \overset{*}{\rightharpoonup} J_1 z$,

$$\begin{aligned}\langle z - x, J_1 x\rangle &= \lim_{i\to\infty} \langle z - x, J_1 x_{n_i}\rangle \\ &= \lim_{n\to\infty} \langle z - x, J_1 x_n\rangle \\ &= \lim_{j\to\infty} \langle z - x, J_1 x_{n_j}\rangle = \langle z - x, J_1 z\rangle.\end{aligned}$$

Thus, we obtain $\langle x - z, J_1 x - J_1 z\rangle = 0$. Since X_1 is uniformly convex, it is strictly convex, and hence by Lemma 2.4.4, we have $x = z$. Thus, we have shown that every subsequence of $\{x_n\}$ converges weakly to x. This implies that $x_n \rightharpoonup x$. On the same lines and weakly sequential continuity of J_2, we can say that there exists $y \in X_2$ such that $y_n \rightharpoonup y$.

Now we show that $Tx = x$ and $Sy = y$. In view of inequality (6.2.9) and (6.2.2), we obtain

$$\lim_{n\to\infty} \phi(p, u_n) \le \lim_{n\to\infty} \phi(p, x_n). \tag{6.2.10}$$

Since T is firmly nonexpansive type mapping and has a fixed point p, therefore strongly relative nonexpansive [110, Theorem 5.2], and from inequality (6.2.10), we have

$$\begin{aligned}0 &\le \phi(p, u_n) - \phi(p, Tu_n) \\ 0 &\le \lim_{n\to\infty} (\phi(p, u_n) - \phi(p, Tu_n)) \\ &= \lim_{n\to\infty} \phi(p, u_n) - \lim_{n\to\infty} \phi(p, Tu_n) \\ &\le \lim_{n\to\infty} \phi(p, x_n) - \lim_{n\to\infty} \phi(p, x_{n+1}) = 0.\end{aligned} \tag{6.2.11}$$

From (6.2.10), $\{\phi(p, u_n)\}$ is bounded, therefore, in view of (2.4.3), we have the boundedness of $\{u_n\}$. Since T is firmly nonexpansive type mapping having a fixed point, by [110, Theorem 5.2], it is strongly relative nonexpansive. Therefore, by inequality (6.2.11), we get

$$\lim_{n\to\infty} \phi(Tu_n, u_n) = 0. \tag{6.2.12}$$

Thus by inequality (6.2.12) and Lemma 2.4.8, we have

$$\lim_{n\to\infty} \|Tu_n - u_n\| = 0. \tag{6.2.13}$$

Similarly, by using (6.2.9) in (6.2.3), we obtain $\lim_{n\to\infty} \phi(q, v_n) \le \lim_{n\to\infty} \phi(q, y_n)$. Further, in view of boundedness of $\{\phi(q, v_n)\}$, we have the boundedness of $\{v_n\}$. Thus by strongly relative nonexpansiveness of S, we have

$$\lim_{n\to\infty} \phi(Sv_n, v_n) = 0. \tag{6.2.14}$$

Thus by inequality (6.2.14) and Lemma 2.4.8, we have

$$\lim_{n\to\infty} \|Sv_n - v_n\| = 0. \tag{6.2.15}$$

From equation (6.2.1a), we have

$$\begin{aligned}
&\phi(x_n, u_n)\\
&= \phi(x_n, J_1^{-1}(J_1x_n - \gamma A^* J_3(Ax_n - By_n))\\
&= \|x_n\|^2 - 2\langle x_n, J_1x_n - \gamma A^* J_3(Ax_n - By_n)\rangle + \|J_1x_n - \gamma A^* J_3(Ax_n - By_n)\|^2\\
&\leq \|x_n\|^2 - 2\langle x_n, J_1x_n\rangle + 2\gamma\langle Ax_n, J_3(Ax_n - By_n)\rangle\\
&\quad + \|x_n\|^2 + \gamma^2\|A\|^2\|Ax_n - By_n\|^2 - 2\gamma\langle Ax_n, J_3(Ax_n - By_n)\rangle\\
&= \phi(x_n, x_n) + \gamma^2\|A\|^2\|Ax_n - By_n\|^2.
\end{aligned} \tag{6.2.16}$$

In view of (2.4.4) and (6.2.9), we have

$$\lim_{n\to\infty} \phi(x_n, u_n) = 0. \tag{6.2.17}$$

Similarly, we can have

$$\lim_{n\to\infty} \phi(y_n, v_n) = 0. \tag{6.2.18}$$

Further, boundedness of $\{x_n\}$ and $\{y_n\}$, inequality (6.2.17) and (6.2.18) and Lemma 2.4.8 imply that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0, \tag{6.2.19}$$

and

$$\lim_{n\to\infty} \|y_n - v_n\| = 0. \tag{6.2.20}$$

Since $x_n \rightharpoonup x$, so in view of inequality (6.2.19), we have $u_n \rightharpoonup x$ as follow for all $f \in X_1^*$,

$$\begin{aligned}
\|f(u_n) - f(x)\| &= \|f(u_n) - f(x_n) + f(x_n) - f(x)\|\\
&\leq \|f(u_n) - f(x_n)\| + \|f(x_n) - f(x)\|\\
&\leq \|f\|\|u_n - x_n\| + \|f(x_n) - f(x)\| \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{6.2.21}$$

Thus, $u_n \rightharpoonup x$. Similarly, by the inequality (6.2.20) and $y_n \rightharpoonup y$, we have $v_n \rightharpoonup y$. Therefore, by the inequalities (6.2.13) and (6.2.15), the weak convergence of $u_n \rightharpoonup x$ and $v_n \rightharpoonup y$ and by the relative nonexpansiveness of T and S, we have $Tx = x$ and $Sy = y$. Thus, we have shown that $x_n \rightharpoonup x$, $y_n \rightharpoonup y$, $Tx = x$ and $Sy = y$.

Now we are left to show that $Ax = By$. Since $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ and since $A : X_1 \to X_3$ and $B : X_2 \to X_3$ are bounded linear operators, we have $Ax_n \rightharpoonup Ax$ and $By_n \rightharpoonup By$.

Indeed, for all $f \in X_3^*$,

$$\|fAx_n - fAx\| \to 0 \quad \text{as} \quad n \to \infty, \tag{6.2.22}$$

because $x_n \rightharpoonup x$ and $fA \in X_1^*$. Thus in view of (6.2.22), we have $Ax_n \rightharpoonup Ax$. Similarly, we can have $By_n \rightharpoonup By$. Thus $Ax_n - By_n \rightharpoonup Ax - By$, therefore by lower semicontinuity of squared normed, we have

$$\|Ax - By\|^2 \leq \liminf_{n\to\infty} \|Ax_n - By_n\|^2 = \lim_{n\to\infty} \|Ax_n - By_n\|^2 = 0.$$

Thus, $Ax = By$. This completes the proof. □

Remark 6.2.1. If X_1, X_2 and X_3 are Hilbert spaces, then Algorithm 6.2.1 studied in [134]. Also, Theorem 6.2.1 is the extension of Theorem 2.1 in [134] to Banach space setting.

Observe that every firmly nonexpansive type mapping having a fixed point is relative-nonexpansive (see [110, Theorem 5.2]), but converse need not be true (see, [137, Example 3.1]).

We now propose the following simultaneous iterative method for relative nonexpansive mappings T and S to solve SEFPP (6.1.1).

Algorithm 6.2.2. Initialization: *Choose* $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$. *Take arbitrary* $x_1 \in X_1$ *and* $y_1 \in X_2$.
Iterative Steps: *For a given current* $x_n \in X_1$, $y_n \in X_2$ *compute*

$$\begin{aligned}
u_n &= J_1^{-1}(J_1x_n - \gamma_n A^* J_3(Ax_n - By_n)), && (6.2.23a)\\
x_{n+1} &= J_1^{-1}(\alpha_n J_1 x_n + (1-\alpha_n) J_1 T u_n), && (6.2.23b)\\
v_n &= J_2^{-1}(J_2 y_n + \gamma_n B^* J_3(Ax_n - By_n)), && (6.2.23c)\\
y_{n+1} &= J_2^{-1}(\alpha_n J_2 y_n + (1-\alpha_n) J_2 S v_n), \quad n \in \mathbb{N}, && (6.2.23d)
\end{aligned}$$

where $\gamma_n \in (0,k)$, $k = \min\left(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\right)$ *such that* $0 < a \leq \gamma_n \leq b < k$ *and* $0 < c \leq \alpha_n \leq d < 1$.
Last Step: *Update* $n := n+1$.

When X_1, X_2 and X_3 are Hilbert spaces, then Algorithm 6.2.2 is studied in [199].

Next we prove the weak convergence of the sequences generated by Algorithm 6.2.2.

Theorem 6.2.2. *Let* $T : X_1 \to X_1$ *and* $S : X_2 \to X_2$ *be relative nonexpansive mappings. Then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm 6.2.2 converges weakly to an element* $(x,y) \in \Theta$.

Proof. Let $(p,q) \in \Theta$. Then $p \in \mathrm{Fix}(T)$, $q \in \mathrm{Fix}(S)$ and $Ap = Bq$. As in the beginning of the proof of Theorem 6.2.1, we have

$$\phi(p,u_n) \leq \phi(p,x_n) - 2\gamma_n\langle Ax_n - Ap, J_3(Ax_n - By_n)\rangle + \gamma_n^2\|A\|^2\|Ax_n - By_n\|^2, \quad (6.2.24)$$

and

$$\phi(q,v_n) \leq \phi(q,y_n) + 2\gamma_n\langle By_n - Bq, J_3(Ax_n - By_n)\rangle + \gamma_n^2\|B\|^2\|Ax_n - By_n\|^2. \quad (6.2.25)$$

From (6.2.23b), in view of inequality (6.2.24) and relative-nonexpansiveness of T, we have

$$\begin{aligned}
&\phi(p,x_{n+1})\\
&= \phi(p, J_1^{-1}(\alpha_n J_1 x_n + (1-\alpha_n)J_1 T u_n)\\
&\leq \|p\|^2 - 2\langle p, \alpha_n J_1 x_n + (1-\alpha_n)J_1 T u_n\rangle + \|\alpha_n J_1 x_n + (1-\alpha_n)J_1 T u_n\|^2\\
&\leq \|p\|^2 - 2\langle p, \alpha_n J_1 x_n\rangle - 2\langle p, (1-\alpha_n)J_1 T u_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Tu_n\|^2\\
&\leq \alpha_n(\|p\|^2 - 2\langle p, J_1 x_n\rangle + \|x_n\|^2) + (1-\alpha_n)(\|p\|^2 - 2\langle p, J_1 T u_n\rangle + \|Tu_n\|^2)\\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p,Tu_n)\\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p,u_n)\\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\left(\phi(p,x_n) - 2\gamma_n\langle Ax_n - Ap, J_3(Ax_n - By_n)\rangle\right)\\
&\quad + (1-\alpha_n)\gamma_n^2\|A\|^2\|Ax_n - By_n\|^2\\
&\leq \phi(p,x_n) - 2(1-\alpha_n)\gamma_n\langle Ax_n - Ap, J_3(Ax_n - By_n)\rangle\\
&\quad + (1-\alpha_n)\gamma_n^2\|A\|^2\|Ax_n - By_n\|^2. \qquad (6.2.26)
\end{aligned}$$

Similarly, in view of (6.2.25), relative nonexpansiveness of S and (6.2.23d), we get

$$\begin{aligned}
\phi(q,y_{n+1}) \leq \phi(q,y_n) &+ 2(1-\alpha_n)\gamma_n\langle By_n - Bq, J_3(Ax_n - By_n)\rangle\\
&+ (1-\alpha_n)\gamma_n^2\|B\|^2\|Ax_n - By_n\|^2. \qquad (6.2.27)
\end{aligned}$$

By adding inequalities (6.2.26) and (6.2.27), and with the help of (6.2.24) and (6.2.25), we obtain

$$\begin{aligned}
&\phi(p,x_{n+1}) + \phi(q,y_{n+1})\\
&\leq \phi(p,x_n) + \phi(q,y_n) - 2(1-\alpha_n)\gamma_n\langle Ax_n - Ap, J_3(Ax_n - By_n)\rangle\\
&\quad + (1-\alpha_n)\gamma_n^2\|A\|^2\|Ax_n - By_n\|^2 + 2(1-\alpha_n)\gamma_n\langle By_n - Bq, J_3(Ax_n - By_n)\rangle\\
&\quad + (1-\alpha_n)\gamma_n^2\|B\|^2\|Ax_n - By_n\|^2\\
&\leq \phi(p,x_n) + \phi(q,y_n) + 2(1-\alpha_n)\gamma_n\langle By_n - Ax_n + Ap - Bq, J_3(Ax_n - By_n)\rangle\\
&\quad + (1-\alpha_n)\gamma_n^2\|A\|^2\|Ax_n - By_n\|^2 + (1-\alpha_n)\gamma_n^2\|B\|^2\|Ax_n - By_n\|^2\\
&\leq \phi(p,x_n) + \phi(q,y_n) - 2(1-\alpha_n)\gamma_n\|Ax_n - By_n\|\\
&\quad + (1-\alpha_n)\gamma_n^2\|A\|^2\|Ax_n - By_n\|^2 + (1-\alpha_n)\gamma_n^2\|B\|^2\|Ax_n - By_n\|^2\\
&\leq \phi(p,x_n) + \phi(q,y_n) - (1-\alpha_n)\gamma_n\{2 - \gamma_n(\|A\|^2 + \|B\|^2)\}\|Ax_n - By_n\|^2. \quad (6.2.28)
\end{aligned}$$

As in the proof of Theorem 6.2.1, $x_n \rightharpoonup x \in X_1$ and $y_n \rightharpoonup y \in X_2$.

Now we show that $Tx = x$ and $Sy = y$. Since by relative nonexpansiveness of T and S, we have

$$\phi(p, Tu_n) \leq (p, u_n) \quad \text{and} \quad \phi(q, Sv_n) \leq (q, v_n).$$

So, by the boundedness of $\phi(p, x_n)$, $\phi(q, y_n)$ and (6.2.9), we have the boundedness of $\phi(p, Tu_n)$ and $\phi(q, Sv_n)$, respectively, which in turn implies the boundedness of $\{Tu_n\}$ and $\{Sv_n\}$. Let $r_1 = \sup_{n\geq 1}\{x_n, Tu_n\}$. Then, by Lemma 2.4.3 and inequality (6.2.23b), we obtain

$$\begin{aligned}
&\phi(p, x_{n+1}) \\
&= \phi(p, J_1^{-1}(\alpha_n J_1 x_n + (1-\alpha_n) J_1 T u_n) \\
&\leq \|p\|^2 - 2\langle p, \alpha_n J_1 x_n + (1-\alpha_n) J_1 T u_n\rangle + \|\alpha_n J_1 x_n + (1-\alpha_n) J_1 T u_n\|^2 \\
&\leq \|p\|^2 - 2\langle p, \alpha_n J_1 x_n\rangle + 2\langle p, (1-\alpha_n) J_1 T u_n\rangle + \alpha_n \|x_n\|^2 + (1-\alpha_n)\|Tu_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)(g(\|J_1 x_n - J_1 T u_n\|)) \\
&\leq \alpha_n(\|p\|^2 - 2\langle p, J_1 x_n\rangle + \|x_n\|^2) + (1-\alpha_n)(\|p\|^2 - 2\langle p, J_1 T u_n\rangle + \|Tu_n\|^2) \\
&\quad - \alpha_n(1-\alpha_n)(g(\|J_1 x_n - J_1 T u_n\|)) \\
&\leq \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, Tu_n) - \alpha_n(1-\alpha_n)(g(\|J_1 x_n - J_1 T u_n\|)) \\
&\leq \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, u_n) - \alpha_n(1-\alpha_n)(g(\|J_1 x_n - J_1 T u_n\|)) \\
&\leq \alpha_n \phi(p, x_n) + (1-\alpha_n)\left(\phi(p, x_n) - 2\gamma_n\langle Ax_n - Ap, J_3(Ax_n - By_n)\rangle\right) \\
&\quad + (1-\alpha_n)\gamma_n^2\|A\|^2\|Ax_n - By_n\|^2 - \alpha_n(1-\alpha_n)(g(\|J_1 x_n - J_1 T u_n\|)) \\
&\leq \phi(p, x_n) - 2(1-\alpha_n)\gamma_n\langle Ax_n - Ap, J_3(Ax_n - By_n)\rangle \\
&\quad + (1-\alpha_n)\gamma_n^2\|A\|^2\|Ax_n - By_n\|^2 - \alpha_n(1-\alpha_n)(g(\|J_1 x_n - J_1 T u_n\|)). \qquad (6.2.29)
\end{aligned}$$

Since $\{Ax_n - By_n\} \to 0$ as $n \to \infty$, $\gamma_n \in (0, k)$ such that $0 < a \leq \gamma_n \leq b < k$ and $0 < c \leq \alpha_n \leq d < 1$ and by the existence of the limit of the sequence $\phi(p, x_n)$, we have

$$\lim_{n\to\infty} g(\|J_1 x_n - J_1 T u_n\|) = 0. \qquad (6.2.30)$$

Since $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$, we have

$$\lim_{n\to\infty} \|J_1 x_n - J_1 T u_n\| = 0. \qquad (6.2.31)$$

Since X_1 is uniformly convex and uniformly smooth, it is smooth, strictly convex and reflexive Banach space and J_1 is a single-valued bijection mapping. In this case, the duality mapping ${J_1}^*$ from X_1^* onto $X_1^{**} = X_1$ coincides with the inverse of the duality mapping J_1 from X_1 onto X_1^*, that is, ${J_1}^* = {J_1}^{-1}$. Since X_1 is uniformly convex, therefore X_1^* is uniformly smooth (see [61]). By uniformly smoothness of X_1^*, ${J_1}^{-1}$ is uniformly norm-to-norm continuous on bounded sets (see [61, 167]). Thus, we obtain

$$\lim_{n\to\infty} \|x_n - Tu_n\| = \lim_{n\to\infty} \left\| {J_1}^{-1}(J_1(x_n)) - {J_1}^{-1}(J_1(Tu_n)) \right\| = 0, \tag{6.2.32}$$

Similarly, we have

$$\lim_{n\to\infty} \|y_n - Sv_n\| = 0. \tag{6.2.33}$$

As in the proof of Theorem 6.2.1, we have

$$\lim_{n\to\infty} \phi(x_n, u_n) = 0, \tag{6.2.34}$$

and

$$\lim_{n\to\infty} \phi(y_n, v_n) = 0. \tag{6.2.35}$$

By Lemma 2.4.8, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0, \tag{6.2.36}$$

and

$$\lim_{n\to\infty} \|y_n - v_n\| = 0. \tag{6.2.37}$$

Thus, in view of (6.2.32) and (6.2.36), we have

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0. \tag{6.2.38}$$

Also, in view of (6.2.33) and (6.2.37), we have

$$\lim_{n\to\infty} \|v_n - Sv_n\| = 0. \tag{6.2.39}$$

Since $x_n \rightharpoonup x$, so in view of inequality (6.2.36), we have $u_n \rightharpoonup x$ as shown in the proof of Theorem 6.2.1.

Similarly by the inequality (6.2.37) and $y_n \rightharpoonup y$, we have $v_n \rightharpoonup y$. Therefore, by the inequalities (6.2.38) and (6.2.39), weak convergence of $u_n \rightharpoonup x$ and $v_n \rightharpoonup y$, and the relative nonexpansiveness of T and S, we have $Tx = x$ and $Sy = y$. Finally, as in the proof of Theorem 6.2.1, we have $Ax = By$. □

6.3 Some Related Problems

In this section, we present some well known split-type problems which are special cases of SEFPP (6.1.1) and can be solved by using Algorithm 6.2.1.

6.3.1 Split Equality Problems

Let C and Q be nonempty closed convex subsets of X_1 and X_2, respectively. Consider $T = \Pi_C$ and $S = \Pi_Q$, where Π_C and Π_Q are generalized projections onto C and Q, respectively. Then, we have $\text{Fix}(T) = C$ and $\text{Fix}(S) = Q$. Now, we can recover split equality problem (in short, SEP) in the setting of Banach spaces as follow:

$$\text{Find } x \in C \text{ and } Ax \in Q \text{ such that } Ax = By. \tag{6.3.1}$$

This problem is introduced and studied by Moudafi in [132] in setting of Hilbert spaces. We denote by Θ_1 the set of solutions of SEP (6.3.1).

When $B \equiv I$ and $X_2 = X_3$, SEP (6.3.1) is studied by Takahashi [169] and Schöpfer et al. [161].

Under the above setting, Algorithm 6.2.1 reduces to the following algorithm for finding the solutions of SEP (6.3.1).

Algorithm 6.3.1. Initialization: *Take arbitrary $x_1 \in X_1$ and $y_1 \in X_2$.*
Iterative Steps: *For a given current $x_n \in X_1$, $y_n \in X_2$ compute*

$$\begin{aligned} x_{n+1} &= \Pi_C(J_1^{-1}(J_1 x_n - \gamma A^* J_3(Ax_n - By_n))), \\ y_{n+1} &= \Pi_Q(J_2^{-1}(J_2 y_n + \gamma B^* J_3(Ax_n - By_n))), \quad n \in \mathbb{N}, \end{aligned}$$

where $\gamma \in \left(0, \min\left(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\right)\right)$.
Last Step: *Update $n := n + 1$.*

It is well-known that the generalized projections are firmly nonexpansive type (see [110, Lemma 2.4]). Thus, by taking $T = \Pi_C$ and $S = \Pi_Q$ in the proof of Theorem 6.2.1, we obtain the following weak convergence result for Algorithm 6.3.1.

Corollary 6.3.1. *Let $C \subseteq X_1$ and $Q \subseteq X_2$ be nonempty closed convex subsets of X_1 and X_2, respectively. Let $A : X_1 \to X_3$ and $B : X_2 \to X_3$ be bounded linear operators. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 6.3.1 converges weakly to an element $(x, y) \in \Theta_1$.*

6.3.2 Split Equality Null Point Problems

For a given maximal monotone operator $M : X_1 \rightrightarrows X_1^*$, it is well known that its associated resolvent mapping $J_\lambda^M = (J + \lambda M)^{-1} J$ is firmly nonexpansive type [110, Lemma 2.3] for $\lambda > 0$, and $0 \in M(x) \Leftrightarrow J_\lambda^M(x) = x$; See, for example, [102, 123]. This means that the zeroes of M are exactly fixed points of its resolvent mapping.

Let $T = J_\lambda^M$ and $S = J_\lambda^N$ where $N : X_2 \rightrightarrows X_2^*$ is a maximal monotone operator. We consider the following split equality null point problem (in short, SENPP) of finding

$$x \in M^{-1}(0) \text{ and } y \in N^{-1}(0) \quad \text{such that} \quad Ax = By. \tag{6.3.2}$$

The solution set of this problem is denoted by Θ_2.

Now we derive the following algorithm for solving SENPP (6.3.2) from Algorithm 6.2.1.

Algorithm 6.3.2. Initialization: *Take arbitrary $x_1 \in X_1$ and $y_1 \in X_2$.*
Iterative Steps: *For a given current $x_n \in X_1$, $y_n \in X_2$ compute*

$$\begin{aligned} x_{n+1} &= J_\lambda^M(J_1^{-1}(J_1 x_n - \gamma A^* J_3(Ax_n - By_n))), \\ y_{n+1} &= J_\lambda^N(J_2^{-1}(J_2 y_n + \gamma B^* J_3(Ax_n - By_n))), \quad n \in \mathbb{N}, \end{aligned}$$

where $\gamma \in \left(0, \min\left(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\right)\right)$.
Last Step: *Update $n := n + 1$.*

We can easily derive the following weak convergence result by taking $T = J_\lambda^M$ and $S = J_\lambda^N$ in the proof of Theorem 6.2.1 and following the idea that resolvent of maximal monotone operators are firmly nonexpansive type.

Corollary 6.3.2. *Let $M : X_1 \rightrightarrows X_1^*$ and $N : X_2 \rightrightarrows X_2^*$ be set-valued maximal monotone operators such that $M^{-1}(0) \neq \emptyset$ and $N^{-1}(0) \neq \emptyset$. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 6.3.2 converges weakly to an element $(x, y) \in \Theta_2$.*

In addition, if $B \equiv I$ and $X_2 = X_3$ in SENPP (6.3.2), then we have the following split common null point problem (in short, SCNPP) which is studied by Byrne et al. [33] in the setting of Hilbert spaces.

$$\text{Find } x \in M^{-1}(0) \quad \text{such that} \quad Ax \in N^{-1}(0). \tag{6.3.3}$$

It is further studied by Takahashi and Yao [170, 172, 175] in setting of Banach spaces. We denote by Θ_3 the solution set of SCNPP (6.3.3), that is,

$$\Theta_3 = \{(x, y) : x \in M^{-1}(0) \text{ and } Ax \in N^{-1}(0) \text{ such that } y = Ax\}.$$

Byrne et al. [33] showed that SCNPP (6.3.3) contains the split variational inequality problem, introduced and studied by Censor et al. [46].

Now we have simultaneous iterative method for solving SCNPP (6.3.3) as follows.

Algorithm 6.3.3. INITIALIZATION: *Take arbitrary* $x_1 \in X_1$ *and* $y_1 \in X_2$.
ITERATIVE STEPS: *For a given current* $x_n \in X_1$, $y_n \in X_2$ *compute*

$$\begin{aligned} x_{n+1} &= J_\lambda^M J_1^{-1}(J_1 x_n - \gamma A^* J_2(Ax_n - y_n)), \\ y_{n+1} &= J_\lambda^N J_2^{-1}(J_2 y_n + \gamma J_2(Ax_n - y_n)), \quad n \in \mathbb{N}, \end{aligned}$$

where $\gamma \in \left(0, \frac{1}{1+\|A\|^2}\right)$.
LAST STEP: *Update* $n := n + 1$.

Taking $T = J_\lambda^M$ and $S = J_\lambda^N$, using the idea that resolvent of maximal monotone operators are firmly nonexpansive type and by setting $B \equiv I$ and $X_2 = X_3$ in Theorem 6.2.1, we have the following weak convergence result for Algorithm 6.3.3.

Corollary 6.3.3. *Let* $M : X_1 \rightrightarrows X_1^*$ *and* $N : X_2 \rightrightarrows X_2^*$ *be set valued maximal monotone operators such that* $M^{-1}(0) \neq \emptyset$ *and* $N^{-1}(0) \neq \emptyset$. *Then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm 6.3.3 converges weakly to an element* $(x, y) \in \Theta_3$.

Since split variational inequality problem studied in [46] is a particular case of SCNPP (6.3.3), by using the above algorithm and weak convergence result, we can easily define the algorithm for finding the solutions of split variational inequality problems in the setting of Banach spaces. Also, we can derive the weak convergence of such algorithm.

6.3.3 Split Equality Equilibrium Problems

Let T_r^F and S_r^G denote the resolvents of bifunctions F and G, respectively, as defined in (2.7.2). It is well known that T_r^F is firmly nonexpansive type mapping and its fixed-points are exactly the equilibria of F, that is, $\mathrm{Fix}(T_r^F) = \mathrm{EP}(C, F)$. let C and Q be nonempty closed convex subsets of uniformly smooth uniformly convex Banach spaces X_1 and X_2, respectively. We consider the following split equality equilibrium problem (in short, SEEP):

$$\text{Find } x \in \mathrm{EP}(C, F) \text{ and } y \in \mathrm{EP}(Q, G) \quad \text{such that} \quad Ax = By. \tag{6.3.4}$$

Let Θ_4 denote the solution set of SEEP (6.3.4).

If $B \equiv I$ and $X_2 = X_3$, then SEEP (6.3.4) reduces to the split equilibrium problem considered and studied in [93] in the setting of Hilbert spaces.

We propose the following algorithm for solving SEEP (6.3.4).

Algorithm 6.3.4. Initialization: *Take arbitrary* $x_1 \in X_1$ *and* $y_1 \in X_2$.
Iterative Steps: *For a given current* $x_n \in X_1$, $y_n \in X_2$ *compute*

$$\begin{aligned} x_{n+1} &= T_r^F \left(J_1^{-1} \left(J_1 x_n - \gamma A^* J_3 (A x_n - B y_n) \right) \right), \\ y_{n+1} &= S_r^G \left(J_2^{-1} \left(J_2 y_n + \gamma B^* J_3 (A x_n - B y_n) \right) \right), \quad n \in \mathbb{N}, \end{aligned}$$

where $\gamma \in \left(0, \min\left(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\right)\right)$.
Last Step: *Update* $n := n + 1$.

Taking $T \equiv T_r^F$ and $S \equiv S_r^G$ in Theorem 6.2.1 and utilizing the fact that T_r^F and S_r^G are firmly nonexpansive type mappings, we have the following weak convergence result for Algorithm 6.3.4.

Corollary 6.3.4. *Let* C *and* Q *be nonempty closed convex subsets of* X_1 *and* X_2 *respectively. Let* $F : C \times C \to \mathbb{R}$ *and* $G : Q \times Q \to \mathbb{R}$ *be bifunctions satisfying (A1)–(A4). Then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm 6.3.4 converges weakly to an element* $(x, y) \in \Theta_4$.